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ON REPRESENTATION OF LINEAR OPERATORS ON $C_0(T, \boldsymbol{X})$

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INTRODUCTION

Let T be a locally compact Hausdorff topological space, let $\boldsymbol{X}$ be a Banach space and let $C_0(T, \boldsymbol{X})$ denote the Banach space of all $\boldsymbol{X}$ valued continuous functions on T tending to zero at infinity with the usual supremum norm. (Throughout the paper we shall suppose that all Banach spaces considered are either real or complex.) In this paper we shall investigate bounded linear operators on $C_0(T, \boldsymbol{X})$ by representing them as integrals with respect to Baire operator valued measures.

The paper is divided into six sections. § 1 is preparatory and collects results on operator valued measures needed for the subsequent parts.

Theorems 2, 2′ and 3 in § 2 are the basic representation theorems. We derive them from their known scalar versions, see VI.7.3. in [15] and Theorem 1 in [27]. Theorem 2 gives a necessary and sufficient condition for a bounded linear operator $\boldsymbol{U}$: : $C_0(T, \boldsymbol{X}) \to \boldsymbol{Y}$ to be expressed as an integral with respect to an $L(\boldsymbol{X}, \boldsymbol{Y})$ valued Baire measure countably additive in the strong operator topology of $L(\boldsymbol{X}, \boldsymbol{Y})$. For such measures extension of the Lebesgue integration theory is possible, see [13] and [37]. Theorem 3 states that for unconditionally converging operators (introduced in [27]), particularly for weakly compact operators $\boldsymbol{U}$ the semivariation $\hat{\boldsymbol{m}}$ of the representing measure $\boldsymbol{m}$ is continuous on the σ-ring of Baire measurable sets of T. This property is of great importance for the integration theory, since then $C_0(T, \boldsymbol{X})$ is a dense subset of $L_p(\boldsymbol{m})$, see [37]. At the same time the space $L_p(\boldsymbol{m})$ shares many good properties of the classical scalar L_p spaces, see [37].

In § 3 we show that various classes of our integrally expressed operators form closed left (in general only left) ideals in the algebra of all bounded linear operators over the space $C_0(T, \boldsymbol{X})$.

Using the results of [18] and Theorem 3 in § 4 we prove the norm equality $|1 + \boldsymbol{U}| = 1 + |\boldsymbol{U}|$ for every unconditionally converging, particularly for every weakly compact operator $\boldsymbol{U} : C_0(T, \boldsymbol{X}) \to C_0(T, \boldsymbol{X})$ (T contains no isolated points). This result was known for compact and majorable operators, see Corollary 4 in [18].

After collecting in § 5 the results on weak convergence in $C_0(T, \boldsymbol{X})$, Theorem 13 in § 6 gives a very partial affirmative answer (for $\boldsymbol{X} = l_1$ or T discrete) to our most important open problem: If $\boldsymbol{X}$ has the Dunford-Pettis property, has also the space $C_0(T, \boldsymbol{X})$ this property?

The first version of this paper contained results on representation of weakly compact and compact linear operators on $C_0(T, \boldsymbol{X})$. These results however were omitted, since coincided with the recent results of J. BATT and J. BERG, see [35] and [36], where we refer the reader to.

1. ON OPERATOR VALUED MEASURES

Operator valued measures were treated in section 1.1 of [13]. For convenience we first state a few basic notions and facts and then we prove some important results about the semivariation $\hat{\boldsymbol{m}}$ of an operator valued measure $\boldsymbol{m}$.

Let T be a locally compact Hausdorff topological space, let $\mathscr{B}_0$ denote the δ-ring of all relatively compact Baire subsets of T, let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be Banach spaces and let $L(\boldsymbol{X}, \boldsymbol{Y})$ denote the Banach space of all bounded linear operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$. We shall suppose that $\boldsymbol{m} : \mathscr{B}_0 \to L(\boldsymbol{X}, \boldsymbol{Y})$ is an operator valued measure countably additive in the strong operator topology of $L(\boldsymbol{X}, \boldsymbol{Y})$, i.e., for every $\boldsymbol{x} \in \boldsymbol{X}$ $\boldsymbol{m}(.)\, \boldsymbol{x}$ is a countably additive vector measure. Let us remind that by the theorem of Orlicz-Pettis, see 3.2.1 in [22], and IV.10.1 in [15], countably additivity in the strong and in the weak operator topology are equivalent conditions.

Let $\mathfrak{S}(\mathscr{B}_0)$ denote the smallest σ-ring containing $\mathscr{B}_0$, i.e., the σ-ring of all Baire subsets of T in the sense of § 51 in [21]. By a $\mathscr{B}_0$-simple function on T with values in $\boldsymbol{X}$ we call a function of the form $\boldsymbol{f} = \sum_{i=1}^{r} \boldsymbol{x}_i \,.\, \chi_{E_i}$ where $\boldsymbol{x}_i \in \boldsymbol{X}$, $E_i \in \mathscr{B}_0$ and $E_i \cap E_j = \emptyset$ for $i \neq j$. Here χ_E denotes the characteristic function of the set E in T. Its integral on a set $E \in \mathfrak{S}(\mathscr{B}_0)$ with respect to the measure $\boldsymbol{m}$ is obviously unambiguously defined by the equality $\int_E \boldsymbol{f}\, \mathrm{d}\boldsymbol{m} = \sum_{i=1}^{r} \boldsymbol{m}(E \cap E_i)\, \boldsymbol{x}_i$. Denote by $\mathfrak{I}_s$ the set of all $\mathscr{B}_0$-simple functions on T with values in $\boldsymbol{X}$. For a function $\boldsymbol{f} : T \to \boldsymbol{X}$ and a set $A \subset T$ put $\|\boldsymbol{f}\|_A = \sup_{t \in A} |\boldsymbol{f}(t)|$ ($|.|$ denotes the norm) and define on $\mathfrak{S}(\mathscr{B}_0)$ the non negative set function $\hat{\boldsymbol{m}}$, called the semivariation of the measure $\boldsymbol{m}$, by the equality: $\hat{\boldsymbol{m}}(E) = \sup \{|\int_E \boldsymbol{f}\, \mathrm{d}\boldsymbol{m}|, \boldsymbol{f} \in \mathfrak{I}_s, \|\boldsymbol{f}\|_E \leqq 1\}$, $E \in \mathfrak{S}(\mathscr{B}_0)$. From this definition it is obvious that $\hat{\boldsymbol{m}}(\emptyset) = 0$ and that $\hat{\boldsymbol{m}}$ is a monotone and countably subadditive set function on $\mathfrak{S}(\mathscr{B}_0)$, see [10, § 4 Proposition 3]. Further, for every function $\boldsymbol{f} \in \mathfrak{I}_s$ and every set $E \in \mathfrak{S}(\mathscr{B}_0)$ we have the important inequality

$$\text{(i)} \qquad \left| \int_E \boldsymbol{f}\, \mathrm{d}\boldsymbol{m} \right| \leqq \|\boldsymbol{f}\|_E \,.\, \hat{\boldsymbol{m}}(E)\,.$$

Denote by $\bar{\mathfrak{I}}_s$ the closure of $\mathfrak{I}_s$ in the norm $\|.\|_T$ in the Banach space of all bounded $\boldsymbol{X}$ valued functions on T and let $\hat{\boldsymbol{m}}(T) = \sup_{E \in \mathscr{B}_0} \hat{\boldsymbol{m}}(E) < +\infty$. Then (i) enables us to extend the integral from $\mathfrak{I}_s$ to $\bar{\mathfrak{I}}_s$. For this purpose it is enough for a function $\boldsymbol{f} \in \bar{\mathfrak{I}}_s$ and for a set $E \in \mathfrak{S}(\mathscr{B}_0)$ to define

$$\int_E \boldsymbol{f} \,\mathrm{d}\boldsymbol{m} = \lim_{n \to \infty} \int_E \boldsymbol{f}_n \,\mathrm{d}\boldsymbol{m}$$

where $\boldsymbol{f}_n \in \mathfrak{I}_s$, $n = 1, 2, \ldots$, and $\|\boldsymbol{f}_n - \boldsymbol{f}\|_T \to 0$. For the extended integral, (i) is clearly valid again. Let us note that since $C_0(T, \boldsymbol{X}) \subset \bar{\mathfrak{I}}_s$, see Theorem 8 in [13], this elementary theory of integration is sufficient for representing a wide class of bounded linear operators $\boldsymbol{U} : C_0(T, \boldsymbol{X}) \to \boldsymbol{Y}$ in the form $\boldsymbol{U}\boldsymbol{f} = \int_T \boldsymbol{f} \,\mathrm{d}\boldsymbol{m}$ with such a measure $\boldsymbol{m}$, see § 2 below. But for investigations of properties of such operators the substantially wider theory developed in [13] and the subsequent parts is needed.

Let $K(T)$ denote the set of all scalar valued continuous functions on T with compact supports and let $\varphi \in K(T)$. Then by Theorem D of § 50 in [21] there is a compact G_δ set C such that $\varphi(t) = 0$ for every $t \in T - C$. Denote by $\mathfrak{I}_s$ the analogue of $\mathfrak{I}_s$ for the scalar valued functions and let $\|\boldsymbol{v}\|$ denote the scalar semivariation of a vector measure $\boldsymbol{v}$, see IV.10.3 in [15]. Since for every $\boldsymbol{x} \in \boldsymbol{X}$ $\sup_{E \in \mathscr{B}_0} \|\boldsymbol{m}(\,)\,\boldsymbol{x}\| (C \cap E) < +\infty$, see IV.10.4 in [15], and since $\varphi \in C_0(T) \subset \bar{\mathfrak{I}}_s$, see Theorem 8 in [13], the function φ is integrable with respect to every measure $\boldsymbol{m}(\,)\,\boldsymbol{x}$, $\boldsymbol{x} \in \boldsymbol{X}$. Let us finally denote by $\mathfrak{Q}$ the set of all functions of the form $\boldsymbol{f} = \sum_{i=1}^{r} \varphi_i \,.\, \boldsymbol{x}_i$ where $\varphi_i \in K(T)$ and $\boldsymbol{x}_i \in \boldsymbol{X}$. Then by the assertion just stated every function $\boldsymbol{f} \in \mathfrak{Q}$ is integrable with respect to the measure $\boldsymbol{m}$. Moreover, we have:

Theorem 1. *$\mathfrak{Q}$ is a dense subset of $C_0(T, \boldsymbol{X})$ and for every set $E \in \mathfrak{S}(\mathscr{B}_0)$*

$$\hat{\boldsymbol{m}}(E) = \sup \{|\textstyle\int_E \boldsymbol{f} \,\mathrm{d}\boldsymbol{m}|,\ \boldsymbol{f} \in \mathfrak{Q},\ \|\boldsymbol{f}\|_E \leqq 1\}.$$

Proof. That $\mathfrak{Q}$ is dense in $C_0(T, \boldsymbol{X})$ follows from Proposition 1 of § 19 in [10]. The inequality $\hat{\boldsymbol{m}}(E) \geqq \sup \{\ldots\}$ is evident from the inclusion $\mathfrak{Q} \subset \bar{\mathfrak{I}}_s$ and from the definition of the semivariation $\hat{\boldsymbol{m}}$. It remains to prove the converse inequality.

Let us have an $\varepsilon > 0$. By the definition of the semivariation $\hat{\boldsymbol{m}}$ choose $\boldsymbol{x}_i \in \boldsymbol{X}$, $|\boldsymbol{x}_i| \leqq 1$ and disjoint $E_i \in \mathscr{B}_0$, $E_i \subset E$, $i = 1, 2, \ldots, r$ in such a way that $\hat{\boldsymbol{m}}(E) - \varepsilon \leqq \leqq |\sum_{i=1}^{r} \boldsymbol{m}(E_i)\,\boldsymbol{x}_i|$. By Theorem D of § 50 in [21] there is a relatively compact open Baire set U such that $\bigcup_{i=1}^{r} \bar{E}_i \subset U$ where $\bar{E}_i$ is the closure in T of the set E_i. Let $\mathscr{S}_1$ denote the σ-ring of sets of the form $U \cap E$, $E \in \mathscr{B}_0$. Since for every $i = 1, 2, \ldots, r$ $\boldsymbol{m}(\,)\,\boldsymbol{x}_i$ is a countably additive vector measure on $\mathscr{S}_1$, Theorem IV.10.5 in [15] implies that for every i there is a finite non negative countably additive measure λ_i on $\mathscr{S}_1$ with the properties: $\lambda_i(E) \leqq \|\boldsymbol{m}(\,)\,\boldsymbol{x}_i\| (E)$ and $\lim_{\lambda_i(E) \to 0} \|\boldsymbol{m}(\,)\,\boldsymbol{x}_i\| (E) = 0$,

$E \in \mathscr{S}_1$. But the measures λ_i, $i = 1, 2, \ldots, r$ are regular on the measure space $(U, \mathscr{S}_1)$, see Theorem G of § 52 in [21], and therefore for every i there is a compact in the relative topology of U set $C_i \subset E_i$ and an open (in T since U is open) set U_i, $E_i \subset \subset U_i \subset U$ with $\|\boldsymbol{m}(\ )\,\boldsymbol{x}_i\|\,(U_i - C_i) \leqq \varepsilon/2r$. Since the sets C_i are relatively closed subsets of U, there are closed subsets $F_i \subset T$ with $C_i = U \cap F_i$ for every i. But $C_i \subset \bar{E}_i \subset U$, so $C_i = \bar{E}_i \cap F_i$, and this is a compact Baire subset of T. According to Theorem B of § 50 in [21] for every i there is a function $\varphi_i \in K(T)$, $0 \leqq \varphi_i(t) \leqq 1$ for every $t \in T$, such that $\varphi_i(t) = 1$ for $t \in C_i$ and $\varphi_i(t) = 0$ for $t \in T - U_i$. But then for every $i = 1, 2, \ldots, r$

$$\left|\int_E \varphi_i \,\mathrm{d}\boldsymbol{m}(\ )\,\boldsymbol{x}_i - \boldsymbol{m}(C_i)\,\boldsymbol{x}_i\right| = \left|\int_{E\cap(U_i - C_i)} \varphi_i \,\mathrm{d}\boldsymbol{m}(\ )\,\boldsymbol{x}_i\right| \leqq$$

$$\leqq \|\varphi_i\|_T \cdot \|\boldsymbol{m}(\ )\,\boldsymbol{x}_i\|\,(U_i - C_i) \leqq \frac{\varepsilon}{2r}$$

and therefore if we put $\boldsymbol{f} = \sum_{i=1}^{r} \varphi_i \,.\, \boldsymbol{x}_i$, then $\boldsymbol{f} \in \mathfrak{Q}$ and

$$\left|\int_E \boldsymbol{f}\,\mathrm{d}\boldsymbol{m} - \sum_{i=1}^{r} \boldsymbol{m}(C_i)\,\boldsymbol{x}_i\right| \leqq \frac{\varepsilon}{2}\,.$$

On the other hand, since $C_i \subset E_i \subset U_i$,

$$\left|\sum_{i=1}^{r} \boldsymbol{m}(E_i)\,\boldsymbol{x}_i - \sum_{i=1}^{r} \boldsymbol{m}(C_i)\,\boldsymbol{x}_i\right| \leqq \sum_{i=1}^{r} \|\boldsymbol{m}(\ )\,\boldsymbol{x}_i\|\,(U_i - C_i) \leqq \frac{\varepsilon}{2}\,.$$

Hence

$$\hat{\boldsymbol{m}}(E) - 2\varepsilon \leqq \left|\sum_{i=1}^{r} \boldsymbol{m}(E_i)\,\boldsymbol{x}_i\right| - \varepsilon \leqq \left|\int_E \boldsymbol{f}\,\mathrm{d}\boldsymbol{m}\right|.$$

Since $\varepsilon > 0$ was arbitrary, the theorem is proved.

By Theorem 8 in [13] $C_0(T, \boldsymbol{X}) \subset \mathfrak{J}_s$ and therefore we immediately have the following

Corollary. *Let* $\hat{\boldsymbol{m}}(T) = \sup_{E \in \mathscr{B}_0} \hat{\boldsymbol{m}}(E) < +\infty$. *Then every function* $\boldsymbol{f} \in C_0(T, \boldsymbol{X})$ *is integrable and for every set* $E \in \mathfrak{S}(\mathscr{B}_0)$

$$\hat{\boldsymbol{m}}(E) = \sup\left\{\left|\int_E \boldsymbol{f}\,\mathrm{d}\boldsymbol{m}\right|,\quad \boldsymbol{f} \in C_0(T, \boldsymbol{X}),\ \|\boldsymbol{f}\|_E \leqq 1\right\}.$$

For the rest of this section we may suppose that T is an arbitrary non empty set, $\mathscr{P}$ a δ-ring of subsets of T, $\mathfrak{S}(\mathscr{P})$ the smallest σ-ring containing $\mathscr{P}$ and $\boldsymbol{m} : \mathscr{P} \to L(\boldsymbol{X}, \boldsymbol{Y})$ an operator valued measure countably additive in the strong operator topology. Denote by $cabv(\mathfrak{S}(\mathscr{P}), \boldsymbol{X}^*)$, $\boldsymbol{X}^*$ being the dual of $\boldsymbol{X}$, the Banach space of all countably additive vector measures $\boldsymbol{\mu} : \mathfrak{S}(\mathscr{P}) \to \boldsymbol{X}^*$ with bounded variations, $|\boldsymbol{\mu}| = v(\boldsymbol{\mu}, T) = \sup_{E \in \mathfrak{S}(\mathscr{P})} v(\boldsymbol{\mu}, E)$. The assertion a) of the next lemma is evident from the paragraph following Theorem 7 in [13], while assertion b) follows from the principle of the uniform boundedness (Banach-Steinhaus Theorem), see [17, Prop. 1, III].

Lemma 1. a) *For every functional* $\mathbf{y}^* \in \boldsymbol{Y}^*$, *for every function* $\mathbf{f} \in \mathfrak{I}_s$ *and for every set* $E \in \mathfrak{S}(\mathscr{P})$ *we have the equality* $\mathbf{y}^* \int_E \mathbf{f} \, d\mathbf{m} = \int_E \mathbf{f} \, d\mathbf{y}^*\mathbf{m}$. *If* $\hat{\mathbf{m}}(T) < +\infty$, *then* $\mathbf{y}^*\mathbf{m} \in cabv(\mathfrak{S}(\mathscr{P}), \boldsymbol{X}^*)$ *for every functional* $\mathbf{y}^* \in \boldsymbol{Y}^*$, *and* $\hat{\mathbf{m}}(E) = \sup_{|\mathbf{y}^*| \leq 1} v(\mathbf{y}^*\mathbf{m}, E)$ *for every set* $E \in \mathfrak{S}(\mathscr{P})$.

b) *If* $v(\mathbf{y}^*\mathbf{m}, E) < +\infty$ *for every functional* $\mathbf{y}^* \in \boldsymbol{Y}^*$, *then also* $\hat{\mathbf{m}}(E) = \sup_{|\mathbf{y}^*| \leq 1} v(\mathbf{y}^*\mathbf{m}, E) < +\infty$, $E \in \mathfrak{S}(\mathscr{P})$.

We say that the semivariation $\hat{\mathbf{m}}$ is continuous on $\mathfrak{S}(\mathscr{P})$ iff for any decreasing sequence of sets $E_n \searrow \emptyset$, $E_n \in \mathfrak{S}(\mathscr{P})$, $n = 1, 2, \ldots$, there is $\lim_{n \to \infty} \hat{\mathbf{m}}(E_n) = 0$. From the theorem of Orlicz-Pettis it follows, see Theorem 5 in [35] and Theorem 5 in [37], that if $\boldsymbol{Y}$ is weakly complete (more generally, if $\boldsymbol{Y}$ contains no subspace isomorphic to the space c_0, see [6]) and $\hat{\mathbf{m}}$ is bounded on $\mathfrak{S}(\mathscr{P})$, then $\hat{\mathbf{m}}$ is continuous on $\mathfrak{S}(\mathscr{P})$. In § 4 we use the following result:

Lemma 2. *The semivariation* $\hat{\mathbf{m}}$ *is continuous on* $\mathfrak{S}(\mathscr{P})$ *if and only if there is a finite non negative countably additive measure* λ *on* $\mathfrak{S}(\mathscr{P})$ *with the properties*: $\lambda(E) \leq \|\mathbf{m}\| (E)$ *and* $\lim_{\lambda(E) \to 0} \hat{\mathbf{m}}(E) = 0$, $E \in \mathfrak{S}(\mathscr{P})$.

Proof. Let the semivariation $\hat{\mathbf{m}}$ be continuous on $\mathfrak{S}(\mathscr{P})$. Since $\|\mathbf{m}\| (E) \leq \hat{\mathbf{m}}(E)$ for every set $E \in \mathfrak{S}(\mathscr{P})$, the measure $\mathbf{m}$ is countably additive in the uniform operator topology on $\mathfrak{S}(\mathscr{P})$. Thus by Theorem IV.10.5 in [15] there is a finite non negative countably additive measure λ on $\mathfrak{S}(\mathscr{P})$ with the properties: $\lambda(E) \leq \|\mathbf{m}\| (E)$ and $\lim_{\lambda(E) \to 0} \|\mathbf{m}\| (E) = 0$, $E \in \mathfrak{S}(\mathscr{P})$. If now $\lambda(N) = 0$, $N \in \mathfrak{S}(\mathscr{P})$, then $\|\mathbf{m}\| (N) = 0$ and therefore also $\hat{\mathbf{m}}(N) = 0$. Suppose that $\lim_{\lambda(E) \to 0} \hat{\mathbf{m}}(E) \neq 0$, $E \in \mathfrak{S}(\mathscr{P})$. Then there is an $\varepsilon > 0$ and a sequence of sets $A_k \in \mathfrak{S}(\mathscr{P})$, $k = 1, 2, \ldots$, with $\lambda(A_k) < 1/2^k$ and $\hat{\mathbf{m}}(A_k) > \varepsilon$. Put $B_k = \bigcup_{i=k}^{\infty} A_i$ and $B = \bigcap_{k=1}^{\infty} B_k$. Then, since λ is a finite non negative countably additive measure on $\mathfrak{S}(\mathscr{P})$, $\lambda(B) = 0$, while $\hat{\mathbf{m}}(B) \geq \hat{\mathbf{m}}(B_k) - \hat{\mathbf{m}}(B_k - B) > \varepsilon$ for sufficiently large k is implied by the monotonocity and continuity of $\hat{\mathbf{m}}$ on $\mathfrak{S}(\mathscr{P})$, a contradiction. Thus we proved the existence of the λ required. The converse assertion is obvious. The lemma is thus completely proved.

Let us note that in the preceding lemma we do not assume the boundedness of the semivariation $\hat{\mathbf{m}}$ on $\mathfrak{S}(\mathscr{P})$. In fact, the boundedness of $\hat{\mathbf{m}}$ on $\mathfrak{S}(\mathscr{P})$ follows from the continuity of $\hat{\mathbf{m}}$ on $\mathfrak{S}(\mathscr{P})$ by this lemma, see the Corollary of Theorem 5 in [37].

Finally, in § 3 we use the following general result:

Lemma 3. *Let* $\mathbf{m}_n : \mathscr{P} \to L(\mathbf{X}, \mathbf{Y})$, $n = 1, 2, \ldots$, *be a sequence of operator valued measures countably additive in the strong (uniform) operator topology and let for each set* $E \in \mathscr{P}$ *and each* $\mathbf{x} \in \mathbf{X}$ *the limit* $\lim_{n\to\infty} \mathbf{m}_n(E)\, \mathbf{x} = \mathbf{m}(E)\, \mathbf{x} \in \mathbf{Y}$ *exist. (For each set* $E \in \mathscr{P}$ *there exists the limit* $\lim_{n\to\infty} \mathbf{m}_n(E) = \mathbf{m}(E) \in L(\mathbf{X}, \mathbf{Y})$.) *Then* $\mathbf{m}$ *is a* $L(\mathbf{X}, \mathbf{Y})$ *valued measure countably additive in the strong (uniform) operator topology on* $\mathscr{P}$ *and for each set* $E \in \mathfrak{S}(\mathscr{P})$ $\hat{\mathbf{m}}(E) \leqq \limsup_n \hat{\mathbf{m}}_n(E)$. *If* $\hat{\mathbf{m}}_n(T) < +\infty$ *for every n and* $(\widehat{\mathbf{m}_{n_1} - \mathbf{m}_{n_2}})(T) \to 0$ *for* $\min(n_1, n_2) \to \infty$, *then* $\hat{\mathbf{m}}(T) < +\infty$ *and* $\lim_{n\to\infty} (\widehat{\mathbf{m}_n - \mathbf{m}})(T) = 0$. *At the same time, if for every n the semivariation* $\hat{\mathbf{m}}_n$ *is continuous on* $\mathscr{P}$, *then the semivariation* $\hat{\mathbf{m}}$ *is also continuous on* $\mathscr{P}$.

Proof. The first assertion of the lemma follows from the theorem of Vitali-Hahn-Saks, see IV.10.6 in [15] and from the Banach-Steinhaus Theorem (uniform boundedness principle). The remaining assertions of the lemma are obvious.

2. REPRESENTATION THEOREMS

In this section we prove the basic theorems on representation of bounded linear operators on $C_0(T, \mathbf{X})$ in the form of an integral with respect to a Baire operator valued measure. Theorem 2 is derived from the following known result, which is in fact its scalar version:

(A) *A bounded linear operator* $\mathbf{U} : C_0(T) \to \mathbf{Y}$ *is weakly compact if and only if it can be uniquely expressed in the form* $\mathbf{U}f = \int_T f \, d\boldsymbol{\mu}$, $f \in C_0(T)$ *where* $\boldsymbol{\mu} : \mathfrak{S}(\mathscr{B}_0) \to \mathbf{Y}$ *is a countably additive vector measure. In that case* $|\mathbf{U}| = \|\boldsymbol{\mu}\|\,(T)$ *and* $\mathbf{U}^*\mathbf{y}^* = \mathbf{y}^*\boldsymbol{\mu} \in ca(\mathfrak{S}(\mathscr{B}_0))$ *for every functional* $\mathbf{y}^* \in \mathbf{Y}^*$.

Let us note that by $\int_T \mathbf{f}\, d\boldsymbol{\mu}$ we understood the integral $\int_F \mathbf{f}\, d\boldsymbol{\mu}$ where $F = \{t \in T, |\mathbf{f}(t)| > 0\} \in \mathfrak{S}(\mathscr{B}_0)$. For compact T this is Theorem VI.7.3 in [15], see also [2] and [19]. For locally compact T it was extended in [24, Lemma 2]. Although in these papers $\boldsymbol{\mu}$ is considered to be a regular Borel measure, each function $\mathbf{f} \in C_0(T, \mathbf{X})$ is Baire measurable and therefore it is sufficient to consider Baire measures.

Theorem 2. *A bounded linear operator* $\mathbf{U} : C_0(T, \mathbf{X}) \to \mathbf{Y}$ *can be uniquely expressed in the form*

$$\mathbf{U}\mathbf{f} = \int_T \mathbf{f}\, d\mathbf{m}\,, \quad \mathbf{f} \in C_0(T, \mathbf{X}) \tag{1}$$

where $\mathbf{m} : \mathfrak{S}(\mathscr{B}_0) \to L(\mathbf{X}, \mathbf{Y})$ *is a Baire operator valued measure countably additive in the strong operator topology with* $\hat{\mathbf{m}}(T) < +\infty$, *if and only if for every* $\mathbf{x} \in \mathbf{X}$ *the bounded linear operator* $\mathbf{U}_{\mathbf{x}} : C_0(T) \to \mathbf{Y}$ *defined by* $\mathbf{U}_{\mathbf{x}}\varphi = \mathbf{U}\varphi.\,\mathbf{x}$, $\varphi \in C_0(T)$ *is weakly compact. In that case* $\mathbf{U}^*\mathbf{y}^* = \mathbf{y}^*\mathbf{m} \in cabv(\mathfrak{S}(\mathscr{B}_0), \mathbf{X}^*)$ *for every functional* $\mathbf{y}^* \in \mathbf{Y}^*$ *and* $|\mathbf{U}| = \hat{\mathbf{m}}(T) = \sup_{|\mathbf{y}^*| \leqq 1} v(\mathbf{y}^*\mathbf{m}, T)$.

Proof. Suppose that the condition of the theorem is fulfilled. Then by (A) for every $\mathbf{x} \in \mathbf{X}$ there is a uniquely determined countably additive vector measure $\mathbf{m}_{\mathbf{x}} : \mathfrak{S}(\mathscr{B}_0) \to \mathbf{Y}$ such that $\mathbf{U}_{\mathbf{x}}\varphi = \int_T \varphi \, \mathrm{d}\mathbf{m}_{\mathbf{x}}$ for every function $\varphi \in C_0(T)$, and $|\mathbf{U}_{\mathbf{x}}| = \|\mathbf{m}_{\mathbf{x}}\| (T) \leqq |\mathbf{U}| \, . \, |\mathbf{x}|$. For every set $E \in \mathfrak{S}(\mathscr{B}_0)$ and every $\mathbf{x} \in \mathbf{X}$ let us put $\mathbf{m}(E) \, \mathbf{x} = \mathbf{m}_{\mathbf{x}}(E)$. Then by the preceding inequality $\mathbf{m}(E) \in L(\mathbf{X}, \mathbf{Y})$ for every set $E \in \mathfrak{S}(\mathscr{B}_0)$. Thus by definition $\mathbf{m}$ is a uniquely determined Baire operator valued measure countably additive in the strong operator topology and for every $\mathbf{x} \in \mathbf{X}$ and every function $\varphi \in C_0(T)$, $\mathbf{U}\varphi\mathbf{x} = \int_T \varphi \, . \, \mathbf{x} \, \mathrm{d}\mathbf{m}$. From here we immediately obtain the equality $\mathbf{U}\mathbf{f} = \int_T \mathbf{f} \, \mathrm{d}\mathbf{m}$ for every function $\mathbf{f} \in \mathfrak{Q}$, see the notation introduced before Theorem 1. But by Theorem 1 $\hat{\mathbf{m}}(T) = \sup \{|\int_T \mathbf{f} \, \mathrm{d}\mathbf{m}|, \ \mathbf{f} \in \mathfrak{Q}, \ \|\mathbf{f}\|_T \leqq 1\} = \sup \{|\mathbf{U}\mathbf{f}|, \ \mathbf{f} \in \mathfrak{Q}, \ \|\mathbf{f}\|_T \leqq 1\} = |\mathbf{U}| < +\infty$, since $\mathfrak{Q}$ is a dense subset of $C_0(T, \mathbf{X})$. Thus by the Corollary of Theorem 1 every function $\mathbf{f} \in C_0(T, \mathbf{X})$ is integrable. Since $\mathbf{U}\mathbf{f} = \int_T \mathbf{f} \, \mathrm{d}\mathbf{m}$ for the dense subset $\mathfrak{Q}$ of $C_0(T, \mathbf{X})$, this expression is valid by inequality (i) for every function $\mathbf{f} \in C_0(T, \mathbf{X})$. The relations $\mathbf{U}^*\mathbf{y}^* = \mathbf{y}^*\mathbf{m} \in cabv(\mathfrak{S}(\mathscr{B}_0), \mathbf{X}^*)$, $\mathbf{y}^* \in \mathbf{Y}^*$, and $\hat{\mathbf{m}}(T) = \sup_{|\mathbf{y}^*| \leqq 1} v(\mathbf{y}^*\mathbf{m}, T)$ follow from Lemma 1. The converse assertion of the theorem is obvious from the elementary properties of the integral and from (A). Thus the theorem is proved.

The following theorem is a generalization of Theorem 2. It may be proved in just the same way. Let us note that $K(D)$, $D \subset T$ denotes the set of all continuous scalar functions on D which have compact supports in D.

Theorem 2'. *A bounded linear operator* $\mathbf{U} : C_0(T, \mathbf{X}) \to \mathbf{Y}$ *can be uniquely expressed in the form*

$$\mathbf{U}\mathbf{f} = \int_T \mathbf{f} \, \mathrm{d}\mathbf{m} \, , \quad \mathbf{f} \in C_0(T, \mathbf{X}) \tag{1'}$$

where $\mathbf{m} : \mathscr{B}_0 \to L(\mathbf{X}, \mathbf{Y})$ *is a Baire operator valued measure countably additive in the strong operator topology with* $\hat{\mathbf{m}}(T) < +\infty$ *if and only if for every* $\mathbf{x} \in \mathbf{X}$ *and for every* G_δ *compact set* $D \subset T$ *the mapping* $\mathbf{U}_{\mathbf{x}} : K(D) \to \mathbf{Y}$ *defined by the equality* $\mathbf{U}_{\mathbf{x}}\varphi = \mathbf{U}\varphi . \, \mathbf{x}$, $\varphi \in K(D)$ *is weakly compact. In that case* $\mathbf{U}^*\mathbf{y}^* = \mathbf{y}^*\mathbf{m} \in cabv(\mathscr{B}_0, \mathbf{X}^*) = cabv(\mathfrak{S}(\mathscr{B}_0), \mathbf{X}^*)$ *for every functional* $\mathbf{y}^* \in \mathbf{Y}^*$ *and* $|\mathbf{U}| = \hat{\mathbf{m}}(T) = \sup_{|\mathbf{y}^*| \leqq 1} v(\mathbf{y}^*\mathbf{m}, T)$.

The identical transformation of the space c_0 into itself is a simple example which fulfils the condition of Theorem 2' but not that of Theorem 2. Namely, the measure $\mathbf{m}$ corresponding to this operator by (1') has no countably additive extension from $\mathscr{B}_0$ to $\mathfrak{S}(\mathscr{B}_0)$.

Obviously the condition of Theorem 2 is fulfilled for every weakly compact linear operator $\mathbf{U} : C_0(T, \mathbf{X}) \to \mathbf{Y}$. Below in Theorem 3 we prove that a necessary condition for the weak compactness of an operator of the form (1) is the continuity of the semivariation $\hat{m}$ on $\mathfrak{S}(\mathscr{B}_0)$. We now consider two examples of bounded linear operators of the form (1) where the semivariation $\hat{m}$ is not continuous on $\mathfrak{S}(\mathscr{B}_0)$; see also examples in [35] and [36].

Let T be the set of natural numbers with the discrete topology. The measure $\boldsymbol{m}$ from example 6 in section 1.1 in [13], which is countably additive only in the strong operator topology defines by (1) a bounded linear operator $\boldsymbol{U} : c_0(l_1) \to c_0$ which is not weakly compact. At the same time it is easy to see that for each $\boldsymbol{x} \in \boldsymbol{X}$ the operator $\boldsymbol{U}_{\boldsymbol{x}} : c_0 \to c_0$ is compact and that for each set $E \in \mathfrak{S}(\mathscr{B}_0)$ $\boldsymbol{m}(E) \in L(l_1, c_0)$ is also compact. Thus in Theorem 2 compactness of operators $\boldsymbol{U}_{\boldsymbol{x}} : C_0(T) \to \boldsymbol{Y}$, $\boldsymbol{x} \in \boldsymbol{X}$ together with compactness of values of the measure $\boldsymbol{m}$ do not imply in general even the countable additivity of the measure $\boldsymbol{m}$ in the uniform operator topology.

Similarly the measure $\boldsymbol{m}$ from example 7 in section 1.1 in [13] defines by (1) a bounded linear operator $\boldsymbol{U} : c_0(l_1) \to c_0$ which is not weakly compact since the semivariation $\hat{m}$ is not continuous on $\mathfrak{S}(\mathscr{B}_0)$. At the same time the measure $\boldsymbol{m}$ is countably additive in the uniform operator topology, its values are compact operators and the operators $\boldsymbol{U}_{\boldsymbol{x}} : c_0 \to c_0$, $\boldsymbol{x} \in \boldsymbol{X}$ are also compact. It is even more remarkable that the set $\{\boldsymbol{m}(E), E \in \mathfrak{S}(\mathscr{B}_0)\}$ is relatively compact in the Banach space $L(\boldsymbol{X}, \boldsymbol{Y}) = L(l_1, c_0)$.

It is not difficult to find whole classes of similar examples. Thus the condition of Theorem 2 is fulfilled in general for a considerably wider class of bounded linear operators than is the class of linear weakly compact operators.

According to [27] a bounded linear operator $\boldsymbol{V} : \boldsymbol{X} \to \boldsymbol{Y}$ is called unconditionally converging if it transforms weakly unconditionally convergent series into (strongly) unconditionally convergent ones. Obviously the class of all unconditionally converging operators from $L(\boldsymbol{X}, \boldsymbol{Y})$ forms a closed linear subspace in $L(\boldsymbol{X}, \boldsymbol{Y})$. Further, a composition of a bounded linear operator with an unconditionally converging bounded linear operator (in any order) is an unconditionally converging bounded linear operator. By the theorem of Orlicz-Pettis, see 3.2.1 in [22], every weakly compact linear operator is unconditionally converging. If $\boldsymbol{Y}$ is a weakly complete Banach space (more generally, if $\boldsymbol{Y}$ contains no subspace isomorphic to c_0), then every bounded linear operator from $\boldsymbol{X}$ to $\boldsymbol{Y}$ is unconditionally converging. Let us note also that according to [38, Proposition 1.9] every completely continuous operator (not the same as compact), see § 6 below, is unconditionally converging. A. Pełczyński in [27, Theorems 1 and 1′] proved that for a reflexive Banach space $\boldsymbol{X}$ a bounded linear operator $\boldsymbol{U} : C_0(T, \boldsymbol{X}) \to \boldsymbol{Y}$ is unconditionally converging if and only if it is weakly compact. The next representation theorem for unconditionally converging bounded linear operators on $C_0(T, \boldsymbol{X})$ is based on this result and on Theorem 2. This theorem obviously generalizes Theorem 5 in [35].

Theorem 3. *Every unconditionally converging bounded linear operator* $\boldsymbol{U} : C_0(T, \boldsymbol{X}) \to \boldsymbol{Y}$ *can be uniquely expressed in the form* (1) *of Theorem* 2 *where the* *values of the measure* $\boldsymbol{m}$ *are unconditionally converging operators from* $L(\boldsymbol{X}, \boldsymbol{Y})$ *and its semivariation* $\hat{\boldsymbol{m}}$ *is continuous on* $\mathfrak{S}(\mathscr{B}_0)$.

Proof. Since a bounded linear operator $\boldsymbol{U}_x : C_0(T) \to \boldsymbol{Y}$ is unconditionally converging if and only if it is weakly compact, see Theorem 1 in [27], from Theorem 2 we immediately have the representation (1). That the values of the measure $\boldsymbol{m}$ are unconditionally converging operators from $L(\boldsymbol{X}, \boldsymbol{Y})$ follows from the facts stated before Theorem 1. By Lemma 1 for each set $E \in \mathfrak{S}(\mathscr{B}_0)$, $\hat{\boldsymbol{m}}(E) = \sup_{|y^*| \leq 1} v(\boldsymbol{y}^*\boldsymbol{m}, E)$, $\boldsymbol{y}^* \in \boldsymbol{Y}^*$. Suppose that the semivariation $\hat{\boldsymbol{m}}$ is not continuous on $\mathfrak{S}(\mathscr{B}_0)$. Then the variations $v(\boldsymbol{y}^*\boldsymbol{m}, .)$, $\boldsymbol{y}^* \in \boldsymbol{Y}^*$, $|\boldsymbol{y}^*| \leqq 1$, are not uniformly countably additive on $\mathfrak{S}(\mathscr{B}_0)$. But this would yield a contradiction in just the same way as in the proof of Theorems 1 and 1′ in [27].

It remains an open problem if the conditions of Theorem 3 are sufficient in general for $\boldsymbol{U}$ to be unconditionally converging.

Remark 1. The first version of this paper contained results on the representation of weakly compact and compact operators $\boldsymbol{U} : C_0(T, \boldsymbol{X}) \to \boldsymbol{Y}$, which however coincided with the recent results of J. Batt and J. Berg, see [35] and [36]. We only note that replacing in the preceding Theorem 3 the unconditional convergence by the weak compactness, we obtain the representation theorem for weakly compact operators on $C_0(T, \boldsymbol{X})$. The conditions obtained on the representing measure $\boldsymbol{m}$ are sufficient for the weak compactness of the operator $\boldsymbol{U}$ in the cases: 1. $\boldsymbol{X}$ is reflexive, 2. $\boldsymbol{X}^{**}$ is separable, and 3. T is discrete, and as Example 3 in [36] demonstrates these conditions are not sufficient in general.

Remark 2. As C. Foias and I. Singer in [17, Theorem 1] deduced, every bounded linear operator $\boldsymbol{U} : C(T, \boldsymbol{X}) \to \boldsymbol{Y}$, T compact can be uniquely represented as a weak integral respect to an additive $L(\boldsymbol{X}, \boldsymbol{Y}^{**})$ valued measure with certain properties, called the representing measure of $\boldsymbol{U}$. Theorem 2 in [35] gives necessary and sufficient conditions for the representing measure to have values in $L(\boldsymbol{X}, \boldsymbol{Y})$. We note that the condition of our Theorem 2 is also a necessary and sufficient condition for this (if $\boldsymbol{m}$ has values in $L(\boldsymbol{X}, \boldsymbol{Y})$, then the regularity implies that $v(\boldsymbol{y}^*\boldsymbol{m}, .)$ is countably additive, so $\boldsymbol{m}$ is countably additive in the weak, equivalently strong (see IV.10.1 in [15]), operator topology of $L(\boldsymbol{X}, \boldsymbol{Y})$, and thus the condition of Theorem 2 is fulfilled). Hence we have the following result, see also Theorem 3 in [35]:

Proposition. *The representing measure* $\boldsymbol{m} : \mathfrak{S}(\mathscr{B}_0) \to L(\boldsymbol{X}, \boldsymbol{Y}^{**})$ *of a bounded linear operator* $\boldsymbol{U} : C(T, \boldsymbol{X}) \to \boldsymbol{Y}$ *has values in* $L(\boldsymbol{X}, \boldsymbol{Y})$ *if and only if it is regular in the weak operator topology of* $L(\boldsymbol{X}, \boldsymbol{Y}^{**})$.

Proof. If $\boldsymbol{m}$ is regular in the weak operator topology of $L(\boldsymbol{X}, \boldsymbol{Y}^{**})$, then $\boldsymbol{m}$ is countably additive in this topology, so it is countably additive in the strong operator topology of $L(\boldsymbol{X}, \boldsymbol{Y}^{**})$. By Theorem 2 the operator $\boldsymbol{U}_x : C(T) \to \boldsymbol{Y}^{**}$ is weakly compact for every $\boldsymbol{x} \in \boldsymbol{X}$. But $\boldsymbol{U}_x[C(T)] \subset \boldsymbol{Y}$ for every $\boldsymbol{x} \in \boldsymbol{X}$, so again by Theorem 2 $\boldsymbol{m}$ has values in $L(\boldsymbol{X}, \boldsymbol{Y})$. The converse assertion is obvious, since every countably additive Baire measure (scalar or vector) is regular, see Theorem G of § 52 in [21] and Lemma 1 in [24].

3. IDEALS OF INTEGRAL OPERATORS

In this section we show that various classes of our integral (integrally expressed) operators over $C_0(T, \boldsymbol{X})$ form closed left (in general only left) ideals in the algebra of all bounded linear operators over $C_0(T, \boldsymbol{X})$.

We say that u is a left sided operator ideal functor if for any two Banach spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$ $uL(\boldsymbol{X}, \boldsymbol{Y})$ is a closed linear subspace of $L(\boldsymbol{X}, \boldsymbol{Y})$ such that if $\boldsymbol{Z}$ is a Banach space $\boldsymbol{U} \in uL(\boldsymbol{X}, \boldsymbol{Y})$ and $\boldsymbol{V} \in L(\boldsymbol{Y}, \boldsymbol{Z})$ then $\boldsymbol{VU} \in uL(\boldsymbol{X}, \boldsymbol{Z})$. Examples of such functors, even two sided, are: a) the identity functor e, $eL(\boldsymbol{X}, \boldsymbol{Y}) = L(\boldsymbol{X}, \boldsymbol{Y})$, b) the weakly compact functor w, $wL(\boldsymbol{X}, \boldsymbol{Y})$ is the subspace of all weakly compact operators in $L(\boldsymbol{X}, \boldsymbol{Y})$, see VI.4.4 and VI.4.5 in [15], c) the compact functor c, $cL(\boldsymbol{X}, \boldsymbol{Y})$ is the subspace of all compact operators in $L(\boldsymbol{X}, \boldsymbol{Y})$, see VI.5.3 and VI.5.4 in [15], d) the unconditionally converging functor uc, $ucL(\boldsymbol{X}, \boldsymbol{Y})$ is the subspace of all unconditionally converging operators in $L(\boldsymbol{X}, \boldsymbol{Y})$, see the end of § 2, e) the completely continuous functor cc, $ccL(\boldsymbol{X}, \boldsymbol{Y})$ is the subspace of all completely continuous operators in $L(\boldsymbol{X}, \boldsymbol{Y})$ (not the same as compact), see § 6 below.

Let us have a closed left sided operator ideal functor u and denote by $I_u^i L(C_0(T, \boldsymbol{X}), \boldsymbol{Y})$, $i = 1, 2, 3$ those operators from $L(C_0(T, \boldsymbol{X}), \boldsymbol{Y})$ which can be expressed in the form (1) of Theorem 2 where the values of the measure $\boldsymbol{m}$ are operators from $uL(\boldsymbol{X}, \boldsymbol{Y})$ and for $i = 1$ the measure $\boldsymbol{m}$ is countably additive in the strong operator topology, for $i = 2$ the measure $\boldsymbol{m}$ is countably additive in the uniform operator topology and for $i = 3$ its semivariation $\hat{\boldsymbol{m}}$ is continuous on $\mathfrak{S}(\mathscr{B}_0)$. If instead of the representation (1) we have the representation (1′) of Theorem 2′, we shall write $I_u^{i'} L(C_0(T, \boldsymbol{X}), \boldsymbol{Y})$. Using this notation we have

Theorem 4. *Let u be a closed left sided operator ideal functor. Then for every $i = 1, 2, 3$ $I_u^i L(C_0(T, \boldsymbol{X}), \boldsymbol{Y})$ and $I_u^{i'} L(C_0(T, \boldsymbol{X}), \boldsymbol{Y})$ is a closed linear subspace of $L(C_0(T, \boldsymbol{X}), \boldsymbol{Y})$. If $\boldsymbol{Z}$ is a Banach space, $\boldsymbol{V} \in L(\boldsymbol{Y}, \boldsymbol{Z})$ and $\boldsymbol{U} \in I_u^i L(C_0(T, \boldsymbol{X}), \boldsymbol{Y})$ or $\boldsymbol{U} \in I_u^{i'} L(C_0(T, \boldsymbol{X}), \boldsymbol{Y})$, then $\boldsymbol{VU} \in I_u^i L(C_0(T, \boldsymbol{X}), \boldsymbol{Z})$, $\boldsymbol{VU} \in I_u^{i'} L(C_0(T, \boldsymbol{X}), \boldsymbol{Z})$ respectively for every $i = 1, 2, 3$.*

Proof. The first assertion of the theorem follows immediately from Lemma 3, Theorems 2 and 2′ and from the fact that $uL(\boldsymbol{X}, \boldsymbol{Y})$ is a closed linear subspace of $L(\boldsymbol{X}, \boldsymbol{Y})$. The second assertion of the theorem follows from the facts: a) if $\boldsymbol{Uf} = \int_T \boldsymbol{f} \, \mathrm{d}\boldsymbol{m}$, $\boldsymbol{f} \in C_0(T, \boldsymbol{X})$, then $\boldsymbol{VUf} = \int_T \boldsymbol{f} \, \mathrm{d}\boldsymbol{Vm}$, $|\boldsymbol{Vm}(E)| \leqq |\boldsymbol{V}| \, . \, |\boldsymbol{m}(E)|$ and $\widehat{\boldsymbol{Vm}}(E) \leqq |\boldsymbol{V}| \, . \, \hat{\boldsymbol{m}}(E)$, $E \in \mathfrak{S}(\mathscr{B}_0)$, see the paragraph following Theorem 7 in [13], and b)

$\boldsymbol{Vm}(E) \in uL(\boldsymbol{X}, \boldsymbol{Z})$ for every set $E \in \mathfrak{S}(\mathscr{B}_0)$, since u is a closed left sided operator ideal functor.

Corollary. *Let u be a closed left sided operator ideal functor. Then for every $i = 1, 2, 3$ $I_u^i L(C_0(T, \boldsymbol{X}), C_0(T, \boldsymbol{X}))$ and $I_u^{i'} L(C_0(T, \boldsymbol{X}), C_0(T, \boldsymbol{X}))$ is a closed left ideal in the algebra of all operators $L(C_0(T, \boldsymbol{X}), C_0(T, \boldsymbol{X}))$.*

The identity functor e is a closed two sided operator ideal functor. Nevertheless, in the following simple example we show that $I_e^1 L(c_0(l_1), c_0(l_1))$ is not a two sided ideal in the algebra $L(c_0(l_1), c_0(l_1))$.

Example. Let us define the bounded linear operator $\boldsymbol{V}$, $\boldsymbol{V} : c_0(l_1) \to c_0(l_1)$ in the following way: if $\boldsymbol{x} = [x_1, x_2, \ldots, x_n, \ldots] \in l_1$ and $\varphi = [a_1, a_2, \ldots, a_n, \ldots] \in c_0$, then we put $\boldsymbol{Vx}\varphi = \psi \in c_0(l_1)$, $\psi = [\psi_1, \psi_2, \ldots, \psi_n, \ldots]$ where $\psi_1 = [x_1 a_1, 0, 0, \ldots] \in l_1$, $\psi_2 = [0, x_1 a_2, 0, 0, \ldots] \in l_1, \ldots, \psi_n = [0, 0, \ldots, 0, x_1 a_n, 0, 0, \ldots] \in l_1$. Obviously $\boldsymbol{V}$ may be linearly extended to the set $\mathfrak{Q}_1$ of all functions of the form $\boldsymbol{f} = \sum_{i=1}^{r} \boldsymbol{x}_i \varphi_i$, r finite, and $|\boldsymbol{Vf}| \leqq \|\boldsymbol{f}\|_T$ for every function $\boldsymbol{f} \in \mathfrak{Q}_1$. Since $\mathfrak{Q}_1$ is a dense subset of $c_0(l_1)$, we may extend $\boldsymbol{V}$ to the whole $c_0(l_1)$ without increasing its norm.

Further we define a bounded linear operator $\boldsymbol{U} : c_0(l_1) \to c_0(l_1)$ in the following way: for $\boldsymbol{x} = [x_1, x_2, \ldots, x_n, \ldots] \in l_1$ and $\varphi = [a_1, a_2, \ldots, a_n, \ldots] \in c_0$ we put $\boldsymbol{Ux}\varphi = \boldsymbol{x} \,.\, \varphi'$ where $\varphi' = [x_1 a_1, x_2 a_2, \ldots, x_n a_n, \ldots] \in c_0$. Obviously $\boldsymbol{U}$ may be extended to a bounded linear operator on $\mathfrak{Q}_1$ with $|\boldsymbol{U}| \leqq 1$, and therefore also onto $c_0(l_1)$ without increasing its norm. Clearly for every $\boldsymbol{x} \in l_1$ the operator $\boldsymbol{U}_{\boldsymbol{x}} : c_0 \to c_0(l_1)$ defined by the equality $\boldsymbol{U}_{\boldsymbol{x}}\varphi = \boldsymbol{Ux}\varphi$, $\varphi \in c_0$ is weakly compact. Therefore by Theorem 2 $\boldsymbol{U} \in I_e^1 L(c_0(l_1), c_0(l_1))$.

We now show that $\boldsymbol{UV} \notin I_e^1 L(c_0(l_1), c_0(l_1))$. In view of Theorem 2 it is enough to find an $\boldsymbol{x}_1 \in \boldsymbol{X}$ for which the operator $(\boldsymbol{UV})_{\boldsymbol{x}_1} : c_0 \to c_0(l_1)$ is not weakly compact. Let $\boldsymbol{x}_1 = [1, 0, 0, \ldots] \in l_1$ and let us have a $\varphi = [a_1, a_2, \ldots, a_n, \ldots] \in c_0$. Then $\boldsymbol{UVx}_1\varphi = [\bar{\boldsymbol{a}}_1, \bar{\boldsymbol{a}}_2, \ldots, \bar{\boldsymbol{a}}_n, \ldots] \in c_0(l_1)$ where $\bar{\boldsymbol{a}}_1 = [a_1, 0, 0, \ldots] \in l_1, \ldots, \bar{\boldsymbol{a}}_n = [0, 0, \ldots, 0, a_n, 0, \ldots] \in l_1$. If we now take the sequence $\varphi_1 = [1, 0, 0, \ldots] \in c_0, \ldots, \varphi_n = [\overbrace{1, 1, \ldots, 1}^{n}, 0, 0, \ldots] \in c_0$, then every φ_n, $n = 1, 2, \ldots$ is in the unit sphere of c_0, but the sequence $\boldsymbol{UVx}_1\varphi_n \in c_0(l_1)$, $n = 1, 2, \ldots$, by Theorem 9 below obviously contains no weakly convergent subsequence in $c_0(l_1)$. Hence the operator $(\boldsymbol{UV})_{\boldsymbol{x}_1} : c_0 \to c_0(l_1)$ is not weakly compact, which is what we wanted.

We say that u is a closed operator functor iff for any two Banach spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$ $uL(\boldsymbol{X}, \boldsymbol{Y})$ is a closed linear subspace of $L(\boldsymbol{X}, \boldsymbol{Y})$. Obviously each left sided closed operator ideal functor is a closed operator functor. Examples of such functors were given before Theorem 4. Using this notion we have:

Theorem 5. *Let λ be a finite or infinite countably additive scalar measure on $\mathfrak{S}(\mathscr{B}_0)$, let u be a closed operator functor and let $\boldsymbol{F} : T \to uL(\boldsymbol{X}, \boldsymbol{Y})$ be a measurable* *function (in the norm topology of* $uL(\mathbf{X}, \mathbf{Y})$*) with* $\int_T |\mathbf{F}| \, \mathrm{d}v(\lambda) < +\infty$*. Then the mapping* $\mathbf{U} : C_0(T, \mathbf{X}) \to \mathbf{Y}$ *defined by the equality* $\mathbf{U}f = \int_T \mathbf{F}f \, \mathrm{d}\lambda$, $f \in C_0(T, \mathbf{X})$ *is an element of* $uL(C_0(T, \mathbf{X}), \mathbf{Y})$ *and* $|\mathbf{U}| \leqq \int_T |\mathbf{F}| \, \mathrm{d}v(\lambda)$.

Proof. If we put $\mathbf{m}(E) = \int_E \mathbf{F} \, \mathrm{d}\lambda$, $E \in \mathfrak{S}(\mathscr{B}_0)$, then clearly $\mathbf{m}$ is an operator valued Baire measure with a finite variation, $\hat{\mathbf{m}}(E) \leqq v(\mathbf{m}, E) = \int_E |\mathbf{F}| \, \mathrm{d}v(\lambda) < +\infty$ for each set $E \in \mathfrak{S}(\mathscr{B}_0)$. Thus by Theorem 2 $\mathbf{U}$ is a bounded linear operator on $C_0(T, \mathbf{X})$ with $|\mathbf{U}| \leqq \int_T |\mathbf{F}| \, \mathrm{d}v(\lambda)$. In view of the elementary properties of the Bochner integral, for every $n = 1, 2, \ldots$ there is a $\mathfrak{S}(\mathscr{B}_0)$-simple function $\mathbf{F}_n$ with values in $uL(\mathbf{X}, \mathbf{Y})$ such that $\int_T |\mathbf{F}_n - \mathbf{F}| \, \mathrm{d}v(\lambda) < 1/n$. But for every n the operator $\mathbf{U}_n$, $\mathbf{U}_n f = \int_T \mathbf{F}_n f \, \mathrm{d}\lambda$, $f \in C_0(T, \mathbf{X})$ is an element of $uL(C_0(T, \mathbf{X}), \mathbf{Y})$ (if $\mathbf{F}_n = \sum_{i=1}^{r} \mathbf{F}_{n,i} \cdot \chi_{E_i}$, $\mathbf{F}_{n,i} \in uL(\mathbf{X}, \mathbf{Y})$, $E_i \in \mathfrak{S}(\mathscr{B}_0)$ being disjoint, then $\mathbf{U}_n f = \int_T \mathbf{F}_n f \, \mathrm{d}\lambda = \sum_{i=1}^{r} \mathbf{F}_{n,i} \int_{E_i} f \, \mathrm{d}\lambda$, $f \in C_0(T, \mathbf{X})$, see the paragraph after Theorem 7 in [13]), and for every n $|\mathbf{U}_n - \mathbf{U}| \leqq \int_T |\mathbf{F}_n - \mathbf{F}| \, \mathrm{d}v(\lambda) < 1/n$. Since $uL(C_0(T, \mathbf{X}), \mathbf{Y})$ is a closed linear subspace of $L(C_0(T, \mathbf{X}), \mathbf{Y})$, $\mathbf{U}$ is an element of $uL(C_0(T, \mathbf{X}), \mathbf{Y})$. Thus the theorem is proved.

4. ON ALMOST DIFFUSE OPERATORS

C. Foias and I. Singer introduced in [18] interesting classes of so called almost diffuse and countably almost diffuse operators $\mathbf{U} : C_0(T, \mathbf{X}) \to \mathbf{Y}$. According to Definition 1 in [18] a point $t \in T$ is a diffuse point of an operator $\mathbf{U} : C_0(T, \mathbf{X}) \to \mathbf{Y}$, if the infimum over the open sets V, $t \in V \subset T$ of $\sup \{|\mathbf{U}f|, f \in K(V, \mathbf{X}), \|f\|_V \leqq 1\}$ is equal to zero. Here $K(V, \mathbf{X})$ denotes the set of all $\mathbf{X}$ valued continuous functions on V with compact supports in V. We denote by $D(\mathbf{U})$ the set of all diffuse points of an operator $\mathbf{U} : C_0(T, \mathbf{X}) \to \mathbf{Y}$. A point $t \in T - D(\mathbf{U})$ is called a point of concentration of the operator $\mathbf{U}$. An operator $\mathbf{U} : C_0(T, \mathbf{X}) \to \mathbf{Y}$ is called almost diffuse or countably almost diffuse, if $D(\mathbf{U})$ is a dense subset of T or if $T - D(\mathbf{U})$ is a countably set, respectively.

Let now $\mathbf{m}$ be a Baire operator valued measure. Then a point $t \in T$ is called a diffuse point of the measure $\mathbf{m}$, if $\inf \hat{\mathbf{m}}(V) = 0$ where the infimum is taken over all open Baire sets V containing t. For example, if λ is a finite non negative countably additive Baire measure, then each point $t \in T$ for which $\bar{\lambda}(\{t\}) = 0$ where $\bar{\lambda}$ is the regular Borel extension of λ, see Theorem D of § 54 in [21], is a diffuse point of λ. This follows from the regularity of $\bar{\lambda}$ and from Theorem D of § 50 in [21]. In this example the points of concentration of λ are exactly the atoms of $\bar{\lambda}$, of which there is at most a countable number.

Let a bounded linear operator $\mathbf{U} : C_0(T, \mathbf{X}) \to \mathbf{Y}$ be represented in the form (1) of Theorem 2 or in the form (1') of Theorem 2'. Then by Theorem D of § 50 in [21] a point $t \in T$ is a diffuse point of the operator $\mathbf{U}$ if and only if it is a diffuse point of the corresponding representing Baire measure $\mathbf{m}$.

Theorem 2 in [18] states that each compact and each majorable linear operator $\mathbf{U} : C_0(T, \mathbf{X}) \to \mathbf{Y}$ is countably almost diffuse. The following theorem substantially extends this result, see its Corollary.

Theorem 6. *Each bounded linear operator $\mathbf{U} : C_0(T, \mathbf{X}) \to \mathbf{Y}$ which can be represented in the form (1) of Theorem 2 where the semivariation $\hat{\mathbf{m}}$ of the measure $\mathbf{m}$ is continuous on $\mathfrak{S}(\mathscr{B}_0)$ is countably almost diffuse.*

Proof. We follow the idea of the Remark of A. Pelczyński on p. 441 in [18]. Namely, if the semivariation $\hat{\mathbf{m}}$ is continuous on $\mathfrak{S}(\mathscr{B}_0)$, then by Lemma 2 there is a finite non negative countably additive measure λ on $\mathfrak{S}(\mathscr{B}_0)$ with $\lim_{\lambda(E) \to 0} \hat{\mathbf{m}}(E) = 0$, $E \in \mathfrak{S}(\mathscr{B}_0)$. Hence it is obvious that a point $t \in T$ is a diffuse point of the measure $\mathbf{m}$ if it is a diffuse point of the measure λ. But the concentration points of the measure λ are exactly the atoms of its Borel extension $\bar{\lambda}$, of which there is at most a countable number. This proves the theorem.

From here and from Theorem 3 we obtain the next

Corollary. *Every unconditionally converging bounded linear operator, particularly each weakly compact linear operator $\mathbf{U} : C_0(T, \mathbf{X}) \to \mathbf{Y}$ is countably almost diffuse.*

By this occasion let us note that the operator u from Example 2 in [18] is a bounded linear operator with one concentration point that cannot be represented in the form (1) of Theorem 2, which in this case reduces to Theorem (A). Supposing that this were possible we obtain that u is a weakly compact operator, so by Theorem VI.7.5 in [15] its square u^2 is a compact linear operator, a contradiction.

In a similar way as Theorem 6 we may prove

Theorem 7. *Every bounded linear operator $\mathbf{U} : C_0(T, \mathbf{X}) \to \mathbf{Y}$ which can be represented in the form (1') of Theorem 2' where the semivariation $\hat{\mathbf{m}}$ of the measure $\mathbf{m}$ is continuous on $\mathscr{B}_0$ is almost diffuse. More precisely, each set $E \in \mathfrak{S}(\mathscr{B}_0)$ contains at most a countable number of concentration points of the operator $\mathbf{U}$. If T is a σ-compact set, then the operator $\mathbf{U}$ is countably almost diffuse.*

The main result of [18] is the following theorem, Theorem 3 in [18] (its proof remains valid for locally compact Hausdorff T):

(B) *T contains no isolated points if and only if for each almost diffuse bounded linear operator $\mathbf{U} : C_0(T, \mathbf{X}) \to C_0(T, \mathbf{X})$ the equality $|1 + \mathbf{U}| = 1 + |\mathbf{U}|$ holds.*

From here and from the Corollary of Theorem 6 we immediately have the important:

Theorem 8. *T contains no isolated points if and only if for each unconditionally* *converging bounded linear operator, particularly for each weakly compact linear operator,* $\mathbf{U} : C_0(T, \mathbf{X}) \to C_0(T, \mathbf{X})$ *the norm equality* $|1 + \mathbf{U}| = 1 + |\mathbf{U}|$ *holds.*

In [18] this is proved for compact and majorable operators, see the Corollary 4 in [18]. In this occasion see also [25].

5. WEAK CONVERGENCE IN $C_0(T, \mathbf{X})$

As it is well known, see [30], and as it also follows immediately from Theorem 2, the dual of $C_0(T, \mathbf{X})$ is isometrically isomorphic to the space $cabv(\mathfrak{S}(\mathscr{B}_0), \mathbf{X}^*)$ of countably additive $\mathbf{X}^*$ valued vector measures with bounded variations. Hence we have the following result:

Theorem 9. *A sequence* $\mathbf{f}_n \in C_0(T, \mathbf{X})$, $n = 1, 2, \ldots$ *weakly converges to a function* $\mathbf{f}_0 \in C_0(T, \mathbf{X})$ *if and only if it is bounded in* $C_0(T, \mathbf{X})$ *and* $\lim_{n \to \infty} \mathbf{x}^* \mathbf{f}_n(t) = \mathbf{x}^* \mathbf{f}_0(t)$ *for each point* $t \in T$ *and each functional* $\mathbf{x}^* \in \mathbf{X}^*$.

For T being a compact interval of reals this theorem was first proved in [7, Theorem 4.3]. For $\mathbf{X}$ reflexive or separable it is stated in [8, Theorem 5]. The general case can be reduced to the case when $\mathbf{X}$ is separable. Since no proof is given in [8] we note that a proof may be given using the deep result of VI.8.7 in [15].

Let us denote by $\mathscr{R}$ the σ-ring of all countable subsets of T and let $cabv(\mathscr{R}, \mathbf{X}^*)$ denote the set of all those measures from $cabv(\mathfrak{S}(\mathscr{B}_0), \mathbf{X}^*)$ which are of the form $\mathbf{m} = \sum_{t \in E} \mathbf{x}_t^* \,.\, \mu_t$ where $E \in \mathscr{R}$, $\mathbf{x}_t^* \in \mathbf{X}^*$ and $\mu_t(A) = \chi_A(t)$ for $A \in \mathfrak{S}(\mathscr{B}_0)$. Then similarly as Theorem 4.4 in [7] we may prove the following interesting

Theorem 10. *Let* $\mathbf{f}_n \in C_0(T, \mathbf{X})$, $n = 1, 2, \ldots$ *and let for every measure* $\mathbf{m} \in cabv(\mathscr{R}, \mathbf{X}^*)$, $\lim_{n \to \infty} \int_T \mathbf{f}_n \, d\mathbf{m} = 0$. *Then the sequence* $\{\mathbf{f}_n\}_{n=1}^{\infty}$ *weakly converges to zero in* $C_0(T, \mathbf{X})$.

From here and from Eberlein-Šmulian's Theorem, see [34] or V.6.1 in [15] we immediately obtain

Theorem 11. *A subset* $F \subset C_0(T, \mathbf{X})$ *is weakly relatively compact if and only if it is sequentially compact in the weak topology* $\sigma(C_0(T, \mathbf{X}), cabv(\mathscr{R}, \mathbf{X}^*))$.

Concerning the $C_0(T, \mathbf{X})$ convergence of sequences in its dual, similarly as Theorem IV.9.15 in [15] we now prove its following generalization, see also Theorem 6 in [8].

Theorem 12. *Let* $\{\mathbf{m}_n\}_{n=0}^{\infty}$ *be a bounded sequence in the space* $cabv(\mathfrak{S}(\mathscr{B}_0), \mathbf{X}^*)$ *and let* $\lim_{n \to \infty} \mathbf{m}_n(G)\, \mathbf{x} = \mathbf{m}_0(G)\, \mathbf{x}$ *for each* $\mathbf{x} \in \mathbf{X}$ *and each open set* $G \in \mathfrak{S}(\mathscr{B}_0)$ *with* $v(\mathbf{m}_0, \bar{G} - G) = 0$ *where* $\bar{G}$ *is the closure of* G *in* T. *Then* $\lim_{n \to \infty} \int_T \mathbf{f} \, d\mathbf{m}_n = \int_T \mathbf{f} \, d\mathbf{m}_0$ *for every function* $\mathbf{f} \in C_0(T, \mathbf{X})$.

Proof. In view of the boundedness of the sequence $\{\boldsymbol{m}_n\}_{n=0}^{\infty}$ in the dual space $C_0^*(T, \boldsymbol{X}) = cabv(\mathfrak{S}(\mathscr{B}_0), \boldsymbol{X}^*)$ it is sufficient to prove the theorem for the dense subset $\mathfrak{Q} \subset C_0(T, \boldsymbol{X})$ ($\mathfrak{Q}$ was defined before Theorem 1). But for $\boldsymbol{f} \in \mathfrak{Q}$ the assertion of the theorem may be proved in a similar way as Theorem IV.9.15 in [15].

6. ON THE DUNFORD-PETTIS PROPERTY OF $C_0(T, \boldsymbol{X})$

We say that a bounded linear operator $\boldsymbol{V} : \boldsymbol{X} \to \boldsymbol{Y}$ is a completely continuous operator, shortly a *cc* operator, if it transforms weakly fundamental sequences into convergent ones. It is easy to see that the *cc* functor is a closed two sided operator ideal functor. Evidently each compact linear operator is a *cc* operator. By definition, see [9] and [19], we say that a Banach space $\boldsymbol{X}$ has the Dunford-Pettis property, shortly the D-P property, if each weakly compact linear operator $\boldsymbol{V} : \boldsymbol{X} \to \boldsymbol{Y}$, $\boldsymbol{Y}$ being arbitrary, is a *cc* operator. If $\boldsymbol{U} : \boldsymbol{Z} \to \boldsymbol{X}$ is a weakly compact linear operator and $\boldsymbol{V} : \boldsymbol{X} \to \boldsymbol{Y}$ is a bounded linear *cc* operator, then by Eberlein-Šmulian's Theorem, see [34] or V.6.1 in [15], their product $\boldsymbol{VU}$ is a compact linear operator. Therefore, if $\boldsymbol{X}$ has the D-P property, the product of two weakly compact linear operators $\boldsymbol{U} : \boldsymbol{Z} \to \boldsymbol{X}$ and $\boldsymbol{V} : \boldsymbol{X} \to \boldsymbol{Y}$, $\boldsymbol{VU}$ is a compact linear operator. Particularly for $\boldsymbol{X}$ with the D-P property the square of any weakly compact linear operator over $\boldsymbol{X}$ is a compact operator. These facts make it important to establish that a given Banach space has the D-P property. As it is well known any space of scalar functions $C_0(T)$ has the D-P property, as well as any scalar space L_1, see [9], [19] or VI.7.5 and VI.8.12 in [15]. The following theorem may be considered as the first very partial result on investigation of the D-P property of the space $C_0(T, \boldsymbol{X})$.

Theorem 13. a) *Let T be a discrete topological space and let $\boldsymbol{X}$ have the D-P property. Then $C_0(T, \boldsymbol{X})$ has also this property.* b) *Let the weak and the strong convergences of sequences coincide in $\boldsymbol{X}$ (then evidently each bounded linear operator $\boldsymbol{V} : \boldsymbol{X} \to \boldsymbol{Y}$ is a cc operator), for example let $\boldsymbol{X} = l_1$. Then for any locally compact Hausdorff topological space T $C_0(T, \boldsymbol{X})$ has the D-P property.*

Proof. Let $\boldsymbol{f}_n \in C_0(T, \boldsymbol{X})$, $n = 1, 2, \dots$ be a weakly fundamental sequence, let $\|\boldsymbol{f}_n\|_T \leqq K$, $n = 1, 2, \dots$ for some finite K, and let $\boldsymbol{U} : C_0(T, \boldsymbol{X}) \to \boldsymbol{Y}$ be a weakly compact linear operator. By Remark 1 in § 2 $\boldsymbol{U}$ can be represented in the form $\boldsymbol{Uf} = \int_T \boldsymbol{f}\, \mathrm{d}\boldsymbol{m}$, $\boldsymbol{f} \in C_0(T, \boldsymbol{X})$ where $\boldsymbol{m}$ is a Baire operator valued measure on $\mathfrak{S}(\mathscr{B}_0)$ with $\hat{\boldsymbol{m}}(T) = |\boldsymbol{U}|$ whose values are weakly compact operators from $L(\boldsymbol{X}, \boldsymbol{Y})$ and its semivariation $\hat{\boldsymbol{m}}$ is continuous on $\mathfrak{S}(\mathscr{B}_0)$. Using this representation we extend the operator $\boldsymbol{U}$ from $C_0(T, \boldsymbol{X})$ onto $\overline{\mathfrak{J}}_s$ without increasing its norm, see the beginning of § 1. Let $\varepsilon > 0$ and put

$$A = \bigcup_{n=1}^{\infty} \left\{ t \in T,\ |\boldsymbol{f}_n(t)| \geqq \frac{\varepsilon}{6(1 + |\boldsymbol{U}|)} \right\}.$$

Then $A \in \mathfrak{S}(\mathscr{B}_0)$ and from elementary properties of the integral we obtain the inequality $|\boldsymbol{U}\boldsymbol{f}_n \cdot \chi_{T-A}| \leqq \varepsilon/6$ for every n.

a) In this case A is clearly a countable set, $A = \{t_1, t_2, \ldots\}$. Since the semivariation $\hat{\boldsymbol{m}}$ is continuous on $\mathfrak{S}(\mathscr{B}_0)$, there is a k_0 such that for $B_{k_0} = \{t_{k_0+1}, t_{k_0+2}, \ldots\}$, and for every n $|\int_{B_{k_0}} \boldsymbol{f}_n \, \mathrm{d}\boldsymbol{m}| \leqq \varepsilon/6$. Thus for every n, $p = 1, 2, \ldots$ we have the inequality $|\boldsymbol{U}\boldsymbol{f}_n - \boldsymbol{U}\boldsymbol{f}_p| \leqq |\int_{\{t_1, \ldots, t_{k_0}\}} (\boldsymbol{f}_n - \boldsymbol{f}_p) \, \mathrm{d}\boldsymbol{m}| + 2\varepsilon/3$. Since for every $i = 1, 2, \ldots, k_0$ the sequence $\{\boldsymbol{f}_n(t_i)\}_{n=1}^{\infty}$ is weakly fundamental, see Theorem 9, since $\boldsymbol{m}(\{t_i\})$ is a weakly compact operator from $L(\boldsymbol{X}, \boldsymbol{Y})$ and since $\boldsymbol{X}$ has the D-P property, there is an n_0 such that for $n, p \geqq n_0$ it is $|\boldsymbol{U}\boldsymbol{f}_n - \boldsymbol{U}\boldsymbol{f}_p| < \varepsilon$. Thus we proved that in this case $C_0(T, \boldsymbol{X})$ has the D-P property.

b) Since the semivariation $\hat{\boldsymbol{m}}$ is continuous on $\mathfrak{S}(\mathscr{B}_0)$, Lemma 2 implies that there is a finite non negative countably additive measure λ on $\mathfrak{S}(\mathscr{B}_0)$ with $\lim_{\lambda(E) \to 0} \hat{\boldsymbol{m}}(E) = 0$, $E \in \mathfrak{S}(\mathscr{B}_0)$. Choose $\delta > 0$ such that $\lambda(E) < \delta$, $E \in \mathfrak{S}(\mathscr{B}_0)$ implies $\hat{\boldsymbol{m}}(E) < \varepsilon/6K$. Since the sequence $\{\boldsymbol{f}_n\}_{n=1}^{\infty}$ is weakly fundamental in $C_0(T, \boldsymbol{X})$, by Theorem 9 for each functional $\boldsymbol{x}^* \in \boldsymbol{X}^*$ and each point $t \in T$ there is a finite limit $\lim_{n \to \infty} \boldsymbol{x}^* \boldsymbol{f}_n(t)$. But by the assumption the weak and the strong convergences of sequences coincide in $\boldsymbol{X}$, and therefore for each point $t \in T$ there exists a limit $\lim_{n \to \infty} \boldsymbol{f}_n(t) = \boldsymbol{f}(t) \in \boldsymbol{X}$. By Egoroff's Theorem for the measure λ there is a set $F \in \mathfrak{S}(\mathscr{B}_0)$ with $\lambda(F) < \delta$ such that on $A - F$ the sequence $\{\boldsymbol{f}_n\}_{n=1}^{\infty}$ converges uniformly to the function $\boldsymbol{f}$. Choose n_0 such that for $n, p \geqq n_0$ it is $\|\boldsymbol{f}_n - \boldsymbol{f}_p\|_{A-F} \leqq \varepsilon/[6(1 + \hat{\boldsymbol{m}}(T))]$. Then from the inequality $|\boldsymbol{U}\boldsymbol{f}_n - \boldsymbol{U}\boldsymbol{f}_p| \leqq |\int_{A-F} (\boldsymbol{f}_n - \boldsymbol{f}_p) \, \mathrm{d}\boldsymbol{m}| + |\int_F (\boldsymbol{f}_n - \boldsymbol{f}_p) \, \mathrm{d}\boldsymbol{m}| + \varepsilon/3$, $n, p = 1, 2, \ldots$ we immediately obtain that for $n, p \geqq n_0$ it is $|\boldsymbol{U}\boldsymbol{f}_n - \boldsymbol{U}\boldsymbol{f}_p| \leqq \varepsilon$. Thus we proved that in this case $C_0(T, \boldsymbol{X})$ has the D-P property.

The theorem just proved is a very partial solution of our perhaps most important and at the same time most difficult problem, which reads as follows: If $\boldsymbol{X}$ has the D-P property, has also $C_0(T, \boldsymbol{X})$ this property? We note that the affirmative answer, besides the cases just proved, is obtained obviously also when $\boldsymbol{X}$ is isometrically isomorphic to some $C_0(T_1)$, since then $C_0(T, C_0(T_1)) = C_0(T \times T_1)$, see Example 1 on p. 89 in [20].

Let us note that from the proof of assertion b) of Theorem 13 it is evident that if an operator $\boldsymbol{U} : C_0(T, \boldsymbol{X}) \to \boldsymbol{Y}$ can be represented in the form (1) of Theorem 2 where the semivariation $\hat{\boldsymbol{m}}$ of the measure $\boldsymbol{m}$ is continuous on $\mathfrak{S}(\mathscr{B}_0)$, then this operator transforms bounded and almost everywhere $\boldsymbol{m}$ convergent sequences in $C_0(T, \boldsymbol{X})$ into convergent ones in $\boldsymbol{Y}$. In this respect see also Theorems 15 and 16 in [13] and [37].

Finally, from Theorem 5 we immediately have

Theorem 14. *Let us have a weakly compact linear operator $\boldsymbol{U} : C_0(T, \boldsymbol{X}) \to C_0(T, \boldsymbol{X})$ defined in Theorem 5 and let $\boldsymbol{X}$ have the D-P property. Then $\boldsymbol{U}^2 = \boldsymbol{U}\boldsymbol{U}$ is a compact linear operator.*

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