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COMMUTATIVE CANCELLATIVE SEMIGROUPS WITH TWO GENERATORS

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A semigroup S is power joined if for each pair $a, b \in S$ there exist positive integers m and n such that $a^m = b^n$. In [3] PETRICH determined all power joined, commutative, cancellative semigroups, without an identity, that can be generated by two elements. In this note we find the remaining commutative cancellative semigroups that can be generated by two elements.

In what follows, N will denote the additive semigroup of positive integers, N_0 will denote the additive semigroup of nonnegative integers, Z will denote the additive group of integers, and Z_m will denote the additive group of integers modulo m. Elements of Z_m will be written in the form k_m. For concepts not defined in this note the reader is referred to [1] and [2].

Lemma. *Let S be a commutative cancellative semigroup. Then*

i) *S has an identity if and only if there is an $a \in S$ such that $a^n = a$ for some $n \in N$ where $n \geqq 2$.*

ii) *If there exist $a, b \in S$ with $a^m b^n = b$ where $m, n \in N$, S has an identity.*

iii) *If S has an identity e, and we denote by S_e the group of units of S, then $ab \in S_e$ implies that $a \in S_e$ and $b \in S_e$.*

Proof. i) If S has an identity e then $e^2 = e$. Conversely, let $a \in S$ satisfy $a^n = a$ where $n \geqq 2$. If $n = 2$ then $a^2 = a$, and if $n > 2$

$$(a^{n-1})^2 = a^{2n-2} = a^{n+n-2} = a^n a^{n-2} = a a^{n-2} = a^{n-1} .$$

Thus in either case a^{n-1} is an idempotent. We complete the proof by noting that if b is any idempotent in S and x is any element of S, $bx = bbx$, whence $x = bx$ by cancellation, and b is an identity for S.

ii) Assume $n > 1$. Then

$$(a^m b^{n-1})^2 = a^{2m} b^{2n-2} = a^m (a^m b^n)\, b^{n-2} = a^m b b^{n-2} = a^m b^{n-1} .$$

If $n = 1$ then $a^{2m}b = a^m(a^mb) = a^mb$ whence $a^{2m} = a^m$ by cancellation. In either case S has an idempotent and hence an identity by i).

iii) Recall that S_e consists of all those elements $x \in S$ for which there is a $y \in S$ such that $xy = e$. If $ab \in S_e$ there is a $c \in S$ with $(ab)\,c = e$. Then $a(bc) = e$ and $b(ac) = e$ whence $a \in S_e$ and $b \in S_e$.

We first determine the commutative cancellative semigroups with identity that can be generated by two elements. When considering those that are groups it is important to keep in mind that we are looking at them as semigroups. Hence two elements a and b of a semigroup S generate S if every element of S can be written in the form a^mb^n where $m, n \in N_0$ and $m + n > 0$. By a^mb^n where $m = 0$ and $n > 0$ we mean b^n.

Theorem 1. *Let S be one of the following types of groups: Z_m, Z, $Z_m \times Z_n$, $Z_m \times Z$. Then S is an abelian group that has a semigroup generating set of two elements. Conversely, every such group is isomorphic to one of these types of groups.*

Proof. It is clear that Z_m, Z and $Z_m \times Z_n$ possess semigroup generating sets of two elements. In $Z_m \times Z$ consider the set $A = \{(1_m, 1), (0_m, -1)\}$. Let $(k_m, n) \in Z_m \times Z$ where $n \geqq 0$. Then

$$(k_m, n) = p(1_m, 1) + (p - n)(0_m, -1),$$

where $p \equiv k \pmod m$ and $p > n$. For $(k_m, -n) \in Z_m \times Z$ where $n > 0$ we have

$$(k_m, -n) = k(1_m, 1) + (n + k)(0_m, -1).$$

Since the coefficients in both cases are nonnegative, and at least one is positive, $Z_m \times Z$ is generated by A as a semigroup. Conversely, if S is any abelian group having a semigroup generating set of two elements, then S is certainly generated by two elements as a group. It follows from the fundamental theorem on finitely generated abelian groups that S is limited to the following: Z_m, Z, $Z_m \times Z_n$, $Z_m \times Z$, $Z \times Z$. To complete the proof we show that $Z \times Z$ cannot be generated by two elements as a semigroup. Suppose, on the contrary that (a, b) and (c, d) generate $Z \times Z$ as a semigroup.

It follows that (a, b) and (c, d) generate $Q \times Q$ as a vector space over Q, where Q is the field of rational numbers. But any generating set of two elements of $Q \times Q$ is a basis, and hence linearly independent over Q. Since (a, b) and (c, d) generate $Z \times Z$ as a semigroup there are $m, n \in N_0$ where $m + n > 0$ and $m(a, b) + n(c, d) = (0, 0)$, contradicting the linear independence of (a, b) and (c, d). Hence no such generating set exists.

Theorem 2. *Let $S = Z_m \times N_0$. Then S is a commutative cancellative semigroup with identity having a semigroup generating set of two elements, that is not a group. Conversely, every such semigroup is isomorphic to $Z_m \times N_0$ for some $m \in N$.*

Proof. That $Z_m \times N_0$ has the desired properties is clear. We show the converse. Let S have the stated properties. Let a and b generate S. Since S is not a group at most one of these elements may be in the group of units. It follows from part iii) of the lemma that exactly one of these elements, say a, is in the group of units and that a generates the group of units. Thus the group of units is isomorphic to Z_m for some $m \in N$. We will show that every element of S can be uniquely written in the form $a^i b^j$ where $1 \leqq i \leqq m$ and $j \in N_0$. It is clear that every element can be written in this form so suppose $a^i b^j = a^k b^n$ where $1 \leqq i, k \leqq m$ and $j, n \in N_0$. If $j > n$ cancellation yields $a^i b^{j-n} = a^k$ from which it follows from part iii) of the lemma that b^{j-n} and hence b is in the group of units. Since S is not an abelian group this is impossible. Similar remarks apply to the case where $n > j$. Hence $n = j$. We may cancel to get $a^i = a^k$ which implies that $i = k$ since a has order m and $1 \leqq i, k \leqq m$. Hence the representation is unique. Then mapping θ defined by $(a^i b^j)\,\theta = (i, j)$ is clearly an isomorphism of S onto $Z_m \times N_0$.

We next determine the commutative cancellative semigroups without an identity that can be generated by two elements.

Theorem 3. *Let F_2 denote the set $\{(m, n) \mid m, n \in N_0, m + n > 0\}$ with the operation*

$$(m, n) + (r, s) = (m + r, n + s)\,.$$

Then F_2 is a non power joined, commutative, cancellative semigroup without an identity that can be generated by two elements. Conversely, every such semigroup is isomorphic to F_2.

Proof. It is clear that F_2 is a commutative cancellative semigroup without an identity that is generated by $(1, 0)$ and $(0, 1)$. Since $(1, 0)^m \neq (0, 1)^n$ for all $m, n \in N$, F_2 is not power joined. Conversely, let S be any such semigroup, and let a and b generate S. If $a^m = b^n$ for $m, n \in N$, the fact that a and b generate S would imply that S is power joined. Hence a and b are not power joined. We claim that every element of S can be written uniquely in the form $a^i b^j$ where $i, j \in N_0$ and $i + j > 0$. It is clear that every element can be written in this form, so we need only show uniqueness. Suppose $a^i b^j = a^m b^n$ where $i, j, m, n \in N_0$ with $i + j > 0$ and $m + n > 0$. We consider cases.

1) Assume $i > m$ and $n > j$. Then cancellation yields $a^{i-m} = b^{n-j}$ which contradicts the fact that a and b are not power joined. Similar remarks apply to the case where $m > i$ and $j > n$.

2) Assume $i > m$ and $j = n$. If $m \neq 0$ cancellation yields $a^i = a^m$ whence $a^{i-m+1} = a$, and by part i) of the lemma S has an identity, contrary to hypothesis. If $m = 0$ then $a^i b^j = b^n$ whence $a^i b = b$ and S has an identity by part ii) of the lemma, contrary to hypothesis. If $i = m$ and $n > j$ similar remarks apply.

3) Assume $i > m$ and $j > n$. If $m > 0$ then $a^{i-m+1}b^{j-n} = a$ whence again by part ii) of the lemma, S has an identity, contrary to hypothesis. If $n > 0$ we have $a^{i-m}b^{j-n+1} = b$ and a similar contradiction arises. Similar remarks apply to the case where $m > i$ and $n > j$.

These cases exhaust all possibilities and hence the representation is unique. The mapping θ defined by $(a^i b^j)\,\theta = (i, j)$ is clearly isomorphism of S onto F_2.

It remains to find all power joined, commutative, cancellative semigroups without identity that can be generated by two elements. It is clear that any such semigroup that can be generated by one element must be isomorphic to N, the semigroup of positive integers under addition. The remaining semigroups of this type are determined by the following theorem due to Petrich.

Theorem 4. *Let $m, n \in N$ with $2 \leqq m \leqq n$. Let $N(m, n)$ denote the set $\{(k_1, k_2) \mid \mid k_1, k_2 \in N_0, 0 \leqq k_2 < n, k_1 + k_2 > 0\}$ with the operation*

$$(j_1, j_2) + (k_1, k_2) = (j_1 + k_1 + im,\ j_2 + k_2 - in)$$

where i is the integer such that $0 \leqq j_2 + k_2 - in < n$.

Then $N(m, n)$ is a power joined, commutative, cancellative semigroup without an identity that has a minimal generating set of two elements. Conversely, every such semigroup is isomorphic to $N(m, n)$ for some m and n.

We summarize with the following list of all commutative cancellative semigroups that can be generated by two elements: Z, Z_m, $Z_m \times Z_n$, $Z_m \times N_0$, F_2, N, and $N(m, n)$.

References

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