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AN OPERATOR CONNECTED WITH THE THIRD BOUNDARY VALUE PROBLEM IN POTENTIAL THEORY

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Introduction. As in [16], we suppose that G is an open set in R^m, the Euclidean space of dimension $m > 2$, and that the boundary B of G is non-void and compact. $\mathfrak{B}$ will denote the Banach space of all finite signed Borel measures with support in B; the norm of an element $\mu \in \mathfrak{B}$ is its total variation $\|\mu\|$. Following J. KRÁL [9], a point $x \in R^m$ will be termed a hit of a half-line or an open segment S on G provided $x \in S$ and each neighborhood of x meets both $S \cap G$ and $S - G$ in a set of positive linear measure. Given $y \in R^m$, $0 < r \leqq \infty$ and $\theta \in \Gamma = \{z \in R^m; |z| = 1\}$, we shall denote by $n_r(\theta, y)$ the total number of all the hits of $\{y + \varrho\theta; 0 < \varrho < r\}$ on G. For fixed $r > 0$ and $y \in R^m$, $n_r(\theta, y)$ is a Baire function of the variable θ on Γ (see [9], proposition 1.6) and one may define

$$v_r(y) = \int_\Gamma n_r(\theta, y) \, \mathrm{d}H_{m-1}(\theta)$$

where H_{m-1} stands for the $(m-1)$-dimensional Hausdorff measure in R^m.

With each $\mu \in \mathfrak{B}$ we associate its potential

$$U\mu(x) = \int_B p(x - y) \, \mathrm{d}\mu(y)$$

corresponding to the Newtonian kernel $p(z) = |z|^{2-m}/(m-2)$.

Throughout this paper we shall assume that λ is a fixed non-negative element of $\mathfrak{B}$ and we agree to impose

$$\sup_{y \in B} [v_\infty(y) + U\lambda(y)] < \infty \tag{1}$$

on G and λ.

Then, for each $\mu \in \mathfrak{B}$, the distribution $\mathscr{T}\mu$ defined in [16] by

$$\mathscr{T}\mu(\varphi) = \int_G \operatorname{grad} \varphi(x) \, . \operatorname{grad} U\mu(x) \, \mathrm{d}x + \int_B \varphi(x) \, . \, U\mu(x) \, \mathrm{d}\lambda(x) \tag{2}$$

over the class $\mathscr{D}$ of all infinitely differentiable functions with compact support in R^m, can be identified with a uniquely determined element $\mathscr{T}\mu$ of $\mathfrak{B}$ and the operator $\mathscr{T} : \mu \longmapsto \mathscr{T}\mu$ acting on $\mathfrak{B}$ is a bounded linear operator (see [16], theorem 5 and remark 9). As mentioned in [16], $\mathscr{T}\mu$ is closely connected with the third boundary value problem in potential theory.

It is natural to investigate the applicability of the Riesz-Schauder theory to the third boundary value problem formulated as follows: Given $v \in \mathfrak{B}$, determine $\mu \in \mathfrak{B}$ with $\mathscr{T}\mu = v$. For this purpose we shall consider the decomposition

$$\mathscr{T} = \alpha A \mathscr{I} + \mathscr{T}_\alpha$$

where α is a real number, $A = H_{m-1}(\Gamma)$ and $\mathscr{I}$ stands for the identity operator on $\mathfrak{B}$ and investigate the quantity

$$\omega' \mathscr{T}_\alpha = \inf_{\mathscr{Q}} \|\mathscr{T}_\alpha - \mathscr{Q}\|$$

where $\mathscr{Q}$ runs over the set $\mathscr{F}'$ of all operators acting on $\mathfrak{B}$ of the form

$$\mathscr{Q} \ldots = \sum_{j=1}^{n} \langle f_j, \ldots \rangle \, m_j$$

where n is a positive integer, $m_j \in \mathfrak{B}$ and f_j's are bounded Baire functions on B. In a similar way as in [10] it is possible to determine the optimal value γ of the parameter α and evaluate the quantity

$$(3) \qquad a' = \frac{\omega' \mathscr{T}_\gamma}{A|\gamma|} = \inf_{\alpha \neq 0} \frac{\omega' \mathscr{T}_\alpha}{A|\alpha|}$$

in geometric terms connected with G and λ.

Denote by I_B the set of all isolated points of B and put $E = B - I_B$ or $E = B$ according as I_B is finite or not and write B_1 for the set of all points $y \in E$ that have a neighborhood $\Omega(y)$ such that $\Omega(y) - G$ has Lebesgue measure zero. Let B_2 stand for the set of those $y \in B$ at which the m-dimensional density of G equals $\frac{1}{2}$. Then B_2 is a Borel set with $H_{m-1}(B_2) < \infty$ and one may consider the Lebesgue decomposition $\lambda = \lambda_1 + \hat{\lambda}$ with respect to the restriction H of H_{m-1} to B_2; here λ_1 is absolutely continuous (H) and $\hat{\lambda}$ and H are mutually singular. If $r > 0$ and $y \in R^m$, denote by $\Omega_r(y)$ the open ball with center y and radius r and put

$$\hat{v}_r(y) = \frac{\hat{\lambda}[\Omega_r(y)]}{(m-2)\, r^{m-2}} + \int_0^r \varrho^{1-m} \hat{\lambda}[\Omega_\varrho(y)] \, d\varrho \, .$$

(Note that $\hat{v}_r(y)$ is just the value of the potential induced at y by the restriction of $\hat{\lambda}$ to $\Omega_r(y)$.)

For $j = 1, 2$ set $k_j = 0$ or

$$k_j = \lim_{r \to 0+} \sup_{y \in B_j} [v_r(y) + \hat{v}_r(y)]$$

according as $B_j = \emptyset$ or not. With this notation we have the following theorem (announced without proof in [15]) which we state here for the simplest case when $U\lambda_1$ is continuous.

Theorem. *If a' and γ are defined by* (3), *then $a' < 1$ if and only if, simultaneously,*

$$k_1 < A\,, \quad k_2 < \tfrac{1}{2}A\,.$$

If these inequalities hold, then one of the following cases must take place:

(i*) $B_1 = \emptyset$,

(ii) $B_2 = \emptyset$ *or* $k_1 \geqq \frac{1}{2}A + k_2$,

(iii) $B_1 \neq \emptyset \neq B_2$ *and* $|k_1 - k_2| < \frac{1}{2}A$.

In the case (i*)

$$a' = 2k_2/A\,, \quad \gamma = \tfrac{1}{2}\,;$$

if (ii) *occurs, then*

$$a' = k_1/A\,, \quad \gamma = 1\,,$$

while in the case (iii)

$$a' = \frac{k_1 + k_2 + \frac{1}{2}A}{k_1 - k_2 + \frac{3}{2}A}\,, \quad \gamma = \frac{3}{4} + \frac{k_1 - k_2}{2A}\,.$$

Under suitable conditions the corresponding theorem for discontinuous $U\lambda_1$ is the same, only the definition of the constants k_1, k_2 must be generalized and becomes more complicated. On the other hand, if $U\lambda$ happens to be continuous on B (in particular, if $\lambda = 0$), then $\hat{v}_r(y)$ can be omitted in the definition of k_1, k_2.

The methods employed here are similar to those developed by J. Král in [9], [10]. Results of this paper will be useful in connection with the solution of the third boundary value problem investigated in [17].

1. Notation. In what follows we shall keep the notation from the introduction. We briefly recall the necessary notation occurring in [16]. As usual, for $M \subset R^m$ we shall denote by $\operatorname{cl} M$, $\operatorname{fr} M$ and $\operatorname{diam} M$ the closure, boundary and diameter of M, respectively. H_k will stand for the k-dimensional Hausdorff measure in R^m defined in the usual manner (see [16]); thus H_m coincides with the Lebesgue measure in R^m.

For $x \in R^m$ and $r > 0$ put

$$\Omega_r(x) = \{z \in R^m;\ |z - x| < r\}\,, \quad \Gamma_r(x) = \operatorname{fr} \Omega_r(x)\,,$$
$$\Gamma = \Gamma_1(0)\,, \quad A = H_{m-1}(\Gamma)\,;$$

$\chi_{r,x}$ denotes the characteristic function of $\Omega_r(x)$.

As mentioned in [16] (see section 2), results of [9] imply, for each $y \in R^m$, the existence of a unique element $v_y \in \mathfrak{B}$ such that

$$A\,d(y)\,\varphi(y) + \langle \varphi, v_y \rangle = \int_G \operatorname{grad} \varphi(x) \, . \, \operatorname{grad} U\delta_y(x) \, dx \, , \quad \varphi \in \mathscr{D} \, , \tag{4}$$

where $d(y)$ is m-dimensional density of G at y and δ_y denotes the Dirac measure concentrated at y. Moreover, for the indefinite variation $|v_y|$ of v_y holds

$$|v_y| \, (\Omega_r(y)) = v_r(y) \, .$$

Denoting by $n(y)$ the exterior normal of G at y in the sense of Federer (for definition see [16], section 2), we have from lemma 2.12 in [9] (letting $C = R^m - G$)

$$v_y(M) = - \int_M \frac{n(x) \, . \, (x - y)}{|x - y|^m} \, dH_{m-1}(x) \, , \quad y \in B \, , \tag{5}$$

whenever $M \subset B$ is a Borel set.

Let $\mathscr{B}$ denote the Banach space of all bounded Baire functions on B with the usual supremum norm; $\mathscr{C}$ will stand for the subspace of continuous functions. It is known (see [3]) that the dual space $\mathscr{B}^*$ of $\mathscr{B}$ consists precisely of all additive set functions with bounded variation defined on the class of all Borel subsets of B. Clearly, $\mathfrak{B}$ is a closed subspace in $\mathscr{B}^*$. It is also easy to see that $\mathscr{B}^*$ is a direct sum of $\mathfrak{B}$ and the space $\mathfrak{B}_0$ which consists of all elements of $\mathscr{B}^*$ vanishing on $\mathscr{C}$. The Hahn-Banach theorem may be used to assert that $\mathfrak{B}$ is a proper subspace of $\mathscr{B}^*$ if and only if B is infinite.

If $\mu \in \mathfrak{B}$ and g is integrable (μ), then $g\mu \in \mathfrak{B}$ is defined by

$$\langle f, g\mu \rangle = \langle fg, \mu \rangle \, , \quad f \in \mathscr{B} \, .$$

In [16] the bounded operators $\tilde{W}, V$ acting on $\mathscr{B}$ were introduced as follows:

$$\tilde{W} f(y) = A\, d(y) f(y) + \langle f, v_y \rangle \, , \quad V f(y) = U f\lambda(y) \, , \quad f \in \mathscr{B} \, , \quad y \in B \, .$$

The importance of these operators lies in the fact that the restriction to $\mathfrak{B}$ of the dual operator to $T = \tilde{W} + V$ coincides with the operator $\mathscr{T}$ (see [16], proposition 8). We also know that

$$\tilde{W} \mathscr{C} \subset \mathscr{C} \tag{6}$$

(see (16) in [16]).

Let us observe a special case here. If B is finite, then (1) implies $\lambda = 0$ so that $\mathscr{T}$ reduces to the operator NU introduced and investigated in [9]. For this reason, in what follows we exclude the case of a finite B.

Let us denote by $\mathscr{G}$ the class of all compact operators acting on $\mathscr{B}$ and by $\mathscr{F}$ the class of all operators Q acting on $\mathscr{B}$ of the form

$$Q \ldots = \sum_{j=1}^{n} \langle \ldots, m_j \rangle f_j$$

where n is a positive integer, $m_j \in \mathfrak{B}$ and $f_j \in \mathscr{B}$. Clearly, $\mathscr{F} \subset \mathscr{G}$.

For any bounded linear operator X on $\mathscr{B}$ put

$$\omega X = \inf_{Q \in \mathscr{F}} \|X - Q\| , \quad \tilde{\omega} X = \inf_{Q \in \mathscr{G}} \|X - Q\| .$$

It follows immediately that $0 \leqq \tilde{\omega} X \leqq \omega X \leqq \|X\|$. An example can be constructed to show that, in general, the equality $\tilde{\omega} X = \omega X$ does not hold.

Let us also recall the following terminology. A point $y \in R^m$ is termed a discontinuity for a $\mu \in \mathfrak{B}$ provided $\mu(\{y\}) \neq 0$. Every $\mu \in \mathfrak{B}$, being finite, has at most countable set of discontinuities. Finally, given $\varepsilon > 0$ and $\mu \in \mathfrak{B}$, there is a $\mu' \in \mathfrak{B}$ such that $\|\mu - \mu'\| < \varepsilon$ and μ' has only a finite number of discontinuities.

The following lemma is an easy consequence of the well-known compactness criterion of a set in $\mathscr{B}$.

2. Lemma. *Let an $\eta_y \in \mathfrak{B}$ be associated with each $y \in B$ in such a way that the equality*

$$Z f(y) = \langle f, \eta_y \rangle , \quad f \in \mathscr{B} ,$$

defines a compact operator on $\mathscr{B}$. Then $\omega Z = 0$.

Proof. Denote $Y = \{f \in \mathscr{B};\ \|f\| \leqq 1\}$ and fix an arbitrary $\varepsilon > 0$. By hypothesis, ZY is relatively compact in $\mathscr{B}$. Using compactness criterion (see [3]) we may assert the existence of pairwise disjoint Borel sets $B_1, \ldots, B_n$ with $B = \bigcup_{j=1}^{n} B_j$ and points $s_j \in B_j$ such that

$$\sup_{s \in B_j} |Z f(s) - Z f(s_j)| < \varepsilon$$

whenever $f \in Y$ and $1 \leqq j \leqq n$.

Let f_j denote the characteristic function of B_j and put

$$Z_\varepsilon f = \sum_{j=1}^{n} \langle f, \eta_{s_j} \rangle f_j , \quad f \in \mathscr{B} .$$

Then $Z_\varepsilon \in \mathscr{F}$ and one easily verifies that $\|Z - Z_\varepsilon\| < \varepsilon$. Consequently, $\omega Z = 0$ and the proof is complete.

The following lemma is in fact a more general version of theorem 3.6 in [9]. It will enable us the investigations of properties of the operator T.

3. Lemma. *Let R be a subset of the real line such that*

$$\inf R = 0 . \tag{7}$$

For each $y \in B$, ξ_y is an element of $\mathfrak{B}$ such that the relation

$$X f(y) = \langle f, \xi_y \rangle , \quad f \in \mathscr{B} , \tag{8}$$

defines a bounded operator X acting on $\mathscr{B}$. Suppose that, for each $r \in R$,

$$|\xi_y - \xi_z| (B - \Omega_r(z)) \to 0 \tag{9}$$

as $|y - z| \to 0$ uniformly with respect to $z \in B$ and

$$|\xi_y| (\Gamma_r(y)) = 0 \tag{10}$$

whenever $y \in B$ and $r \in R$. Let for each $y \in B$ be

$$\xi_y(\{z\}) = 0 \tag{11}$$

provided $z \neq y$.

If $K_1 \subset B$ is a finite set, then

$$\omega X \leqq \lim_{r \to 0+} \sup_{y \in B - K_1} |\xi_y| (\Omega_r(y)) . \tag{12}$$

Given an arbitrary $\varepsilon > 0$, there is a finite $K \subset B$ such that

$$\omega X + \varepsilon \geqq \lim_{r \to 0+} \sup_{y \in B - K} |\xi_y| (\Omega_r(y)) . \tag{13}$$

If, in addition, the equality

$$\xi_y(\{y\}) = 0 \tag{14}$$

holds for each $y \in B$, then

$$\omega X = \lim_{r \to 0+} \sup_{y \in B} |\xi_y| (\Omega_r(y)) \tag{15}$$

and

$$X\mathscr{B} \subset \mathscr{C} \tag{16}$$

provided X is compact.

Proof. Fix an arbitrary $\varepsilon > 0$. By definition of ωX, one easily constructs $f_j \in \mathscr{B}$, $m_j \in \mathfrak{B}$ such that the operator X_ε defined by

$$X_\varepsilon f = \sum_{j=1}^{n} \langle f, m_j \rangle f_j , \quad f \in \mathscr{B} ,$$

satisfies

$$\|X - X_\varepsilon\| \leqq \omega X + \varepsilon \tag{17}$$

and, in addition, m_j have only a finite number of discontinuities each. Fix a finite set $K \subset B$ in such a manner that all the discontinuities of each m_j belong to K. Every m_j splits into m_j^1 having no discontinuities and a finite combination of Dirac measures, to be denoted by m_j^2. According to (11), y is the only possible discontinuity for ξ_y, so that we have for $y \in B - K$

$$\|\xi_y - \sum_{j=1}^{n} f_j(y)\, m_j\| = \|\xi_y - \sum_{j=1}^{n} f_j(y)\, m_j^1\| + \|\sum_{j=1}^{n} f_j(y)\, m_j^2\|$$

whence, for $r > 0$,

$$\|X - X_\varepsilon\| \geqq \sup_{y \in B - K} \|\xi_y - \sum_{j=1}^{n} f_j(y)\, m_j\| \geqq$$

$$\geqq \sup_{y \in B - K} |\xi_y - \sum_{j=1}^{n} f_j(y)\, m_j^1| (\Omega_r(y)) .$$

Putting $\beta = \max_{1 \leqq j \leqq n} \|f_j\|$, the norm $\|X - X_\varepsilon\|$ admits the estimate

$$\|X - X_\varepsilon\| \geqq \sup_{y \in B - K} |\xi_y| (\Omega_r(y)) - \beta \sum_{j=1}^{n} \sup_{y \in B} |m_j^1| (\Omega_r(y)) . \tag{18}$$

Since B is compact and m_j^1 has no discontinuities,

$$\lim_{r \to 0+} \sup_{y \in B} |m_j^1| (\Omega_r(y)) = 0 . \tag{19}$$

Letting $r \to 0+$ in (18) and using (19) and (17) we obtain (13).

As for the proof of (12), fix first an arbitrary finite set $K_1 \subset B$ and an $r \in R$ and for $\delta \in (0, r)$ put

$$\alpha(\delta) = \sup_{y \in B} |\xi_y| (\mathrm{cl}\, \Omega_{r+\delta}(y) - \Omega_{r-\delta}(y)) .$$

Using (10), (7), (9) and compactness of B, we verify that

$$\lim_{\delta \to 0+} \alpha(\delta) = 0 . \tag{20}$$

For $r \in R$ define the operator X_r acting on $\mathscr{B}$ by

$$X_r f(y) = \langle f, (1 - \chi_{r,y})\, \xi_y \rangle , \quad y \in B .$$

If $f \in \mathscr{B}$ with $\|f\| \leqq 1$ and $y, z \in B$ with $0 < |y - z| = \delta$, then

$$|X_r f(y) - X_r f(z)| \leqq \alpha(\delta) + |\xi_z - \xi_y| (B - \Omega_r(z)) .$$

Consequently, by virtue of (20) and (9), the functions in

$$\{X_r f;\ f \in \mathscr{B},\ \|f\| \leqq 1\}$$

are equicontinuous and, of course, uniformly bounded, so that the operator X_r is compact. Applying lemma 2 we conclude that there is a $Z_r \in \mathscr{F}$ such that

$$\|X_r - Z_r\| \leqq \frac{1}{r} \tag{21}$$

whenever $r \in R$.

The above considerations show, in particular,

$$X_r \mathscr{B} \subset \mathscr{C}, \quad r \in R. \tag{22}$$

Fix now an arbitrary finite set K_1, $K_1 = \{y_1, \ldots, y_k\}$. Let c_j denote the characteristic function of $\{y_j\}$ and for $r \in R$ put

$$\begin{aligned} Y_r f &= \sum_{j=1}^{k} \langle f, \chi_{r,y_j} \xi_{y_j} \rangle \, c_j, \quad f \in \mathscr{B}, \\ \tilde{X}_r &= X - X_r - Y_r, \\ \tilde{Z}_r &= X - Z_r - Y_r. \end{aligned}$$

The inequality (21) yields $\|\tilde{X}_r - \tilde{Z}_r\| < 1/r$ and we have

$$\omega X \leqq \|\tilde{Z}_r\| \leqq \|\tilde{X}_r\| + r^{-1}, \tag{23}$$

because $Y_r \in \mathscr{F}$.

Since for $f \in \mathscr{B}$ and $y \in B - K_1$ we have

$$\tilde{X}_r f(y) = \langle f, \chi_{r,y} \xi_y \rangle,$$

while $\tilde{X}_r f(y) = 0$ provided $y \in K_1$, we conclude that

$$\|\tilde{X}_r\| \leqq \sup_{y \in B - K_1} |\xi_y| \, (\Omega_r(y)).$$

This together with (23) and (7) yields

$$\omega X \leqq \inf_{r \in R} \|\tilde{X}_r\| \leqq \lim_{r \to 0+} \sup_{y \in B - K_1} |\xi_y| \, (\Omega_r(y)). \tag{24}$$

Now (12) is established.

In the rest of the proof we shall assume (14). Then

$$\sup_{y \in B - K'} |\xi_y| \, (\Omega_r(y)) = \sup_{y \in B} |\xi_y| \, (\Omega_r(y))$$

for any finite $K' \subset B$, so that (15) follows from (12) and (13) immediately.

If X is compact, then $\omega X = 0$ by lemma 2. Going back to (15), we see that

$$\lim_{r \to 0+} \sup_{y \in B} |\xi_y| \, (\Omega_r(y)) = 0.$$

Putting $K_1 = \emptyset$ in (24) we have $\tilde{X}_r = X - X_r$ and (24) implies

$$\inf_{r \in R} \|X - X_r\| = 0 .$$

Combining this with (22) we get (16) and the proof is complete.

4. Notation. The symbol $\mathfrak{B}^+$ will stand for the set of all non-negative elements of $\mathfrak{B}$. In other words, $\mathfrak{B}^+$ consists of all finite Borel measures with support in B. Recall that the reduced boundary $\hat{B}$ of G is the set of all y with $n(y) \neq 0$. As quoted in [16] (see section 11),

$$H_{m-1}(\hat{B}) < \infty . \tag{25}$$

Denote by R_1 the set of all $r > 0$ for which there is a spherical shell with radius r such that $H_{m-1}(S \cap \hat{B}) > 0$. Analogously, given $\lambda_0 \in \mathfrak{B}^+$, let $R_0(\lambda_0)$ stand for the set of all $r > 0$ such that $\lambda_0(S)$ is positive for at least one spherical shell S with radius r. We shall write, for the sake of brevity, R_0 in place of $R_0(\lambda)$.

5. Lemma. *The set R_1 is countable. If $\lambda_0 \in \mathfrak{B}^+$ and the potential $U\lambda_0$ is bounded, then the set $R_0(\lambda_0)$ is countable as well.*

Proof. Let S_1, S_2 be the spherical shells with different radii r_1, r_2, respectively, and $S' = S_1 \cap S_2$. Since $H_{m-2}(S') < \infty$, S' is a polar set and λ_0, having bounded potential, possesses finite energy so that λ_0 does not charge S' (see [12], theorems 3.14 and 2.1). Consequently, for any positive integer n and each choice of shells $S_1, \ldots, S_n$ with mutually different radii we have

$$\sum_{i=1}^{n} \lambda_0(S_i) = \lambda_0\left(\bigcup_{i=1}^{n} S_i\right) \leqq \lambda_0(B) < \infty ,$$

$$\sum_{i=1}^{n} H_{m-1}(\hat{B} \cap S_i) = H_{m-1}\left(\hat{B} \cap \bigcup_{i=1}^{n} S_i\right) \leqq H_{m-1}(\hat{B}) < \infty .$$

Now we conclude easily that R_1 and $R_0(\lambda_0)$ are countable.

6. Proposition. *Let $\lambda_0 \in \mathfrak{B}^+$ and suppose that $U\lambda_0$ is bounded. Fix an arbitrary $y_0 \in B$.*

The potential $U\lambda_0$ is continuous at y_0 with respect to B if and only if the following condition is fulfilled: For any $\varepsilon > 0$ there is an $r > 0$ such that the inequality

$$U\chi_{r,y}\, \lambda_0(y) < \varepsilon \tag{26}$$

holds for each $y \in \Omega_r(y_0) \cap B$.

In order that the potential $U\lambda_0$ be continuous on B (with respect to B) it is necessary and sufficient that

$$\lim_{r \to 0+} \sup_{y \in B} U\chi_{r,y}\, \lambda_0(y) = 0 . \tag{27}$$

Proof. For $r > 0$, $y \in B$ denote $c_{r,y}$ the characteristic function of $R^m - \Omega_r(y)$. Suppose that $r \notin R_0(\lambda_0)$ and let $y_1, y_2, \ldots$ be points of B with $\lim_{n \to \infty} y_n = y_0$. Since $\lambda_0(\Gamma_r(y_0)) = 0$, the Lebesgue dominated convergence theorem may be used to assert

$$\lim_{n \to \infty} Uc_{r,y_n} \lambda_0(y_n) = Uc_{r,y_0} \lambda_0(y_0) .$$

Consequently, the function

$$y \longmapsto Uc_{r,y} \lambda_0(y)$$

is continuous at y_0 with respect to B whenever $r \notin R_0(\lambda_0)$. The set $R_0(\lambda_0)$ being countable by lemma 5, we can choose numbers $r_n \notin R_0(\lambda_0)$ such that $r_n \searrow 0$. Fix now $y \in B$. Then

$$c_{r_n,y} \nearrow 1$$

almost everywhere (λ_0) because $\lambda_0(\{y\}) = 0$. Making use of the monotone convergence theorem we obtain

$$\lim_{n \to \infty} Uc_{r_n,y} \lambda_0(y) = U\lambda_0(y) .$$

Since $U\lambda_0$ is a limit of functions continuous at y_0, the first part of the proposition follows immediately by the well-known theorem.

The latter assertion is an easy consequence of (26) and compactness of B.

7. Remark. Referring to the Evans theorem (see [12], theorem 1.7), it should be noted that conditions occurring in the above proposition characterize continuity $U\lambda_0$ not only on B but in R^m as well. Let us also observe the following corollary of the last proposition: If $\tilde{\lambda}_1, \tilde{\lambda}_2 \in \mathfrak{B}^+$, the potential $U\tilde{\lambda}_2$ is bounded and $\tilde{\lambda}_1 \leqq \tilde{\lambda}_2$ (i.e. $\tilde{\lambda}_2 - \tilde{\lambda}_1 \in \in \mathfrak{B}^+$), then continuity of $U\tilde{\lambda}_2$ implies the continuity of $U\tilde{\lambda}_1$.

Note here that a condition similar to (26) occurs in [1] in connection with the investigations of properties of the logarithmic potential.

8. Lemma. *For each $y \in B$ define*

$$\mathrm{d}\xi_y(x) = p(x - y)\, \mathrm{d}\lambda(x) . \tag{28}$$

Then $\xi_y \in \mathfrak{B}$ and

$$\langle f, \xi_y \rangle = Vf(y) , \quad y \in B , \quad f \in \mathscr{B} . \tag{29}$$

If $R = (0, \infty) - R_0$ where R_0 has the meaning defined in 4, *then* (7) *is true and ξ_y satisfies the assumptions* (9), (10), (11), (14) *of lemma* 3; *in particular,*

$$\omega V = \lim_{r \to 0+} \sup_{y \in B} U\chi_{r,y} \lambda(y) . \tag{30}$$

Proof. Clearly, $\xi_y \in \mathfrak{B}$ and (29) holds. Since R_0 is countable by lemma 5, the equality $\inf R = 0$ is obvious. Suppose that $r \in R$ and $y, z \in B$ and denote $|y - z| = \delta$.

If $\delta < \frac{1}{2}r$ and $x \in B - \Omega_r(z)$, then

$$\left| |x - y|^{2-m} - |x - z|^{2-m} \right| \leqq (m - 2)\, \delta \cdot (\tfrac{1}{2}r)^{1-m}.$$

Consequently,

$$|\xi_y - \xi_z| \, (B - \Omega_r(z)) \leqq \delta \|\lambda\| \, (\tfrac{1}{2}r)^{1-m}$$

and (9) is verified. The rest is easy.

As for (30), it suffices to observe that

$$|\xi_y| \, (\Omega_r(y)) = U\chi_{r,y} \, \lambda(y)$$

and apply (15) of lemma 3.

9. Proposition. *The following statements are equivalent each to other:*

(i) $V\mathscr{B} \subset \mathscr{C}$.

(ii) $V\mathscr{C} \subset \mathscr{C}$.

(iii) $T\mathscr{C} \subset \mathscr{C}$.

(iv) *The potential* $U\lambda$ *is continuous.*

(v) $\omega V = 0$.

(vi) *The operator* V *is compact.*

Proof. Trivially, (i) implies (ii). Going back to (6) and recalling that $T = V + \tilde{W}$ we see that (ii) and (iii) are equivalent each to other. In particular, if $f = 1$ on B, then (iii) together with the equality $Vf = U\lambda$ implies (iv). The implication (iv) $\Rightarrow$ (v) follows from proposition 6 and lemma 8 (see (27) and (30)). Clearly, (v) implies (vi).

It remains to prove (vi) $\Rightarrow$ (i). According to lemma 8 it is possible to apply lemma 3 to the operator V in place of X. Since V is compact by hypothesis, it is $V\mathscr{B} \subset \mathscr{C}$ by (16) and this completes the proof.

10. Lemma. *Put* $R = (0, \infty) - R_1$ *and for* $y \in B$, $\alpha \in R^1$, *define*

(31) $$\xi_y = A(d(y) - \alpha)\, \delta_y + \nu_y\,.$$

Then (7) *holds and* ξ_y *satisfies the assumptions* (9), (10), (11) *of lemma* 3. *Further*

(32) $$\langle f, \xi_y \rangle = \tilde{W} f(y) - A\alpha f(y)$$

whenever $y \in B$ *and* $f \in \mathscr{B}$.

Proof. We have $\inf R = 0$ by lemma 5. Returning to (5) and to the definition of R_1 we easily verify (10) and (11). The equality (32) is obvious by (31) and the definition of $\tilde{W}$.

If $r \in R$ and y, z are arbitrary points in B with $0 < |y - z| = \delta < \frac{1}{2}r$, then

(33) $$|\xi_y - \xi_z| \, (B - \Omega_r(z)) = |\nu_y - \nu_z| \, (B - \Omega_r(z))$$

because v_y and δ_y are mutually singular. Simple calculation shows

$$\left| \frac{x-y}{|x-y|^m} - \frac{x-z}{|x-z|^m} \right| \leqq \delta \cdot (\tfrac{1}{2}r)^{-m} (m+1)$$

provided $x \in B - \Omega_r(z)$, whence by (31), (33) and (5)

$$|\xi_y - \xi_z| (B - \Omega_r(z)) = \int_{B-\Omega_r(z)} \left| \frac{n(x) . (x-y)}{|x-y|^m} - \frac{n(x) . (x-z)}{|x-z|^m} \right| \mathrm{d}H_{m-1}(x) \leqq$$
$$\leqq \delta \cdot (\tfrac{1}{2}r)^{-m} (m+1) H_{m-1}(\hat{B}) .$$

Now (9) follows immediately and the proof of the lemma is complete.

11. Notation. As in the introduction, the symbol I_B will stand for the set of all isolated points of B and put $E = B - I_B$ or $E = B$ according as I_B is finite or not. For $y \in B$ we put

$$\mathrm{d}\pi_y(x) = p(x-y)\, \mathrm{d}\lambda(x) , \quad \tau_y = \pi_y + v_y .$$

In what follows we fix a real number α and for $r > 0$ and $y \in B$ define

$$a_r(y) = A|d(y) - \alpha| + |\tau_y| (\Omega_r(y)) .$$

Finally, I stands for the identity operator on $\mathscr{B}$ and

$$T_\alpha = T - \alpha A I . \tag{34}$$

12. Theorem. *Let ε be an arbitrary positive number. Then there is a finite set K_0 such that*

$$K_0 \subset B - I_B \; (\subset E) \tag{35}$$

and ωT_α admits the following estimates:

$$\lim_{r \to 0+} \sup_{y \in E - K_0} a_r(y) \leqq \omega T_\alpha + \varepsilon , \tag{36}$$

$$\lim_{r \to 0+} \sup_{y \in E} a_r(y) \geqq \omega T_\alpha . \tag{37}$$

Proof. Denoting

$$\xi_y = A(d(y) - \alpha)\, \delta_y + \tau_y , \quad y \in B ,$$

we have for $f \in \mathscr{B}$

$$T_\alpha f(y) = \langle f, \xi_y \rangle .$$

If we put in lemma 3 $R = (0, \infty) - (R_0 \cup R_1)$, $X = T_\alpha$, it follows from lemmas 5, 8, 10 that hypotheses (7), (9), (10), (11) are satisfied. Using lemma 3 with $K_1 =$

$= B - E$, we obtain (37) from (12), because

$$(38) \qquad |\xi_y| (\Omega_r(y)) = a_r(y) .$$

By (13) of lemma 3 we conclude that there exists a finite set $K \subset E$ such that

$$(39) \qquad \lim_{r \to 0+} \sup_{y \in E - K} a_r(y) \leqq \omega T_\alpha + \varepsilon .$$

Put now $K_0 = K \cap (B - I_B)$. Then (35) is true and we are going to show (36). This is particularly clear whenever I_B is finite. Indeed, then $E - K = E - K_0$ and (39) yields (36).

Therefore we limit ourselves to the case of I_B infinite, so that $E = B$. To prove (36) it is sufficient to show that

$$(40) \qquad \sup_{y \in B - K} a_r(y) = \sup_{y \in B - K_0} a_r(y)$$

holds for $r > 0$ small enough. Putting $B_1 = B - K_0$ we have $B - K = B_1 - (I_B \cap K)$ and (40) will follow if we verify the inequality

$$(41) \qquad \sup_{y \in B_1} a_r(y) \leqq \sup_{y \in B_1 - (I_B \cap K)} a_r(y)$$

for small $r > 0$.

Fix an arbitrary $y \in I_B \cap K \subset B_1$ and

$$(42) \qquad 0 < r < \operatorname{dist} (I_B \cap K, B_1 - (I_B \cap K))$$

where dist (...) stands for the distance of sets. Then

$$(43) \qquad a_r(y) = A|1 - \alpha|$$

because $d(y) = 1$ and $|\tau_y| (\Omega_r(y)) = 0$. On the other hand, we have for any $z \in I_B$

$$(44) \qquad A|1 - \alpha| \leqq a_r(z) .$$

Observing that $I_B - K \neq \emptyset$ and $B_1 \cap (I_B - K) \subset B_1 - (I_B \cap K)$ we obtain from (43) and (44)

$$a_r(y) \leqq \sup_{z \in B_1 \cap (I_B - K)} a_r(z) \leqq \sup_{z \in B_1 - (I_B \cap K)} a_r(z) .$$

We have thus (41) for r satisfying (42). This completes the proof of the theorem.

13. Notation. Let us denote by B^* the set of all $y_0 \in R^m$ with the following property: there are $y_n \in B - \{y_0\}$ such that $y_n \to y_0$ and $d(y_n) \to d(y_0)$. A point $y_0 \in R^m$ is said to belong to Z if there exists $r > 0$ such that $H_m(\Omega_r(y_0) - G) = 0$. Put $\tilde{B} = B - B^*$ and for $\gamma \in \langle 0, 1 \rangle$ denote

$$B(\gamma) = \{y \in B;\ d(y) = \gamma\} .$$

Clearly, $\hat{B} \subset B(\frac{1}{2})$, $Z \subset B(1)$ and $I_B \subset \tilde{B}$. The set $B(\gamma) \cap \tilde{B}$ is isolated for each $\gamma \in \langle 0, 1 \rangle$.

14. Lemma. *The following statements hold:*

(i) *The set $\tilde{B}$ is countable.*

(ii) *The set $Z \cup \hat{B}$ is dense in B.*

(iii) *The sets $\tilde{B}$ and $\hat{B}$ are disjoint and*

$$Z \cap \tilde{B} = I_B .$$

Proof. Put

$$D = \{[x, d(x)] \in R^{m+1};\ x \in B\}$$

and observe that $y_0 \in \tilde{B}$ implies that $[y_0, d(y_0)]$ is an isolated point of D. Now (i) follows from the fact that any isolated set in R^{m+1} is countable.

As for the proof of (ii), choose an arbitrary $y \in B - Z$. Then we have $H_m(\Omega_r(y) - G) > 0$ for each $r > 0$ and also $H_m(\Omega_r(y) \cap G) > 0$ because G is open. Hence it follows by the relative isoperimetric inequality for sets with finite perimeter (see Theorem (4.3) in [14]) that

$$H_{m-1}(\Omega_r(y) \cap \hat{B}) > 0 . \tag{45}$$

In particular, $y \in \operatorname{cl} \hat{B}$. The proof of (ii) is complete.

Since $\hat{B} \subset B(\frac{1}{2})$, it follows from (45) that each $y \in \hat{B}$ belongs to B^*. In other words, $\tilde{B} \cap \hat{B} = \emptyset$. Considering the definition of Z and the inclusion $Z \subset B(1)$ we easily verify that $Z - I_B \subset B^*$. Consequently, $Z \cap \tilde{B} \subset I_B$. The opposite inclusion is obvious.

The proof of the lemma is complete.

15. Notation. As in [16], denote by H the restriction of H_{m-1} to $\hat{B}$. In view of (25) we have $H(B) < \infty$ and as quoted in [16], section 11, $H \in \mathfrak{B}^+$.

For each $y \in B$ put

$$l(y) = \lim_{r \to 0+} \frac{\lambda(\Omega_r(y))}{H(\Omega_r(y))}$$

provided the last limit is meaningful and finite; otherwise set $l(y) = 0$. Letting $\hat{\lambda} = \lambda - lH$ we conclude that $\hat{\lambda}$ is the singular part of λ with respect to H (compare [5], section 6).

16. Lemma. *If Q is an arbitrary finite subset of E and $y_0 \in B^*$, then*

$$\lim_{r \to 0+} \sup_{y \in E - Q} a_r(y) = \lim_{r \to 0+} \sup_{y \in E - (Q \cup \{y_0\})} a_r(y) . \tag{46}$$

Proof. Fix an $\varepsilon > 0$ and a sequence $\{y_n\}$ of points of $E - \{y_0\}$ such that $y_n \to y_0$ and $d(y_n) \to d(y_0)$. There is a positive integer n_0 such that

$$A|d(y_0) - \alpha| \leqq A|d(y_n) - \alpha| + \varepsilon \tag{47}$$

provided $n \geqq n_0$.

Let R_0, R_1 be the sets defined in 4. According to the definition of $a_r(y)$ and (47)it will be sufficient to prove

$$|\tau_{y_0}|\,(\Omega_r(y_0)) \leqq \liminf_{n\to\infty} |\tau_{y_n}|\,(\Omega_r(y_n)) \tag{48}$$

for $r \in R = (0, \infty) - (R_1 \cup R_0)$.

Recalling that $\lambda = \hat{\lambda} + lH$ we may write

$$|\tau_y|\,(\Omega_r(y)) = (m-2)^{-1} \int_B \chi_{r,y}(z)\,|z - y|^{2-m}\,\mathrm{d}\hat{\lambda} + \tag{49}$$

$$+ \int_B \chi_{r,y}(z) \left| \frac{1}{m-2} \cdot \frac{l(z)}{|z-y|^{m-2}} - \frac{n(z)\,.\,(z-y)}{|z-y|^m} \right| \mathrm{d}H(z)$$

for all $y \in B$ and $r > 0$.

If $r \in R$, the Fatou's lemma may be applied to assert (48).

The proof is complete.

17. Corollary. *Theorem* 12 *remains valid if* (35) *is replaced by*

$$K_0 \subset \tilde{B} - I_B\,. \tag{35'}$$

In particular, if $B - I_B = B^*$ *holds, then*

$$\omega T_\alpha = \lim_{r\to 0+} \sup_{y\in E} a_r(y)\,.$$

Proof. It follows immediately from the last lemma and theorem 12.

18. Lemma. *Let* $Q \subset E$ *be a finite set and* $y_0 \in B - I_B$. *Suppose that the potential* $U(\lambda - \hat{\lambda})$ *is continuous at* y_0. *Then*

$$\lim_{r\to 0+} \sup_{y\in E-Q} a_r(y) = \lim_{r\to 0+} \sup_{y\in E-(Q\cup\{y_0\})} a_r(y)\,. \tag{50}$$

Proof. For $r > 0$, $y \in B$ put

$$q_r(y) = A|d(y) - \alpha| + \int_B \chi_{r,y}(z)\,.\,\frac{|n(z)\,.\,(z-y)|}{|z-y|^m}\,\mathrm{d}H(z)\,,$$

$$s_r(y) = (m-2)^{-1} \int_B \chi_{r,y}(z)\,|z-y|^{2-m}\,\mathrm{d}\hat{\lambda}\,,$$

$$\bar{v}_y = A(d(y) - \alpha)\,\delta_v + v_v\,.$$

Defining the operator $\overline{W}$ acting on $\mathscr{B}$ by

$$\overline{W}f(y) = \langle f, \bar{\nu}_y \rangle , \quad f \in \mathscr{B} , \quad y \in B ,$$

we see immediately that $\overline{W} = \tilde{W} - \alpha AI$. Going back to (6) we have $\overline{W}\mathscr{C} \subset \mathscr{C}$. Consequently, since for $r > 0$

$$|\bar{\nu}_y| (\Omega_r(y)) = \sup \{\overline{W}f(y)\}$$

where supremum is taken over all continuous functions with $\|f\| \leqq 1$ having support in $\Omega_r(y)$, we conclude that the function

$$y \longmapsto |\bar{\nu}_y| (\Omega_r(y))$$

is lower semicontinuous on B for every $r > 0$.

Choose now $y_n \in E - (Q \cup \{y_0\})$ with $y_n \to y_0$. The above consideration and the equality

$$|\bar{\nu}_y| (\Omega_r(y)) = q_r(y)$$

yield for every $r > 0$

$$q_r(y_0) \leqq \liminf_{n \to \infty} q_r(y_n) . \tag{51}$$

Employing the Fatou's lemma we obtain

$$s_r(y_0) \leqq \liminf_{n \to \infty} s_r(y_n) \tag{52}$$

provided $r \notin R_0$. For the sake of brevity put $\lambda_1 = \lambda - \hat{\lambda}$. By (49) we have

$$s_r(y) + q_r(y) - U\chi_{r,y} \lambda_1(y) \leqq a_r(y) \leqq s_r(y) + q_r(y) + U\chi_{r,y} \lambda_1(y) \tag{53}$$

for every $r > 0$ and $y \in B$.

Consequently, combining (51), (52), (53), we obtain for $r \notin R_0$

$$a_r(y_0) \leqq \liminf_{n \to \infty} a_r(y_n) + \limsup_{n \to \infty} U\chi_{r,y_n} \lambda_1(y_n) + U\chi_{r,y_0} \lambda_1(y_0) . \tag{54}$$

Fix an arbitrary $\varepsilon > 0$. By hypothesis, $U\lambda_1$ is continuous at y_0. Using proposition 6 we conclude from (54)

$$a_r(y_0) \leqq \liminf_{n \to \infty} a_r(y_n) + \varepsilon$$

for all $r \notin R_0$ small enough.

The rest of the proof is easy.

19. Remark. Observe that in the course of the last proof we established (54) in fact for an arbitrary point of E, $y_n \in E - \{y_0\}$ with $y_n \to y_0$ and $r \in (0, \infty) - R_0$. This will be useful for us later.

In the following theorem we require continuity of $U(\lambda - \hat{\lambda})$ at every point of $\tilde{B}$. This assumption does not seem to be too strong because the set $\tilde{B}$ is at most countable by lemma 14. A sufficient condition for continuity of the potential $U(\lambda - \hat{\lambda})$ at a point is stated in corollary 22.

20. Theorem. *Suppose that the potential $U(\lambda - \hat{\lambda})$ is continuous at every point of $\tilde{B}$. Then*

$$\omega T_\alpha = \lim_{r \to 0+} \sup_{y \in E} a_r(y) . \tag{55}$$

In particular, if $U\lambda$ is continuous, then

$$\omega T_\alpha = \lim_{r \to 0+} \sup_{y \in E} \left[A|d(y) - \alpha| + |v_y| \left(\Omega_r(y)\right) \right] . \tag{56}$$

Proof. Returning to lemma 18 we have

$$\lim_{r \to 0+} \sup_{y \in E - K_0} a_r(y) = \lim_{r \to 0+} \sup_{y \in E} a_r(y)$$

for any finite $K_0 \subset \tilde{B} - I_B$. Combining now corollary 17 and (36) of theorem 12 we obtain easily

$$\lim_{r \to 0+} \sup_{y \in E} a_r(y) \leqq \omega T_\alpha .$$

Since the opposite inequality was established in (37) we see that (55) holds.

Suppose now that $U\lambda$ is continuous. Recalling that $\tau_y = v_y + \pi_y$ and $|\pi_y| \left(\Omega_r(y)\right) = U\chi_{r,y} \lambda(y)$ we can write

$$|v_y| \left(\Omega_r(y)\right) - U\chi_{r,y} \lambda(y) \leqq |\tau_y| \left(\Omega_r(y)\right) \leqq |v_y| \left(\Omega_r(y)\right) + U\chi_{r,y} \lambda(y) . \tag{57}$$

By proposition 6, continuity of $U\lambda$ implies

$$\lim_{r \to 0+} \sup_{y \in B} U\chi_{r,y} \lambda(y) = 0 .$$

Recalling the definition of $a_r(y)$, (56) now follows from (57) and (55).

21. Lemma. *Let $r > 0$, $y_0 \in B$ and $\beta \in R^1$. Suppose that l_1 is a non-negative function measurable (H) on B such that $l_1(x) \leqq \beta$ for almost all (H) points $x \in \Omega_{2r}(y_0)$. Put*

$$\gamma = m(m + 1)^m \left(A + \sup_{y \in B} v_\infty(y)\right) .$$

Then

$$U\chi_{r,y} l_1 \, H(y) \leqq 2\beta\gamma r \tag{58}$$

for all $y \in \Omega_r(y_0)$.

Proof. Employing corollary 2.14 in [9] we establish the inequality

$$l_1 H(\Omega_\varrho(y)) \leqq \beta\gamma\varrho^{m-1}$$

for any $y \in \Omega_r(y_0)$ and $\varrho \in (0, r)$. By lemma 10 of [16] we have

$$U\chi_{r,y} l_1 H(y) = \frac{l_1 H(\Omega_r(y))}{(m-2)\, r^{m-2}} + \int_0^r \varrho^{1-m}[l_1 H(\Omega_\varrho(y))]\, d\varrho \leqq$$

$$\leqq \beta\gamma r(1 + (m-2)^{-1}) \leqq 2\beta\gamma r\,, \quad y \in \Omega_r(y_0)\,.$$

The proof is complete.

22. Corollary. *If $y_0 \in B$ and the function l is bounded almost everywhere (H) in a neighborhood of y_0, then the potential $U(\lambda - \hat{\lambda})$ is continuous at y_0. The potential $Ul_1 H$ is continuous provided l_1 is a non-negative function measurable (H) and bounded on B almost everywhere (H).*

Proof. This is an easy consequence of (58) and proposition 6.

The formula (56) together with the equality $v_r(y) = |v_y|\,(\Omega_r(y))$ (see section 1) represents a good geometrical interpretation of the quantity ωT_α (and, as we shall see later, of $\omega' \mathscr{T}_\alpha$ as well). We are going to give a similar geometrical meaning to $|\tau_y|\,(\Omega_r(y))$.

23. Notation and terminology. A function f defined on a non-void set $M \subset R^m$ is said to be of the class C_1 provided M is open and partial derivatives of the first order of f are continuous on M. A set $Q \subset R^m$ will be called a smooth surface if there is a function f of the class C_1 on $M \subset R^m$ such that

$$Q = \{x \in M;\ f(x) = 0,\ \operatorname{grad} f(x) \neq 0\}\,. \tag{59}$$

Let Q be a smooth surface and $x_0 \in Q$. An $h \in R^m$ is said to be a tangent vector of Q at x_0 provided there exists a mapping ψ of an interval $(-\delta, \delta)$ $(\delta > 0)$ into Q such that $\psi(0) = x_0$ and $\psi'(0) = h$. The set of all tangent vectors of Q at x_0 will be denoted by T_{x_0} and called the tangent space of Q at x_0. It is well-known (compare [8]) that T_{x_0} is an $(m-1)$-dimensional linear subspace of R^m. Each $\theta \in \Gamma$ orthogonal to T_{x_0} is called a normal of Q at x_0. If Q satisfies (59), then

$$T_{x_0} = \{h \in R^m;\ \operatorname{grad} f(x_0) \,.\, h = 0\}$$

and the vector $\operatorname{grad} f(x_0)/|\operatorname{grad} f(x_0)|$ is a normal of Q at x_0.

The mapping $\Phi = [\Phi_1, \ldots, \Phi_m]$ of $M \subset R^m$ into R^m belongs to the class C_1 provided each Φ_i is of the class C_1 on M. Fix an arbitrary $x \in M$. The linear mapping

$d\Phi_x$ defined by

$$d\Phi_x(h) = \lim_{t \to 0} t^{-1}[\Phi(x + th) - \Phi(x)], \quad h \in R^m,$$

will be called the differential of Φ at x.

Suppose that both S and S' are smooth surfaces and let Φ be a mapping of the class C_1 of a neighborhood of S. Let $x \in S$, $\Phi(x) = y$, $\Phi(S) \subset S'$ and denote by T_x and T_y the tangent space of S at x and of S' at y, respectively. It is known that $d\Phi_x(h) \in T_y$ whenever $h \in T_x$ so that $d\Phi_x$ is a linear mapping of T_x into T_y (compare [13]). Let $\mathscr{V}$ and $\mathscr{V}'$ stand for the orthonormal basis of T_x and T_y, respectively. Denote by M_x the matrix of the mapping $d\Phi_x$ with respect to the bases $\mathscr{V}$, $\mathscr{V}'$. Then the absolute value of the determinant of M_x does not depend upon the choice of the bases in T_x and T_y, respectively, and will be denoted by $J\Phi(x)$.

Using the introduced terminology we are in position to formulate the following theorem which is a very special case of a general transformation theorem stated in [7] (Theorem 3.1.).

Theorem. *Let S and S' be smooth surfaces in R^m and ψ be a Lipschitzian mapping of the class C_1 of a neighborhood of S. Suppose that $\psi(S) \subset S'$ and*

$$\int_S J\psi(x)\, dH_{m-1}(x) < \infty .$$

Then the set $\psi_{-1}(y) \cap S$ is finite for almost all (H_{m-1}) points $y \in S'$. If g is a finite function on S, put

$$N^g(y; S, S', \psi) = \sum_{z \in \psi_{-1}(y) \cap S} g(z)$$

provided the set $\psi_{-1}(y) \cap S$ is finite; otherwise let $N^g(y; S, S', \psi) = 0$.

Then

$$\int_S g(x)\, J\psi(x)\, dH_{m-1}(x) = \int_{S'} N^g(y; S, S', \psi)\, dH_{m-1}(y) \tag{60}$$

provided the integral on the left-hand side converges.

24. Lemma. *Suppose that S is a smooth surface in R^m, $0 \notin S$. For $x \in R^m - \{0\}$ set $\varphi(x) = x/|x|$ and for $x \in S$ denote by $\sigma(x)$ a normal of S at x. Then*

$$J\varphi(x) = \frac{|\sigma(x) \,.\, x|}{|x|^m} \tag{61}$$

whenever $x \in S$.

Proof. Fix an arbitrary $x \in S$ and write, for the sake of brevity, $\sigma(x) = \sigma$, $\varphi(x) = \vartheta$. Denote by T^2 the tangent space of Γ at ϑ and T^1 the tangent space of S at x, respectively. Clearly, ϑ is orthogonal to T^2. Suppose first $T^1 \neq T^2$ so that $T^1 \cap T^2$ is a linear space of dimension $m - 2$. Choose in $T^1 \cap T^2$ an orthonormal basis $e_1, \ldots, e_{m-2}$ and put $e' = \vartheta - (\sigma . \vartheta)\, \sigma$, $f' = (\sigma . \vartheta)\, \vartheta - \sigma$, $\gamma = |e'|$ and $f_i = e_i$ for $i = 1, \ldots, m - 2$. Since $T^1 \neq T^2$ we have $\gamma \neq 0$ and it is easy to verify that $|f'| = \gamma$. Putting $e_{m-1} = \gamma^{-1} e'$, $f_{m-1} = \gamma^{-1} f'$ we see that $e_1, \ldots, e_{m-1}$ is an orthonormal basis in T^1 and $f_1, \ldots, f_{m-1}$ an orthonormal basis in T^2, respectively.

Simple calculation shows

$$\mathrm{d}\varphi_x(u) = |x|^{-1} \left(u - (\vartheta . u)\, \vartheta\right), \quad u \in R^m .$$

Since ϑ is orthogonal to T^2 we have for $j \in \{1, \ldots, m - 2\}$

$$\mathrm{d}\varphi_x(e_j) = |x|^{-1} f_j$$

while

$$\mathrm{d}\varphi_x(e_{m-1}) = |x|^{-1} (\sigma . \vartheta)\, f_{m-1} .$$

Now (61) follows immediately.

It remains to consider the case $T^1 = T^2$. Denote by $e_1, \ldots, e_{m-1}$ an orthonormal basis in T^1. The above consideration yields $J\varphi(x) = |x|^{-m+1}$. Since $\sigma(x) = x/|x|$, (61) holds again.

The proof of the lemma is complete.

25. Proposition. *Let $r > 0$, $y \in B$ and g be a finite non-negative function measurable (H) on B. Then the set*

$$\hat{B} \cap \{y + \varrho\theta;\ 0 < \varrho < r\} \tag{62}$$

is finite for almost all (H_{m-1}) points $\theta \in \Gamma$. For $\theta \in \Gamma$ put

$$n_r^g(y, \theta) = \sum g(z) \tag{63}$$

where the sum is extended over the set (62) provided that set is finite; otherwise put $n_r^g(y, \theta) = 0$. Then the function $n_r^g(y, \theta)$ of the variable θ is measurable (H_{m-1}) on Γ and

$$\int_\Gamma n_r^g(y, \theta)\, \mathrm{d}H_{m-1}(\theta) = \int_{B \cap \Omega_r(y)} g(z) \frac{|n(z) . (z - y)|}{|z - y|^m}\, \mathrm{d}H_{m-1}(z) . \tag{64}$$

Proof. It follows from the results of [6] and [2] that there exist sets $N, S_i, \tilde{S}_i$ $(i = 1, 2, \ldots)$ such that $\hat{B} - N = \bigcup_{j=1}^{\infty} \tilde{S}_j$, $\tilde{S}_j$'s are pairwise disjoint Borel sets, $\tilde{S}_j \subset S_j$, $H_{m-1}(N) = 0$ and S_j is a smooth surface in R^m with the property that for each j and each $z \in \tilde{S}_j$ the Federer normal $n(z)$ is a normal of S_j at z.

We may assume that $y = 0$. Recall here that

$$\int_B \frac{|n(z) . z|}{|z|^m}\, \mathrm{d}H_{m-1}(z) = v_\infty(0) < \infty \tag{65}$$

(compare (1) and lemma 2.12 in [9]).

Suppose first that f is a bounded non-negative function on B measurable (H). Define the function $\hat{f}$ on R^m so as to coincide with f on B and to vanish elsewhere and denote by χ_j the characteristic function of $\tilde{S}_j$. Put $f_j = \hat{f} \,.\, \chi_j$ and choose an arbitrary $\tau \in (0, r)$. The mapping φ is defined in the same way as in lemma 24. Fix a positive integer j and write $\Omega^* = \Omega_r(0) - \mathrm{cl}\, \Omega_\tau(0)$, $S_j^* = S_j \cap \Omega^*$, $\tilde{S}_j^* = \tilde{S}_j \cap \Omega^*$. Since $0 \notin \mathrm{cl}\, S_j^*$, the mapping φ is a Lipschitzian mapping of the class C_1 on a neighborhood of S_j^* and $\varphi(S_j^*) = \Gamma$. Setting $S = S_j^*$, $S' = \Gamma$, $\psi = \varphi$, $g = f_j$ in the theorem quoted in 23 we obtain according to (65), (60) and (61) that the function $N^{f_j}(\theta; S_j^*, \Gamma, \varphi)$ of the variable θ is measurable (H_{m-1}) on Γ and

$$(66) \qquad \int_{\tilde{S}_j^*} f_j(z) \frac{|n(z) \,.\, z|}{|z|^m} \,\mathrm{d}H_{m-1}(z) = \int_\Gamma N^{f_j}(\theta; S_j^*, \Gamma, \varphi) \,\mathrm{d}H_{m-1}(\theta) \,.$$

Put $B_\tau = \hat{B} \cap \Omega^*$, $\Gamma' = \varphi(N)$. Since f is bounded and $H_{m-1}(N) = 0$ we obtain from (66)

$$(67) \qquad \int_{B_\tau} f(z) \frac{|n(z) \,.\, z|}{|z|^m} \,\mathrm{d}H_{m-1}(z) = \int_\Gamma \sum_{j=1}^{\infty} N^{f_j}(\theta; S_j^*, \Gamma, \varphi) \,\mathrm{d}H_{m-1}(\theta) \,.$$

Consider first the function $f = 1$ on B. Since the integral on the left-hand side of (67) converges, the sum $\sum_{j=1}^{\infty} N^{f_j}(\theta; S_j^*, \Gamma, \varphi)$ is finite for almost all (H_{m-1}) points $\theta \in \Gamma$. Choose a $\theta \in \Gamma - \Gamma'$ such that the mentioned sum is finite. Since $\theta \notin \Gamma'$ we see that this sum equals the number of the points of

$$(68) \qquad \hat{B} \cap \{\varrho\theta; \tau < \varrho < r\} \,.$$

For such θ's we have (after returning to the original f)

$$(69) \qquad \sum_{j=1}^{\infty} N^{f_j}(\theta; S_j^*, \Gamma, \varphi) = \sum f(z)$$

where the last sum is taken over (68). Since $H_{m-1}(N) = 0$ and φ is a locally Lipschitzian mapping we conclude that $H_{m-1}(\Gamma') = 0$. Consequently, the set (68) is finite for almost all (H_{m-1}) points $\theta \in \Gamma$. Put $N_\tau^f(\theta) = \sum f(z)$ where the sum is taken over the set (68) provided that set is finite while otherwise $N_\tau^f(\theta) = 0$. It follows now from (69) that

$$N_\tau^f(\theta) = \sum_{j=1}^{\infty} N^{f_j}(\theta; S_j^*, \Gamma, \varphi)$$

for almost all (H_{m-1}) points $\theta \in \Gamma$. In particular, the function N_τ^f is measurable (H_{m-1}) on Γ and we have by (67)

$$(70) \qquad \int_{B_\tau} f(z) \frac{|n(z) \,.\, z|}{|z|^m} \,\mathrm{d}H_{m-1}(z) = \int_\Gamma N_\tau^f(\theta) \,\mathrm{d}H_{m-1}(\theta) \,.$$

Fix now a sequence $\tau_k \searrow 0$ and write N_k^f in place of $N_{\tau_k}^f$ for k sufficiently large. Then we obtain by (70) and (65)

$$\int_{\hat{B}\cap\Omega_r(0)} f(z)\,\frac{|n(z)\,.\,z|}{|z|^m}\,\mathrm{d}H_{m-1}(z) = \int_\Gamma \lim_{k\to\infty} N_k^f(\theta)\,\mathrm{d}H_{m-1}(\theta)\,. \tag{71}$$

Considering for a moment $f = 1$ again, we see from (71) that

$$\lim_{k\to\infty} N_k^f(\theta) < \infty$$

for almost all (H_{m-1}) points $\theta \in \Gamma$. Consequently, the set

$$\hat{B} \cap \{\varrho\theta;\ 0 < \varrho < r\} \tag{72}$$

is finite for almost all (H_{m-1}) points $\theta \in \Gamma$ and one easily verifies (we have now returned to our original f)

$$n_r^f(0, \theta) = \lim_{k\to\infty} N_k^f(\theta) \tag{73}$$

provided $\theta \in \Gamma$ is such that the set (72) is finite. In particular, the function $n_r^f(0, \theta)$ of the variable θ is measurable (H_{m-1}) on Γ and by (71), (73)

$$\int_{B\cap\Omega_r(0)} f(z)\,\frac{|n(z)\,.\,z|}{|z|^m}\,\mathrm{d}H_{m-1}(\theta) = \int_\Gamma n_r^f(0, \theta)\,\mathrm{d}H_{m-1}(\theta)\,, \tag{74}$$

which is (64) for $y = 0$ and $f = g$.

Thus (64) is established under the additional assumption that g is bounded. Using standard reasonings one easily extends (64) to any finite non-negative g measurable (H).

The proof of the proposition is complete.

26. Notation. Fix $y \in R^m$ and put

$$B_y = \{z \in \hat{B};\ n(z)\,.\,(z - y) = 0\}$$

and let c_y stand for the characteristic function of B_y. If n_j is the j-th component of the Federer normal n, then n_j is a Baire function. This follows from [4], theorem 4.5 and [11], chap. 2. § 31, VI. Consequently, B_y is a Borel set and $c_y \in \mathscr{B}$.

For $z \in \hat{B} - B_y$ put

$$l_y(z) = \left|1 - \frac{|z - y|^2\, l(z)}{n(z)\,.\,(z - y)}\,.\,\frac{1}{m - 2}\right|$$

and let $l_y(z) = 0$ elsewhere on B. Then l_y is a finite non-negative function on B measurable (H).

Given $y \in B$ and $r > 0$, put

$$\hat{v}_r(y) = \frac{\hat{\lambda}[\Omega_r(y)]}{(m-2)\, r^{m-2}} + \int_0^r \varrho^{1-m} \hat{\lambda}[\Omega_\varrho(y)]\, \mathrm{d}\varrho\,,$$

$$w_r(y) = U c_y \chi_{r,y} (\lambda - \hat{\lambda})(y)\,,$$

$$v_r^l(y) = \int_\Gamma n_r^{l_y}(y, \theta)\, \mathrm{d}H_{m-1}(\theta)\,,$$

where $n_r^{l_y}(y, \theta)$ has the meaning defined in proposition 25. Finally set

$$g_r(y) = \hat{v}(y) + v_r^l(y) + w_r(y)\,.$$

We see that the quantity $g_r(y)$ is connected with the geometrical shape of G and the distribution λ over B. The following theorem expresses ωT_α in terms of $g_r(y)$ and $d(y)$.

27. Theorem. *Suppose that the potential $U(\lambda - \hat{\lambda})$ is continuous at every point of the set $\tilde{B}$. Then*

(75) $$\omega T_\alpha = \lim_{r \to 0+} \sup_{y \in E} \left[A|d(y) - \alpha| + g_r(y)\right].$$

Proof. First we prove the equality

(76) $$|\tau_y| (\Omega_r(y)) = g_r(y)$$

whenever $y \in B$ and $r > 0$. Going back to (49) and to the definitions of l_y and B_y, respectively, we have

$$|\tau_y| (\Omega_r(y)) = U \chi_{r,y} \hat{\lambda}(y) + \int_{B \cap \Omega_r(y)} l_y(z) \frac{|n(z) \,.\, (z - y)|}{|z - y|^m}\, \mathrm{d}H_{m-1}(z) +$$
$$+ (m-2)^{-1} \int_{B_y \cap \Omega_r(y)} |z - y|^{2-m}\, \mathrm{d}H(z)\,.$$

The second summand equals $v_r^l(y)$ by proposition 25 and the third one equals $w_r(y)$. Applying lemma 10 of [16] we obtain the equality

(77) $$U \chi_{r,y}\, \hat{\lambda}(y) = \hat{v}_r(y)$$

so that (76) is established.

Now (75) follows from theorem 20 and (76).

28. Notation. Write, for the sake of brevity, $B_1 = Z \cap E$, $B_2 = \hat{B}$ and for $j = 1, 2$ define $k_j = 0$ or

$$k_j = \lim_{r \to 0+} \sup_{y \in B_j} g_r(y)$$

according as $B_j = \emptyset$ or not.

In the following two theorems the same reasonings are used as in [10] (compare theorems 3.8, 3.9).

29. Theorem. *Suppose that the potential* $U(\lambda - \hat{\lambda})$ *is continuous at every point of* cl $[B - (B_1 \cup B_2)]$. *Let us distinguish the following three cases*:

(i) $B_1 = \emptyset$ *or* $k_2 \geqq \frac{1}{2}A + k_1$,

(ii) $B_2 = \emptyset$ *or* $k_1 \geqq \frac{1}{2}A + k_2$,

(iii) $B_1 \neq \emptyset \neq B_2$ *and* $|k_1 - k_2| < \frac{1}{2}A$.

Then

$$\omega T_\alpha = k_2 + A|\tfrac{1}{2} - \alpha| \quad \text{in the case (i)}, \tag{78}$$

$$\omega T_\alpha = k_1 + A|1 - \alpha| \quad \text{in the case (ii)}, \tag{79}$$

while in the case (iii)

$$\omega T_\alpha = \tfrac{1}{2}(k_1 + k_2) + \tfrac{1}{4}A + \left|\frac{k_1 - k_2}{2} + \tfrac{3}{4}A - \alpha A\right|. \tag{80}$$

Proof. Let us observe that under our assumptions the hypotheses of theorems 20 and 27 are fulfilled because $\tilde{B} - I_B \subset B - (B_1 \cup B_2)$ by lemma 14.

We first prove that

$$\lim_{r \to 0+} \sup_{y \in E} a_r(y) = \lim_{r \to 0+} \sup_{y \in B_1 \cup B_2} a_r(y). \tag{81}$$

Fix an arbitrary $\varepsilon > 0$ and write, for the sake of brevity, $\lambda_1 = \lambda - \hat{\lambda}$ and $B' = B - (B_1 \cup B_2)$. Using proposition 6 and compactness of cl B' we conclude that there is an open set $Q \subset R^m$ and $r_0 > 0$ such that

$$U\chi_{r,y}\,\lambda_1(y) < \varepsilon \tag{82}$$

whenever $y \in B \cap Q$ and $r \in (0, r_0)$.

Suppose that $y_0 \in E - (B_1 \cup B_2)$. The set $B_1 \cup B_2$ is dense in E by lemma 14, so that there exist $y_n \in B_1 \cup B_2$ with $y_n \to y_0$. According to the remark 19 it is possible to go back to (54), which together with (82) yields

$$a_r(y_0) \leqq \sup_{y \in B_1 \cup B_2} a_r(y) + 2\varepsilon$$

for all $r \in (0, r_0) - R_0$. Now (81) follows immediately.

By definition $a_r(y)$ and by (81), (55) and (76) we obtain

$$\omega T_\alpha = \lim_{r \to 0+} \sup_{y \in B_1 \cup B_2} [A|d(y) - \alpha| + g_r(y)]. \tag{83}$$

If $B_1 = \emptyset$, then $d(y) = \frac{1}{2}$ for each $y \in B_1 \cup B_2$ and (78) follows from (83). Similarly, $B_2 = \emptyset$ implies $d(y) = 1$ for each $y \in B_1 \cup B_2$ whence we conclude (79). Suppose now that $B_1 \neq \emptyset \neq B_2$. Then

$$\omega T_\alpha = \max\left(A|1 - \alpha| + k_1,\ A|\tfrac{1}{2} - \alpha| + k_2\right).$$

Calculating this maximum for the cases (i), (ii) and (iii) discussed in the theorem, we obtain (78), (79) and (80), respectively.

The following lemma shows that under additional continuity assumptions it is possible to simplify the expressions for k_j's.

30. Lemma. *Fix $j \in \{1, 2\}$ and suppose that $B_j \neq \emptyset$; then the following assertions hold.*

If $U\hat{\lambda}$ is continuous, then

(84) $$k_j = \lim_{r \to 0+} \sup_{y \in B_j} [v_r^l(y) + w_r(y)].$$

If $U(\lambda - \hat{\lambda})$ is continuous, then

(85) $$k_j = \lim_{r \to 0+} \sup_{y \in B_j} [v_r(y) + \hat{v}_r(y)].$$

Finally, if $U\lambda$ is continuous, then

(86) $$k_j = \lim_{r \to 0+} \sup_{y \in B_j} v_r(y).$$

Proof. Continuity of $U\hat{\lambda}$ implies

(87) $$\lim_{r \to 0+} \sup_{y \in B_j} U\chi_{r,y}\, \hat{\lambda}(y) = 0$$

by proposition 6. Recalling the equality $\hat{v}_r(y) = U\chi_{r,y}\, \hat{\lambda}(y)$ (see (77)) we obtain (84) by definition of k_j and by (87).

If $U(\lambda - \hat{\lambda})$ is continuous, then proposition 6 may be applied again to assert

$$\lim_{r \to 0+} \sup_{y \in B_j} U\chi_{r,y}(\lambda - \hat{\lambda})(y) = 0.$$

For $r > 0$ and $y \in B$ we have by (49) and (76)

$$\hat{v}_r(y) + v_r(y) - U\chi_{r,y}(\lambda - \hat{\lambda})(y) \leqq g_r(y) \leqq$$
$$\hat{v}_r(y) + v_r(y) + U\chi_{r,y}(\lambda - \hat{\lambda})(y)$$

so that (85) is true.

Finally, continuity of $U\lambda$ implies continuity of $U(\lambda - \hat{\lambda})$ and $U\hat{\lambda}$ as well (see remark 7) so that we may use (85), (87) to show the validity of (86).

31. Theorem. *Suppose that the potential* $U(\lambda - \hat{\lambda})$ *is continuous at each point of* $\mathrm{cl}\,[B - (B_1 \cup B_2)]$. *Define the number a by*

(88)
$$a = \inf_{\alpha \neq 0} \frac{\omega T_\alpha}{A|\alpha|}\,.$$

Then

(89)
$$a < 1$$

holds if and only if

(90)
$$k_1 < A\,, \quad k_2 < \tfrac{1}{2}A\,.$$

If the conditions (90) *are fulfilled, then there is exactly one* γ *with*

$$\frac{\omega T_\gamma}{A|\gamma|} = a$$

and one of the following three cases must occur:

(i*) $B_1 = \emptyset\,,$

(ii) $B_2 = \emptyset$ *or* $k_1 \geqq \tfrac{1}{2}A + k_2\,,$

(iii) $B_1 \neq \emptyset \neq B_2$ *and* $|k_1 - k_2| < \tfrac{1}{2}A\,.$

The corresponding values a and γ *are then given as follows*:

$$a = 2k_2/A\,, \quad \gamma = \tfrac{1}{2} \quad \textit{in the case } (\mathrm{i}^*)\,,$$
$$a = k_1/A\,, \quad \gamma = 1 \quad \textit{in the case } (\mathrm{ii})$$

while in the case (iii)

$$a = \frac{k_1 + k_2 + \frac{1}{2}A}{k_1 - k_2 + \frac{3}{2}A}\,, \quad \gamma = \frac{3}{4} + \frac{k_1 - k_2}{2A}\,.$$

Proof. The proof of this theorem reduces to the succesive investigation of cases occurring in theorem 29. Since this calculation is completely the same as in the proof of theorem 3.9 in [10] we omit it here.

32. Remark. In this remark we shall suppose that $U\lambda$ is continuous. The similar case for special domains in R^3 has been investigated by V. D. Sapožnikova in [18]. By lemma 30, the numbers k_1, k_2 are given by (86) and the hypotheses of theorems 29 and 31 are fulfilled. According to proposition 9 we have $T\mathscr{C} \subset \mathscr{C}$, $V\mathscr{C} \subset \mathscr{C}$. Denote by $\hat{T}$, $\hat{V}$, $\hat{\tilde{W}}$ the restriction of T, V, $\tilde{W}$ on $\mathscr{C}$, respectively, and let $\hat{I}$ stand for the identity operator on $\mathscr{C}$. Then

$$\langle \hat{T}f, \mu \rangle = \langle f, \mathscr{T}\mu \rangle$$

whenever $\mu \in \mathfrak{B}$, $f \in \mathscr{C}$ (compare (18) in [16]) so that $\hat{T}^* = \mathscr{T}$ where we have denoted by $\hat{T}^*$ the dual operator to $\hat{T}$. For $\alpha \in R^1$ put $\hat{T}_\alpha = \hat{T} - \alpha A\hat{I}$ and for each bounded linear operator X on $\mathscr{C}$ denote

$$\tilde{\omega}_{\mathscr{C}} X = \inf_Q \|X - Q\|$$

where Q runs over all compact operators acting on $\mathscr{C}$.

Since $\hat{V}$ is compact by proposition 9 we have

$$\tilde{\omega}_{\mathscr{C}} \hat{T}_\alpha = \tilde{\omega}_{\mathscr{C}}(\hat{\hat{W}} - \alpha A\hat{I}) . \tag{91}$$

It is easily seen from the proof of theorem 3.6 in [9] that

$$\tilde{\omega}_{\mathscr{C}}(\hat{\hat{W}} - \alpha A\hat{I}) = \lim_{r\to 0+} \sup_{y\in E} [A|d(y) - \alpha| + v_r(y)] .$$

Comparing this with (56) and recalling the equality $|v_y| (\Omega_r(y)) = v_r(y)$ we arrive at

$$\tilde{\omega}_{\mathscr{C}} \hat{T}_\alpha = \omega T_\alpha . \tag{92}$$

In particular, theorems 29, 31 remain true if we write $\tilde{\omega}_{\mathscr{C}} \hat{T}_\alpha$ in place of ωT_α. If $\lambda = 0$, the corresponding assertions complete the results of § 3 in [9].

With the same notation as in the introduction we have the following lemma.

33. Lemma. *The equality* $\omega' \mathscr{T}_\alpha = \omega T_\alpha$ *holds for every* $\alpha \in R^1$. *In particular,* $a = a'$.

Proof. Fix $\alpha \in R^1$. Choose an arbitrary positive integer n and elements $v_j \in \mathfrak{B}$, $f_j \in \mathscr{B}$, $j = 1, \ldots, n$. Then the operator

$$Q \ldots = \sum_{j=1}^n \langle \ldots, v_j \rangle f_j$$

acting on $\mathscr{B}$ belongs, by definition, to $\mathscr{F}$ and the operator

$$\mathscr{Q} \ldots = \sum_{j=1}^n \langle f_j, \ldots \rangle v_j$$

acting on $\mathfrak{B}$ belongs to $\mathscr{F}'$.

Clearly,

$$\langle (T_\alpha - Q)\, g, v \rangle = \langle g, (\mathscr{T}_\alpha - \mathscr{Q})\, v \rangle$$

provided $g \in \mathscr{B}$ and $v \in \mathfrak{B}$. Consequently,

$$\|T_\alpha - Q\| = \|\mathscr{T}_\alpha - \mathscr{Q}\|$$

The rest is easy.

34. Remark. In view of the above lemma, theorems 29 and 31 can be stated in terms of $\omega'\mathcal{T}_\alpha$ and a' as well. The importance of the inequality $a' < 1$ lies in the fact that, under this assumption, the Riesz-Schauder theory is applicable to the equation

$$\mathcal{T}\mu = \nu \tag{93}$$

over $\mathfrak{B}$.

In other words, the inequalities in (80) express in geometrical terms connected with the shape of G and the distribution of λ the sufficient conditions for applicability of the Riesz-Schauder theory to the solving of the third boundary value problem in the formulation (93). We shall deal with these problems in [17].

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