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TOLERANCE IN ALGEBRAIC STRUCTURES II

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In [1] the concept of tolerance relation was studied. Here we shall add some new results on this topic.

A tolerance relation on a set is a binary relation which is reflexive and symmetric.

Let an algebraic structure $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be given, where A is the set of elements of this structure and $\mathscr{F}$ is the set of operations. If a tolerance ξ on A is given, we say that ξ is compatible with $\mathfrak{A}$, if and only if for any n-ary (n positive integer) operation $f \in \mathscr{F}$ and any $2n$ elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ of A such that $(x_i, y_i) \in \xi$ for $i = 1, \ldots, n$ we have $(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \in \xi$.

We shall prove some theorems.

Theorem 1. *Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebraic structure and ξ a tolerance on A which is compatible with $\mathfrak{A}$. Then the transitive closure $T\xi$ of ξ is a congruence on $\mathfrak{A}$.*

Proof. Let $f \in \mathscr{F}$ be an n-ary operation of $\mathfrak{A}$, where n is a positive integer. Let $x_1, \ldots, x_n, y_1, \ldots, y_n$ be elements of A such that $(x_i, y_i) \in T\xi$ for $i = 1, \ldots, n$. This means that for any $i = 1, \ldots, n$ there exists a finite sequence $z_1^{(i)}, \ldots, z_{k_i}^{(i)}$ such that $x_i = z_1^{(i)}$, $y_i = z_{k_i}^{(i)}$ and $(z_j^{(i)}, z_{j+1}^{(i)}) \in \xi$ for $j = 1, \ldots, k_i - 1$. Let $k = \max_{1 \leqq i \leqq n} k_i$. If for some i the number $k_i < k$, we define $z_{k_i+1}^{(i)}, \ldots, z_k^{(i)}$ so that all these elements are equal to y_i. Therefore for any $i = 1, \ldots, n$ we have the sequence $z_1^{(i)}, \ldots, z_k^{(i)}$ so that $z_1^{(i)} = x_i$, $z_k^{(i)} = y_i$ and $(z_j^{(i)}, z_{j+1}^{(i)}) \in \xi$ for $j = 1, \ldots, k - 1$. As ξ is compatible with $\mathfrak{A}$, we have $(f(z_j^{(1)}, z_j^{(2)}, \ldots, z_j^{(n)}), f(z_{j+1}^{(1)}, z_{j+1}^{(2)}, \ldots, z_{j+1}^{(n)})) \in \xi$ for $j = 1, \ldots, k - 1$ and therefore $(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) = (f(z_1^{(1)}, \ldots, z_n^{(1)}), f(z_1^{(k)}, \ldots, z_n^{(k)}) \in T\xi$. This can be done for any n-ary operation $f \in \mathscr{F}$, where n is a positive integer, therefore $T\xi$ is a congruence on $\mathfrak{A}$.

Theorem 2. *Let $\mathfrak{G}$ be a group, let ξ be a tolerance which is compatible with $\mathfrak{G}$ as with a semigroup. Then ξ is a congruence on $\mathfrak{G}$.*

Remark. We say that ξ is compatible with $\mathfrak{G}$ as with a semigroup, if $(x_1, y_1) \in \xi$, $(x_2, y_2) \in \xi$ imply $(x_1x_2, y_1y_2) \in \xi$, saying nothing about the inversion operation.

Proof. In [1] it was proved that if ξ is compatible with $\mathfrak{G}$ as with a group, i.e. if $(x_1, y_1) \in \xi$, $(x_2, y_2) \in \xi$ imply not only $(x_1x_2, y_1y_2) \in \xi$, but also $(x_1^{-1}, y_1^{-1}) \in \xi$, $(x_2^{-1}, y_2^{-1}) \in \xi$, the tolerance ξ is a congruence. Therefore it remains to prove that each tolerance ξ which is compatible with $\mathfrak{G}$ as with a semigroup is compatible with $\mathfrak{G}$ as with a group. Let $a \in \mathfrak{G}$, $b \in \mathfrak{G}$, $(a, b) \in \xi$, Let a^{-1}, b^{-1} be the inverse elements to a, b respectively. As ξ is reflexive ,we have $(a^{-1}, a^{-1}) \in \xi$, $(b^{-1}, b^{-1}) \in \xi$. The relations $(a, b) \in \xi$, $(a^{-1}, a^{-1}) \in \xi$ imply $(e, a^{-1}b) \in \xi$, where e is the unit element of $\mathfrak{G}$. This relation together with $(b^{-1}, b^{-1}) \in \xi$ implies $(b^{-1}, a^{-1}) \in \xi$ and the symmetry of ξ gives $(a^{-1}, b^{-1}) \in \xi$, q.e.d.

Theorem 3. *Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra, let ξ_1, ξ_2 be two tolerances on A which are compatible with $\mathfrak{A}$. Then the relation $\xi_1 \cap \xi_2$ is also a tolerance compatible with $\mathfrak{A}$.*

Proof. The intersection of two reflexive and symmetric relations is evidently again reflexive and symmetric. Now let $f \in \mathscr{F}$ be an n-ary operation on $\mathfrak{A}$ and let $x_1, \ldots, x_n, y_1, \ldots, y_n$ be elements of A such that $(x_i, y_i) \in \xi_1 \cap \xi_2$ for $i = 1, \ldots, n$. Then $(x_i, y_i) \in \xi_1$ and therefpre $(f(x_1, \ldots, x_n), f(y_1. \ldots, y_n)) \in \xi_1$. But also $(x_i, y_i) \in \xi_2$ for $i = 1, \ldots, n$ and thus $(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \in \xi_2$. We have obtained that $(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \in \xi_1 \cap \xi_2$. As f was chosen arbitrarily, this holds for any $f \in \mathscr{F}$ and $\xi_1 \cap \xi_2$ is a compatible tolerance on $\mathfrak{A}$.

This theorem enables us to formulate the following definition.

Let ξ_0 be a tolerance on the set of elements of an algebra $\mathfrak{A}$. Then the tolerance generated by ξ_0 on $\mathfrak{A}$ is the intersection of all tolerances which are compatible with $\mathfrak{A}$ and contain ξ_0.

In the case when $\mathfrak{A}$ is a semigroup, it is easy to prove that such a tolerance consists of all pairs $(x_1x_2 \ldots x_n, y_1y_2 \ldots y_n)$, where n is a positive integer and $(x_i, y_i) \in \xi_0$ for $i = 1, \ldots, n$.

Theorem 4. *Let $\mathfrak{S}$ be a semigroup with at least there elements. If $\mathfrak{S}$ contains a proper two-side ideal $\mathfrak{J}$, then there exists a tolerance compatible with $\mathfrak{S}$ which is is not a congruence on $\mathfrak{S}$.*

Remark. We must suppose that $\mathfrak{S}$ has at least three elements, because each tolerance on a set with less than three elements is either the equality, or the universal relation, which both are equivalences.

Proof. At first assume that $\mathfrak{S} \dot{-} \mathfrak{J}$ contains at least two elements. Choose two distinct elements a, c of $\mathfrak{S} \dot{-} \mathfrak{J}$ and one element $b \in \mathfrak{J}$. Let ξ_0 consist of the pairs (a, b), (b, c) and of all pairs of equal elements; this is a tolerance on $\mathfrak{S}$. Let ξ be the tolerance on $\mathfrak{S}$ generated by ξ_0. This tolerance ξ contains obviously (a, b) and (b, c); we shall prove that it does not contain (a, c). Assume that $(a, c) \in \xi$. Then $a = x_1 x_2 \dots x_n$, $c = y_1 y_2 \dots y_n$, where $(x_i, y_i) \in \xi_0$ for $i = 1, \dots, n$ and n is a positive integer. This means that either $x_i = y_i$, or one of these two elements is equal to b. But if $x_i = b$, then $a \in \mathfrak{J}$, because $b \in \mathfrak{J}$ and $\mathfrak{J}$ is an ideal of $\mathfrak{S}$. As we have supposed that $a \in \mathfrak{S} \dot{-} \mathfrak{J}$, we obtain a contradiction. Similarly $y_i = b$ implies $c \in \mathfrak{J}$. Thus $x_i = y_i$ must hold for all $i = 1, \dots, n$, which implies $a = c$, which is also a contradiction. Therefore ξ is not transitive and it is not a congruence.

Now assume that $\mathfrak{S} \dot{-} \mathfrak{J}$ consists only of one element a. If a is not idempotent, $a^2 \in \mathfrak{J}$ and a cannot be expressed as a product of two elements of $\mathfrak{S}$. We choose two elements b and c of $\mathfrak{J}$. Let ξ_0 consist of (a, b), (b, c) and all pairs of equal elements of $\mathfrak{S}$, let ξ be the tolerance on $\mathfrak{S}$ generated by ξ_0. Then ξ cannot contain (a, c), because a cannot be expressed as $x_1 x_2 \dots x_n$.

Now suppose that a is idempotent and $a\mathfrak{J}a$ is a proper subset of $\mathfrak{J}$. Choose $b \in a\mathfrak{J}a$, $c \in \mathfrak{J} \dot{-} a\mathfrak{J}a$ and let again ξ_0 consist of (a, b), (b, c) and all pairs of equal elements. If $(a, c) \in \xi$, this means again $a = x_1 \dots x_n$, $c = y_1 \dots y_n$ and $(x_i, y_i) \in \xi_0$ for $i = 1, \dots, n$. As a is the unique element of $\mathfrak{S}$ not belonging to the ideal $\mathfrak{J}$, we must have $x_i = a$ for $i = 1, \dots, n$ and therefore for each i either $y_i = a$, or $y_i = b$. As $b \in a\mathfrak{J}a$ and a is idempotent, we have $ab = ba = b$ and therefore $y_1 \dots y_n$ is equal to a pr to a power of b. But as $ab = ba = b$, we have $b^k = ab^k a$ for any positive integer k and thus all powers of b are in $a\mathfrak{J}a$. None of them can be equal to c, because $c \in \mathfrak{J} \dot{-} a\mathfrak{J}a$; we have a contradiction.

Finally suppose that a is idempotent and $a\mathfrak{J}a = \mathfrak{J}$. Then a is a unit element for all elements of $\mathfrak{J}$. We choose two elements b, c of $\mathfrak{J}$ so that c is no power of b. This can be always done. If $\mathfrak{J}$ contains idempotents, then we choose b idempotent and c will be an arbitrary element of $\mathfrak{J}$ different from b. If $\mathfrak{J}$ does not contain idempotents, it is torsion-free and it suffices to choose c and put $b = c^2$. Now let again ξ_0 consist of (a, b), (b, c) and all pairs of equal elements and let ξ be the tolerance on $\mathfrak{S}$ generated by ξ_0. If $(a, c) \in \xi$, this means again $a = x_1 \dots x_n$, $c = y_1 \dots y_n$ and $(x_i, y_i) \in \xi_0$ for $i = 1, \dots, n$. As in the preceding case we must have $x_i = a$ for $i = 1, \dots, n$ and therefore either $y_i = a$, or $y_i = b$. As a is the unit element for $\mathfrak{J}$, $y_1 \dots y_n$ is a or a power of b. Therefore it cannot be equal to c. Thus we have exhausted all cases and the proof is complete.

Theorem 5. *For each positive integer n there exists a semigroup with n elements such that each tolerance on its element set is compatible with it.*

Proof. We take an arbitrary set S with n elements and for any $x \in S$, $y \in S$ we put $xy = y$. It is easy to prove that the semigroup thus defined is the required semigroup.

This theorem shows that on a semigroup compatible tolerances which are not congruences can exist even if this semigroup has no proper two-side ideal.

Problem 1. Does there exist a semigroup with more than two elements which is not a group and on which each compatible tolerance is a congruence?*)

Problem 2. Does there exist a commutative semigroup such that each tolerance on its element set is compatible with it?

Reference

[1] *B. Zelinka:* Tolerance in algebraic structures. Czech. Math. J. *20* (1970), 179—183.

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*) Added in proof: In the author's paper „*Tolerance on periodic commutative semigroups*" (to appear in this Journal) this question is answered negatively for periodic commutative semigroups and affirmatively for periodic non-commutative ones.