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A MATRIX CHARACTERIZATION OF THE MAXIMAL GROUPS IN β_X

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In establishing some setting of this note in the currently published research, we cite that recently, much work has been done on β_X, the semigroup of relations on a set X. SCHWARZ [6], characterizes the idempotents in this semigroup. Each of these idempotents is then in some maximal group of β_X. SCHWARZ [5] questions whether these groups are in fact isomorphic to symmetric groups on some subset of X. MONTAGUE and PLEMMONS [3] answer this question in the negative by proving the remarkable result that every finite group is isomorphic to a maximal group of β_X for some X. PLEMMONS and SCHEIN [4], as well as CLIFFORD [2], extend the result to arbitrary groups.

An essential tool in the arguments of the above results is the Theorem of BIRKHOFF [1] which states that every group is isomorphic to a group of automorphisms on some partially ordered set (X, α) where α is the partial order on the set X. The pivotal point of argument hinges on showing that Auto (X, α) is isomorphic to the maximal group in β_X containing α as its identity.

This paper then provides a matrix characterization of the maximal groups of β_X. This characterization may be utilized to given an alternate proof of the Montague-Plemmons result and in fact the characterization yields a clear view of the interplay of the roles of the automorphisms of (X, α) and the members of the maximal group in β_X containing α.

Results. Let n be a positive integer and $X = \{1, 2, \ldots, n\}$. It is well known that the semigroup of relations on X, i.e., β_X, is isomorphic to $\mathcal{M}$, the matrices of order n over a Boolean algebra $\mathcal{B}$ of order two. This isomorphism maps the relation R to the matrix A where $a_{ij} = 1$ if and only if $(i, j) \in R$. For the work herein we consider the equivalent matrix problem of characterizing the maximal groups of matrices in $\mathcal{M}$.

Let $\mathcal{G}$ be a maximal group in $\mathcal{M}$ with I, an idempotent, as its identity. Properties concerning I are contained in the following Theorem of Schwarz [6].

Theorem. *If I is idempotent then there is a permutation matrix P so that*

$$P^t I P = \begin{pmatrix} A_1 & 0 & \dots 0 & 0 \\ A_{2,1} & A_2 & \dots 0 & 0 \\ \dots & \dots & \dots & \dots \\ A_{s-1,1} & A_{s-1,2} & \dots A_{s-1} & 0 \\ A_{s,1} & A_{s,2} & \dots A_{s,s-1} & A_s \end{pmatrix}$$

where

(1) *A_k is composed entirely of ones or $A_k = (0)$, the 0-matrix of order one.*

(2) *Each A_{kj} is composed entirely of ones or entirely of zeros.*

(3) *The columns of* $\begin{pmatrix} 0 \\ \dots \\ 0 \\ A_k \\ A_{k+1,k} \\ \dots \\ A_{s,k} \end{pmatrix}$ *are identical.*

(4) *If $A_{kj} > 0$ and a_k a column in* $\begin{pmatrix} 0 \\ \dots \\ 0 \\ A_k \\ \dots \\ A_{s,k} \end{pmatrix}$, *$a_j$ a column in* $\begin{pmatrix} 0 \\ \dots \\ 0 \\ A_j \\ \dots \\ A_{s,j} \end{pmatrix}$,

then $a_j \geqq a_k$.

Without loss of generality, we assume I has the form specified in the above idempotent theorem.

Our characterization of $\mathcal{G}$ is accomplished through a sequence of lemmas. The first of these lemmas utilizes the following notations.

Let $E = \{(x_1, x_2, \dots, x_n)^t$ where $x_k \in \mathcal{B}$ for each $k\}$. For $A \in \mathcal{M}$, let $\mathcal{N}(A) = \{x \in E$ where $Ax = 0\}$ and $R(A) = \{y \in E$ where $Ax = y$ for some $x \in E\}$. An elementary argument provides the initial result.

Lemma 1. *If $A \in \mathcal{G}$ then $\mathcal{N}(A) = \mathcal{N}(I)$ and $R(A) = R(I)$. Moreover, if $x \leqq y$ then $Ax \leqq Ay$.*

Lemma 2. *If $A \in \mathcal{G}$ then A is a permutation on $R(I) = R(A)$.*

Proof. If $Ax = Ay$ for $x, y \in R(I)$ then $Ix = A^{-1}Ax = A^{-1}Ay = Iy$. Since $x, y \in R(I)$, $x = y$.

The next lemma utilizes an elementary result concerning the algebraic system E. For the sake of completeness, we include the necessary background for this result.

If $\mathscr{A} = \{\alpha_1, \ldots, \alpha_r\} \subseteq E$ is such that $\lambda_i \alpha_i = \sum_{k \neq i} \lambda_k \alpha_k$, where each $\lambda_k \in \mathscr{B}$ implies that $\lambda_1 = \lambda_2 = \ldots = \lambda_r = 0$, then $\mathscr{A}$ is said to be *independent*. If $\mathscr{S} \subseteq E$ is closed under addition and $\mathscr{S}$ contains an independent set $\mathscr{A}$ such that $\mathscr{S} = \{\sum_{k=1}^{r} \lambda_k \alpha_k$ where $\lambda_k \in \mathscr{B}$ and $\alpha_k \in \mathscr{A}\}$, then $\mathscr{A}$ is called a *basis* of $\mathscr{S}$. The aforementioned result may now be formulated as follows. The proof is left to the reader.

Lemma 3. *Every set $\mathscr{S} \subseteq E$ which is closed under addition has a unique basis.*

Let $I = (a_1, a_2, \ldots, a_n)$. From the above discussion then, $R(I)$ being closed under addition has a unique basis, say $\mathscr{A} = \{a_{i_1}, \ldots, a_{i_s}\}$. As any $A \in \mathscr{G}$ is one-one and onto $R(I)$, A must map $\mathscr{A}$ onto $\mathscr{A}$. Thus there is a permutation $\bar{\pi}$ on $\{i_1, \ldots, i_s\}$ such that $Aa_{i_k} = a_{\bar{\pi}(i_k)}$. Since A is order preserving on $\mathscr{A}$, $\bar{\pi}$ induces an order automorphism π on $\mathscr{A}$ by defining $\pi(a_{i_k}) = a_{\bar{\pi}(i_k)}$.

Lemma 4. *If $A \in \mathscr{G}$ then there is an order automorphism π of the poset $\mathscr{A} = \{a_{i_1}, \ldots, a_{i_s}\}$ such that $Aa_{i_k} = a_{\bar{\pi}(i_k)}$.*

Out next lemma allows us to determine a form for matrices A in $\mathscr{G}$. For this, let $E_i = \{e_k \mid Ie_k = a_i\}$.

Lemma 5. *Let $A \in \mathscr{G}$ and let π be the order automorphism of $\mathscr{A}$ determined by A. If $a_j = a_{i_{k_1}} + \ldots + a_{i_{k_r}}$ then $E_j = \{e_k \mid Ae_k = a_{\bar{\pi}(i_{k_1})} + \ldots + a_{\bar{\pi}(i_{k_r})}\}$. In particular, if $a_j \in \mathscr{A}$ then $E_j = \{e_k \mid Ae_k = a_{\bar{\pi}(j)}\}$.*

Proof. If $e_k \in E_j$ then $Ie_k = a_j$. Hence $Ae_k = AIe_k = Aa_j = Aa_{i_{k_1}} + \ldots + Aa_{i_{k_r}} = a_{\bar{\pi}(i_{k_1})} + \ldots + a_{\bar{\pi}(i_{k_r})}$. On the other hand, if $Ae_k = a_{\bar{\pi}(i_{k_1})} + \ldots + a_{\bar{\pi}(i_{k_r})}$ then $Ie_k = A^{-1}Ae_k = A^{-1}a_{\bar{\pi}(i_{k_1})} + \ldots + A^{-1}a_{\bar{\pi}(i_{k_r})} = a_{i_{k_1}} + \ldots + a_{i_{k_r}} = a_j$.

Lemma 5 may be utilized to determined a form for each $A \in \mathscr{G}$. For this, partition the columns of I as in the idempotent theorem. Partition the columns of each $A \in \mathscr{G}$ as those of I. Lemma 4 now implies that the columns in each partition of A are identical.

Further, if $a_j \in \mathscr{A}$ column j of A is $a_{\bar{\pi}(j)}$. If $a_j \notin \mathscr{A}$ say $a_j = a_{i_{k_1}} + \ldots + a_{i_{k_r}}$, then column j of A is $a_{\bar{\pi}(i_{k_1})} + \ldots + a_{\bar{\pi}(a_{i_{k_r}})}$. We call any A so determined an order induced form of I or simply an I-form.

It is clear that I and any order automorphism π of $\mathscr{A}$ uniquely determine an I-form A. The identity map on $\mathscr{A}$ of course, uniquely determines I. These I-forms then provide the characterization of $\mathscr{G}$.

Theorem 1. *A matrix $A \in \mathscr{G}$ if and only if A is an I-form.*

Proof. Let $\mathscr{F} = \{A \mid A \text{ is an } I\text{-form}\}$. From the above lemmas, $\mathscr{G} \subseteq \mathscr{F}$.

Conversely, pick $A \in \mathscr{F}$ and let π be the order automorphism of $\mathscr{A}$ associated with A. First note that as $Ix = x$ for each $x \in R(I)$, $IA = A$. Now pick $a_i \in \mathscr{A}$. Suppose $a_i = e_{i_1} + e_{i_2} + \ldots + e_{i_t}$. Then $a_i = Ie_{i_1} + Ie_{i_2} + \ldots + Ie_{i_t}$. Hence $e_{i_k} \in \mathscr{N}(I)$ or $e_{i_k} \in E_i$ which in turn implies that $Ae_{i_k} = 0$ or $Ae_{i_k} = a_{\bar{\pi}(i_k)}$. Thus $Aa_i = a_{\bar{\pi}(i)}$. Hence if $e_l \in E_i$, $AIe_l = Ae_l$ and as A is an I-form, $AI = A$. Finally, let B be the I-form determined by π^{-1}. It follows that $ABa_i = BAa_i$ for each $a_i \in \mathscr{A}$. Thus, as the product of two I-forms is an I-form, $AB = BA = I$. Hence $\mathscr{F}$ is a group with I as identity which implies that $\mathscr{F} \subseteq \mathscr{G}$.

As examples of the utility of this characterization we provide the following.

Examples. Let

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = (a_1 a_2 a_3)\,.$$

Note that $a_1 > a_2$ and $a_1 > a_3$. Thus

$$\mathscr{G} = \left\{ (a_1 a_2 a_3) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},\quad (a_1 a_3 a_2) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \right\}.$$

Let

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} = (a_1 a_2 a_2 a_3)\,.$$

Then $a_1 > a_3$ and $a_2 > a_3$. Thus

$$\mathscr{G} = \left\{ (a_1 a_2 a_2 a_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix},\quad (a_2 a_1 a_1 a_3) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \right\}.$$

As an immediate consequence of Theorem 1 we have the following isomorphism result.

Corollary 1. *$\mathscr{G}$ is isomorphic to* Auto $(\mathscr{A}, \leqq)$.

From this corollary we see that the Montague-Plemmons result is also a consequence of our characterization by showing the following.

Corollary 2. Auto $(\mathscr{A}, \leqq)$ *is isomorphic to* Auto (X, α) *for any partial order* α.

Proof. First note that since α is a partial order, $\mathscr{A} = \{a_1, a_2, \ldots, a_n\}$, i.e. each column of I is a member of the basis. Suppose $(i, j) \in \alpha$. Then by the idempotent theorem $a_j > a_i$. Further if $a_j > a_i$ then again by the idempotent theorem $(i, j) \in \alpha$. Thus $(\mathscr{A}, \leqq)$ is the transpose of (X, α) and hence Auto $(\mathscr{A}, \leqq)$ is isomorphic to Auto (X, α).

This corollary, together with the characterization theorem, then give the reader some indication as to why Auto (X, α) is isomorphic to $\mathscr{G}$ for α a partial order.

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