Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 26 (1976), No. 1, 71–83

Persistent URL: http://dml.cz/dmlcz/101374

Czechoslovak Mathematical Journal, 26 (101) 1976, Praha

GENERIC PROPERTIES OF PARAMETRIZED VECTORFIELDS II

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(Received February 26, 1974)

In [1] we have studied the generic properties of critical points of vectorfields, depending on a parameter. This paper is concerned with generic properties of closed orbits of vectorfields depending on a parameter.

Since this paper is a direct continuation of [1], we shall refer to [1] for definitions and results. We assume that A is a 1-dimensional C^{r+1} compact manifold and X is an n-dimensional C^{r+1} compact manifold ($r \geqq 0$). Denote by $G^r(A, X)$ the set of all parametrized C^r vectorfields on $A \times X$, endowed with the C^r topology defined in [1].

1. Δ-TRANSVERSAL CLOSED ORBITS

Let φ be the parametrized flow of a $\xi \in G^r(A, X)$. We shall use the following notation:

(1) For $a \in A$ the mapping $\varphi_a : X \times R \to X$ is given by $\varphi_a(x, t) = \varphi(a, x, t)$ for $(x, t) \in X \times R$.

(2) For $x \in X$ the mapping $\varphi_x : A \times R \to X$ is given by $\varphi_x(a, t) = \varphi(a, x, t)$ for $(a, t) \in A \times R$.

(3) For $t \in R$, $a \in A$ the mapping $\varphi_{(t,a)} : X \to X$ is given by $\varphi_{(t,a)}(x) = \varphi(a, x, t)$ for $x \in X$.

Let $\xi \in G^r(A, X)$, $a \in A$ and let γ be a closed orbit of the vectorfield ξ_a through x ($\xi_a(x) = \xi(a, x)$ for $x \in X$) of a prime period τ. Then γ is called a Δ-transversal closed orbit, if $\Phi(\xi) \mathrel{\overline{\pitchfork}}_{(a,x,\tau)} \Delta$, where $\Delta = \{(x, t, y) \in X \times R^+ \times X \mid x = y\}$, $R^+ = (0, +\infty)$, $\Phi : G^r(A, X) \to C^r(A \times X \times R^+, X \times R^+ \times X)$ is given by $\Phi(\xi) = \Phi_\xi$ for $\xi \in G^r(A, X)$, $\Phi_\xi(a, x, t) = (x, t, \varphi^\xi(a, x, t))$ for $(a, x, t) \in A \times X \times R^+$, φ^ξ is the parametrized flow of ξ.

Denote by $G^r_\Delta(A, X)$ the set of all $\xi \in G^r(A, X)$ such that if $a \in A$, then all closed orbits of the vectorfield ξ_a are Δ-transversal.

Choose a metric $d_{T(X)}$, d_X on $T(X)$, X respectively. Let L be a positive number.

Denote by $G_L^r(A, X)$ the set of all $\xi \in G^r(A, X)$ such that for arbitrary $(a, x_1), (a, x_2) \in A \times X$, $d_{T(X)}(\xi(a, x_1), \xi(a, x_2)) < L_1 d_X(x_1, x_2)$ where $L_1 < L$. Obviously, the set $G_L^r(A, X)$ is open in $G^r(A, X)$.

Lemma 1. *If $\xi \in G_L^r(A, X)$, $a \in A$, then every closed orbit of the vectorfield ξ_a has a prime period $\geqq 4/L$.*

This lemma follows from [9, Theorem 4].

Lemma 2. *Let $\xi \in G^r(A, X)$, $a \in A$, let γ be a closed orbit of the vectorfield ξ_a of a prime period τ, $x \in \gamma$, $\dot{x} \in T_x X$ and let φ be the parametrized flow of ξ. Then there is a parametrized vectorfield $\eta \in G^r(A, X)$ such that $(\mathrm{d}/\mathrm{d}s)\{\varphi_\tau^s(a, x)\}_{s=0} = \dot{x}$ (φ^s is the parametrized flow of $\xi^s = \xi + s\eta$, $s \in R$).*

Proof. Let the mapping $\psi : X \times R \to X$ be given by $\psi(x, t) = \varphi(a, x, t)$ for $(x, t) \in A \times R$. By [4, Theorem 31.7] there is a $\tilde{\xi} \in \Gamma^r(\tau_X)$ such that $(\mathrm{d}/\mathrm{d}s)\{\psi_\tau^s(x)\}_{s=0} = \dot{x}$, where ψ^s, $s \in R$ is the flow of $\tilde{\xi}^s = \xi_a + s\tilde{\xi}$. It suffices to choose $\eta \in G^r(A, X)$ such that $\eta(a, x) = \tilde{\xi}(x)$.

Lemma 3. *Assume $\xi \in G^r(A, X)$ and $(a, x, \tau) \in A \times X \times R^+$ such that there is a closed orbit of the vectorfield ξ_a through x of a prime period τ. Then $ev_\Phi \,\bar{\pitchfork}_{(a,x,\tau)} \Delta$.*

Proof. $ev_\Phi : G^r(A, X) \times A \times X \times R^+ \to X \times R^+ \times X$, $ev_\Phi(\xi, a, x, t) = \Phi_\xi(a, x, t)$ for $\xi \in G^r(A, X)$, $(a, x, t) \in A \times X \times R^+$. Since $G^r(A, X)$ is a Banach space, we can identify $T_\xi G^r(A, X)$ and $G^r(A, X)$. By virtue of Lemma 2 it is easy to show that the condition of transversality is satisfied.

Let $\{L_i\}_{i=1}^\infty$ be an increasing sequence of positive numbers such that $\lim_{i\to\infty} L_i = +\infty$. Denote $b_i = 4/L_i$. If $\xi \in G_{L_i}^r(A, X)$, $a \in A$, then by Lemma 1 all closed orbits of the vectorfield ξ_a have prime periods $\geqq b_i$. Let $p : A \times X \times R^+ \to A \times X$ be the projection and $Z \subset A \times X \times R^+$. Denote $B(Z, \sigma) = \{(a, x, t) \in A \times X \times R^+ \mid d(Z, (a, x, t)) < \sigma\}$, where $\sigma > 0$ and d is a metric on $A \times X \times R^+$. Denote $B_p(Z, \sigma) = p[B(Z, \sigma)]$ and $N(Z, \sigma) = A \times X - \overline{B_p(Z, \sigma)}$. For $\xi \in G^r(A, X)$, denote $Y_0(\xi) = \{(a, x) \in A \times X \mid \xi(a, x) = 0_x\}$, where 0_x is the zero in $T_x X$, the set of critical points. Let q be a natural number and let $\{\varepsilon_i\}_{i=1}^\infty$ be a sequence of positive numbers such that $\delta_i = \varepsilon_i q^{-1} < \frac{1}{2} b_i$. For $\xi \in G^r(A, X)$, ϱ positive number, define the following mappings: $\Phi_{k,\varrho}(\xi) : N(\bigcup_{s=0}^k Y_s(\xi), \varrho) \times R^+ \to X \times R^+ \times X$, $\Phi_{k,\varrho}(\xi) = \Phi(\xi)/N(\bigcup_{s=0}^k Y_s(\xi), \varrho) \times R^+$, where $Y_j(\xi) = \{[\Phi_{j-1,q^{-1}}(\xi)^{-1}(\Delta)\} \cap [A \times X \times (0, (j + 1) b_i)$, $j = 1, 2, \ldots, k$. Now, define the following sets: $G_{ijq}^r = \{\xi \in G_{L_i}^r(A, X) \mid \Phi_{jq^{-1}}(\xi) \,\bar{\pitchfork}\, \Delta$ on the set $N(\bigcup_{k=0}^j Y_k(\xi), 2q^{-1}) \times [jb_i - \delta_i, (j + 1) b_i - \delta_i]\}$, where $i, j, q = 1, 2, \ldots$

Lemma 4. *The set* G^r_{ijq} $(i, j, q = 1, 2, \ldots)$ *is open and dense in* $G^r_{L_i}(A, X)$.

Proof. Density. Let $\xi_0 \in G^r_{L_i}(A, X)$. From [5, Theorem 3] it follows that there is a $\delta > 0$ and an open neighborhood $N_{ijq}(\xi_0)$ of ξ_0 in $G^r_{L_i}(A, X)$ such that for $\xi \in N_{ijq}(\xi_0)$, $N(\bigcup_{k=0}^{j} Y_k(\xi), q^{-1}) \subset N(\bigcup_{k=0}^{j} Y_k(\xi_0), q^{-1} - \delta)$. Define the mapping $\hat{\Phi}$: $: N_{ijq}(\xi_0) \to C^r(N(\bigcup_{k=0}^{j} Y_k(\xi_0), q^{-1} - \delta) \times (0, (j + 1) b_i), X \times R^+ \times X)$, $\hat{\Phi}(\xi) = \hat{\Phi}_\xi$, where $\hat{\Phi}_\xi = \Phi(\xi)/N(\bigcup_{k=0}^{j} Y_k(\xi_0), q^{-1} - \delta) \times (0, (j + 1) b_i)$. By Lemma 3 $ev_{\hat{\Phi}} \mathrel{\bar{\cap}} \Delta$. Denote $M_{ijq} = \{\xi \in N_{ijq}(\xi_0) \mid \hat{\Phi}(\xi) \mathrel{\bar{\cap}} \Delta\}$. From [4, Theorem 19.1] it follows that the set M_{ijq} is dense in $N_{ijq}(\xi_0)$. Therefore, there is a $\hat{\xi} \in N_{ijq}(\xi_0)$ close enough to ξ_0 such that $\hat{\Phi}(\hat{\xi}) \mathrel{\bar{\cap}} \Delta$. Since $\hat{\Phi}(\hat{\xi})/N(\bigcup_{k=0}^{j} Y_k(\hat{\xi}), 2q^{-1}) \times [jb_i - \delta_i, (j + 1) b_i - - \delta_i] = \Phi_{jq-1}(\hat{\xi})/N(\bigcup_{k=0}^{j} Y_k(\hat{\xi}), 2q^{-1}) \times [jb_i - \delta_i, (j + 1) b_i - \delta_i]$, so $\hat{\xi} \in G^r_{ijq}$ and the density is proved.

From [4, Theorem 18.2] it follows that the set $\hat{M}_{ijq} = \{\xi \in N_{ijq}(\xi_0) \mid \hat{\Phi}(\xi) \mathrel{\bar{\cap}} \Delta$ on the set $N(\bigcup_{k=0}^{j} Y_k(\hat{\xi}), 2q^{-1}) \times [jb_i - \delta_i, (j + 1) b_i - \delta_i]\}$ is open in $N_{ijq}(\xi_0)$ and therefore in $G^r_{L_i}(A, X)$, too. Since the set $G^r_{L_i}$ is open in $G^r(A, X)$, the set $\hat{M}_{ijq}$ is open in $G^r(A, X)$.

Proposition 1. *The set* $G^r_\Delta(A, X)$ $(r \geqq 1)$ *is residual in* $G^r(A, X)$.

Proof. Define the sets $H_{ikq} = \bigcap_{j=1}^{k} G^r_{ijq}$, $K_{kq} = \bigcup_{i=1}^{\infty} H_{ikq}$. The set K_{kq} is open in $G^r(A, X)$. Since $G^r(A, X) = \bigcup_{i=1}^{\infty} G^r_{L_i}(A, X)$, so $\bar{K}_{kq} \supset \bigcup_{i=1}^{\infty} \bar{H}_{ikq} = \bigcup_{i=1}^{\infty} G^r_{L_i}(A, X) = = G^r(A, X)$, i.e. the set K_{kq} is dense in $G^r(A, X)$. Therefore the set $G^r_\Delta(A, X) = = \bigcap_{k,q=1}^{\infty} K_{kq}$ is residual in $G^r(A, X)$.

2. POINCARÉ MAPPING

Let $\xi \in G^r(A, X)$, $a_0 \in A$, $x_0 \in X$ and let γ be a closed orbit of ξ_{a_0} through x_0 of a prime period τ_0. Let $(U \times V, \alpha \times \beta)$ be a chart on $A \times X$ at (a_0, x_0) such that if $\xi_{\alpha \times \beta}$ is the local representation of ξ with respect to this chart, then $\xi_{\alpha \times \beta}(0, 0) = = (1, 0)$, where $\alpha(a_0) = 0$, $\beta(x_0) = 0$. The existence of such a chart follows from [4, Theorem 21.6].

Let $\Sigma \subset X$ be an $(n - 1)$-dimensional submanifold of X such that $\beta(V \cap \Sigma) = = \{(y_1, y_2, \ldots, y_n) \in \beta(V) \mid y_1 = 0\}$. Then $p_1 \circ \beta \circ \varphi[(\alpha \times \beta)^{-1}(0, 0), \tau_0] = 0$, where $p_1 : R \times R^{n-1} \to R$ is the projection. The implicit function theorem implies that there is an open neighborhood $W = V_1 \times V_2$ of (a_0, x_0) in $A \times X$ and a C^r function $\tau : V_1 \times V_2 \to R$ such that $p_1 \circ \beta \circ \varphi^\xi(a, x, \tau(a, x)) = 0$ for all $x \in V_1 \times V_2$ and $\tau(a_0, x_0) = \tau_0$. Define the mapping $L : V_1 \times V_2 \to \Sigma$, $L(a, x) = \varphi^\xi(a, x, \tau(a, x))$ for $(a, x) \in V_1 \times V_2$. Let $H = L/V_1 \times (V_2 \cap \Sigma)$. We shall denote this mapping by $H[\xi, a_0, x_0, \gamma, V_1 \times (V_2 \cap \Sigma)]$, too. The mapping H is called the Poincaré mapping.

Now, define the mapping $\hat{H} : V_1 \times (V_2 \cap \Sigma) \to \Sigma \times \Sigma$ by $\hat{H}(a, x) = (x, H(a, x))$ for $(a, x) \in V_1 \times (V_2 \cap \Sigma)$. Obviously, $\Delta(\Sigma) = \{(x, y) \in \Sigma \times \Sigma \mid x = y\}$ is a closed submanifold of $\Sigma \times \Sigma$ of dimension $n - 1$.

Lemma 5. *If* $\xi \in G^r_{ijq}$, $(a_0, x_0, \tau_0) \in N(\bigcup_{k=0}^{j} Y_k(\xi), 2q^{-1}) \times [jb_i - \delta_i, (j + 1) b_i - - \delta_i]$, *then* $\hat{H} \mathrel{\bar{\pitchfork}}_{(a_0, x_0)} \Delta(\Sigma)$.

Proof. Since $\xi \in G^r_{ijq}$, so $\Phi_{jq^{-1}}(\xi) \mathrel{\bar{\pitchfork}} \Delta$. Let $\hat{H}(a_0, x_0) \in \Delta(\Sigma)$ and let $(U \times V, \alpha \times \beta)$ be a chart on $A \times X$ at (a_0, x_0), $\alpha(a_0) = 0$, $\beta(x_0) = 0$ such that if $\xi_{\alpha \times \beta}$ is the local representation of ξ, then $\xi_{\alpha \times \beta}(0, 0) = (1, 0)$. Using the condition for the transversality of the mapping $\Phi_{jq^{-1}}(\xi)$ in this coordinates, it is easy to prove the assertion of Lemma 5.

Corollary. *Let* $\xi \in G^r_{ijq}$, $(a_0, x_0, \tau_0) \in N(\bigcup_{k=0}^{j} Y_k(\xi), 2q^{-1}) \times [jb_i - \delta_i, (j + 1) b_i - - \delta_i]$ *and let there exist a closed orbit of* ξ_{a_0} *through* x_0 *of a prime period* τ_0. *Then* $\hat{H}^{-1}(\Delta(\Sigma))$ *is a closed 1-dimensional submanifold of* $V_1 \times (V_2 \cap \Sigma)$ *for* $V_1 \times V_2$ *sufficiently small neighborhood of* (a_0, x_0).

3. CONSTRUCTION OF A VECTORFIELD TO A GIVEN PERTURBATION OF POINCARÉ MAPPING

Lemma 6. *Let* $\xi \in G^r(A, X)$, $(a_0, x_0, \tau_0) \in A \times X \times R$ *and let* γ *be a closed orbit of the vectorfield* ξ_{a_0} *of a prime period* τ_0. *Let* $V_1 \times V_2$ *be an open neighborhood of* (a_0, x_0) *in* $A \times X$ *such that the Poincaré mapping* $H = H[\xi, a_0, x_0, \gamma, V_1 \times \times (V_2 \cap \Sigma)]$ $(H(a, x) = \varphi(a, x, \tau(a, x))$ *for* $(a, x) \in V_1 \times (V_2 \cap \Sigma)$, *where* φ *is the parametrized flow of* $\xi)$ *is defined. Let* W_1 *be an open neighborhood of* a_0 *in* A *such that* $\overline{W}_1 \subset V_1$ *and let* W_2 *be an open neighborhood of* x_0 *in* X *such that* $\overline{W}_2 \subset V_2$. *Let* $H_1 = H/\overline{W}_1 \times (\overline{W}_2 \cap \Sigma)$. *Then there is an open neighborhood* $U(H_1)$ *of the mapping* H_1 *in* $C^r(\overline{W}_1 \times (\overline{W}_2 \cap \Sigma), \Sigma)$ *such that for every* $\tilde{H}_1 \in U(H_1)$ *there is a* $\tilde{\xi} \in G^r(A, X)$ *such that* $\tilde{\varphi}(a, x, \tau(a, x)) = \tilde{H}_1(a, x)$ *for all* $(a, x) \in \overline{W}_1 \times (\overline{W}_2 \cap \Sigma)$, *where* $\tilde{\varphi}$ *is the parametrized flow of* $\tilde{\xi}$. *Moreover,* $\tilde{\xi}$ *depends continuously on* $\tilde{H}_1$.

Proof. Let $\varepsilon_1, \varepsilon_2$ be real numbers. Define the following sets:

$$T_1 = T_1(\varepsilon_1, \varepsilon_2) = \{(a, y) \in A \times X \mid y = \varphi(a, x, t),\ (a, x) \in V_1 \times (V_2 \cap \Sigma),$$
$$\varepsilon_1 < t < \tau(a, x) + \varepsilon_2\},$$

$$T_2 = T_2(\varepsilon_1, \varepsilon_2) = \{(\mu, t, z) \mid \beta^{-1}(0, z) \in V_2 \cap \Sigma, \alpha^{-1}(\mu) \in V_1, \varepsilon_1 < t < \tau(a, x) + \varepsilon_2\},$$

where $(U \times V, \alpha \times \beta)$ is a chart as in the definition of H. Let $\tau_1 : \alpha(V_1) \times p \circ \circ \beta(V_2 \cap \Sigma) \to R$ be defined by $\tau_1(\mu, z) = \tau(\alpha^{-1}(\mu), \beta^{-1}(0, z))$, where $p : R^1 \times \times R^{n-1} \to R^{n-1}$ is the projection. Now, define the mapping $\Phi_{\varepsilon_1,\varepsilon_2} : T_2(\varepsilon_1, \varepsilon_2) \to \to T_1(\varepsilon_1, \varepsilon_2)$, $\Phi_{\varepsilon_1,\varepsilon_2}(\mu, t, z) = (\alpha^{-1}(\mu), \varphi(\alpha^{-1}(\mu), \beta^{-1}(0, z), t)$ for $(\mu, t, z) \in T_2(\varepsilon_1, \varepsilon_2)$. If $\varkappa_1 \geqq 0$, $\varkappa_2 \leqq 0$ are chosen small enough, then $(T_2(\varkappa_1, \varkappa_2), id_{R^{n+1}})$ is a chart on R^{n+1} and $(T_1(\varkappa_1, \varkappa_2), \Phi^{-1}_{\varkappa_1,\varkappa_2})$ is a chart on $A \times X$. The local representation $\hat{f}$ of ξ with respect to the chart $(T_1(\varkappa_1, \varkappa_2), \Phi^{-1}_{\varkappa_1,\varkappa_2})$ has the form $\hat{f}(\mu, t, z) = (1, 0)$ for $(\mu, t, z) \in \Phi_{\varkappa_1,\varkappa_2}(T_1(\varkappa_1, \varkappa_2))$. Denote $I_1 = \alpha(\overline{W}_1)$, $I_2 = \{t \mid 0 \leqq t \leqq \tau_1(\mu, z),\ \mu \in I_1,\ \beta^{-1}(0, z) \in \overline{W}_2 \cap \Sigma\}$, $I_4 = \beta(\overline{W}_2 \cap \Sigma)$, $I_3 = \{z \mid (0, z) \in I_4\}$. Let $r_0 = \min \tau_1(\mu, z)$ on $I_1 \times I_3$ and let $\Psi : R^1 \times R^1 \times R^{n-1} \to R^1$ be a C^r function such that $\Psi = 0$ outside $R_1 = I_{11} \times \{t \mid \frac{1}{4}r_0 < t < \frac{3}{4}r_0\} \times I_{31}$, where I_{11} is an open interval in R^1 such that $\bar{I}_{11} \subset I_1$, I_{31} is an open set in R^{n-1} such that $\bar{I}_{31} \subset I_3$, $\Psi = 1$ on the set $R_0 = I_{10} \times \{t \mid \frac{1}{3}r_0 < t < \frac{2}{3}r_0\} \times I_{30}$, where I_{10} is an open interval in R^1 such that $\bar{I}_{10} \subset I_{11}$, I_{30} is an open set in R^{n-1} such that $\bar{I}_{30} \subset I_3$ and $\int_0^{\tau_1(\mu,z)} \Psi(\mu, s, z)\, ds = 1$ for $(\mu, z) \in I_1 \times I_3$. Denote $B = \{g \in C^r(I_1 \times I_2 \times I_3, R^{n-1}) \mid g(\mu, t, z) = \Psi(\mu, t, z)\, h(\mu, z),\ h \in C^r(I_1 \times I_3, R^{n-1})\}$. B is a closed, linear subspace of $C^r(I_1 \times I_2 \times I_3, R^{n-1})$ and hence it is a Banach space.

Let $\varphi_{t,g}(\mu, z) = z + \int_0^t g(\mu, s, \varphi_{s,g}(\mu, z))\, ds$ for $(\mu, t, z) \in I_1 \times I_2 \times I_3$, $g \in B$ ($\varphi_{(\cdot,g)}$ is the flow of g). Define the mapping $\mathscr{F} : B \to C^r(I_1 \times I_3, R^{n-1})$, $\mathscr{F}(g)(\mu, z) = \varphi_{\tau_1(\mu,z),g}(\mu, z)$ for $g \in B$. Let $id \in C^r(I_1 \times I_3, R^{n-1})$ be defined by $id(\mu, z) = z$ for all $(\mu, z) \in I_1 \times I_3$, while $\Pi \in C^r(I_1 \times I_2 \times I_3, R^{n-1})$ is defined by $\Pi(\mu, t, z) = 0$ for all $(\mu, t, z) \in I_1 \times I_2 \times I_3$. Obviously $\mathscr{F}(\Pi) = id$.

Let

$$d\mathscr{F}(g, h) = \lim_{s \to 0} \frac{\mathscr{F}(g + sh) - \mathscr{F}(g)}{s}$$

be the Gateaux differential and let $D\mathscr{F}(g, h)$ be the Frechet differential of $\mathscr{F}$.

Sublemma. *If $g, h \in C^r(I_1 \times I_2 \times I_3, R^{n-1})$, then*

(1) *$d\mathscr{F}(g, h)$ exists.*

(2) *The mapping*

$$d\mathscr{F} : C^r(I_1 \times I_2 \times I_3, R^{n-1}) \times C^r(I_1 \times I_2 \times I_3, R^{n-1}) \to C^r(I_1 \times I_3, R^{n-1})$$

is uniformly continuous in g and continuous in h on the set $K(\sigma) = \{w \in B \mid \|w\| < \sigma\}$, $(\sigma > 0)$ with respect to the C^r metric on $C^r(I_1 \times I_2 \times I_3, R^{n-1})$.

Proof. Denote $Q(t, s, \mu, z, g, h) = \varphi_{t,g+sh}(\mu, z) - \varphi_{t,g}(\mu, z)$. $(d/dt)\, Q(t, s, \mu, z, g, h) = g(\mu, t, \varphi_{t,g+sh}(\mu, z) - g(\mu, t, \varphi_{t,g}(\mu, z) + sh(\mu, t, \varphi_{t,g+sh}(\mu, z))$. Let

$$K_1 = \sup_{I_1 \times I_2 \times I_3} \left\| \frac{\partial g}{\partial z}(\mu, t, z) \right\|, \quad K_2 = \sup_{I_1 \times I_2 \times I_3} \left\| \int_0^t h(\mu, v, \varphi_{v,g+sh}(v, z))\, dv \right\|,$$

$$K_3 = \sup_{I_1 \times I_3} \tau_1(\mu, z).$$

Then by Gronwall's lemma

$$(*) \qquad \|Q(t, s, \mu, z, g, h)\| \leqq sK \quad \text{for} \quad (\mu, t, z) \in I_1 \times I_2 \times I_3 ,$$

where $K = K_2 \exp(K_1 K_3)$. Therefore $Q \to 0$ if $s \to 0$ uniformly with respect to $(\mu, t, z) \in I_1 \times I_2 \times I_3$. Using [7, Theorem 8.6.2] we have

$$\frac{\mathrm{d}}{\mathrm{d}t} Q(t, s, \mu, z, g, h) = \left[\frac{\partial}{\partial z} g(\mu, t, \varphi_{t,g}(\mu, z)) + \omega\right] Q(t, s, \mu, z, g, h) + \\ + sh(\mu, t, \varphi_{t,g+sh}(\mu, z)) ,$$

where $\omega = \omega(Q)$ is a matrix function such that if $\varepsilon > 0$, then there is a $\delta > 0$ such that $\|\omega(Q)\| < \varepsilon$ for $\|Q\| < \delta$ and $(\mu, t, z) \in I_1 \times I_2 \times I_3$.

Denote $X(t, s, \mu, z, g, h) = Q(t, s, \mu, z, g, h)/s$. Then

$$(**) \qquad \frac{\mathrm{d}}{\mathrm{d}t} X(t, s, \mu, z, g, h) = \frac{\partial}{\partial z} g(\mu, t, \varphi_{t,g}(\mu, z)) X(t, s, \mu, z, g, h) + \\ + \gamma + h(\mu, t, \varphi_{t,g+sh}(\mu, z) ,$$

where $\gamma = (\omega/s) Q$.

Using $(*)$ we have $\gamma \leqq K\|\omega\|$ and so $\gamma \to 0$ if $s \to 0$ uniformly. Denote by $Q_0(t, \mu, z, g, h)$ the solution of the equation

$$(***) \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial}{\partial z} g(\mu, t, \varphi_{t,g}(\mu, z)) \, y + h(\mu, t, \varphi_{t,g}(\mu, z))$$

for which the condition $Q_0(0, \mu, z, g, h) = 0$ is satisfied. Since $\gamma \to 0$ if $s \to 0$ uniformly and the equalities $(**)$, $(***)$ are satisfied, so $\lim_{s \to 0} [Q(t, s, \mu, z, g, h) - Q_0(t, s, \mu, z, g, h)] = 0$ uniformly in the C^0 metric. The convergence in the C^r metric can be proved similarly. Since $\mathrm{d}\mathscr{F}(g, h)(\mu, z) = Q_0(\tau_1(\mu, z), \mu, z, g, h)$, so $\mathrm{d}\mathscr{F}(g, h)$ exists. Since $Q_0(t, s, \mu, z, g, h)$ is a solution of the differential equation $(***)$, the form of this equation implies the assertion (2) of Sublemma.

By [8, VIII., Theorem 2] and by Sublemma $D\mathscr{F}(g, h)$ exists and $\mathrm{D}\mathscr{F}(g, h) = \mathrm{d}\mathscr{F}(g, h)$ for $g, h \in K(\sigma)$. $\mathrm{D}\mathscr{F}(g, h) = \mathscr{F}'(g) \, h$, where $\mathscr{F}'(g) \in L(B, C^r(I_1 \times I_3, R^{n-1}))$. The mapping $g \to \mathscr{F}'(g)$ is continuous and bounded in a neighborhood of $\Pi \in B$. Let $h_0 \in B$. Then there is an $h_1 \in C^r(I_1 \times I_3, R^{n-1})$ such that $h_0(\mu, t, z) = \Psi(\mu, t, z) \, h_1(\mu, z)$ for $(\mu, t, z) \in I_1 \times I_2 \times I_3$. $[\mathscr{F}'(\Pi)(h_0)](\mu, z) = \lim_{s \to 0} (1/s) [\mathscr{F}(\Pi + sh_0)(\mu, z) - \mathscr{F}(\Pi)(\mu, z)] = \int_0^{\tau_1(\mu, z)} \Psi(\mu, \sigma, z) \, h_1(\mu, z) \, \mathrm{d}\sigma = h_1(\mu, z)$ and so $\mathscr{F}'(\Pi)$ is a linear isomorphism of B onto $C^r(I_1 \times I_3, R^{n-1})$. $\mathscr{F}(\Pi) = id$. The conditions of [8, Theorem 10.2.5] are satisfied. By this theorem there is an open neighborhood N of the mapping id in $C^r(I_1 \times I_3, R^{n-1})$ and an open neighborhood N of the mapping Π in $C^r(I_1 \times I_2 \times I_3, R^{n-1})$ such that $\mathscr{F}/N$ is a diffeomorphism of N onto M. $U_\varkappa = \{(a, \varphi(a, x, t)| - \varkappa < t < \varkappa, (a, x) \in V_1 \times$

$\times (V_2 \cap \Sigma)\}$, $\Psi_{1\varkappa} : U_\varkappa \to R^{n+1}$, $\psi_{1\varkappa}(a, \varphi(a, x, t)) = (\alpha(a), t, z)$, where $\beta^{-1}(0, z) = x$, $V_\varkappa = \{\varphi(a_0, x, t) \mid \tau(a_0, x) - \varkappa < t < \tau(a_0, x) + \varkappa, (a_0, x) \in V_1 \times (V_2 \cap \Sigma)\}$, $\Psi_{2\varkappa} : : V_\varkappa \to R^n$, $\Psi_{2\varkappa}(\varphi(a_0, x, t)) = (t, z)$, $\beta^{-1}(0, z) = x$, $\varkappa > 0$. If $\varkappa$ is chosen small enough, then $(U_\varkappa, \Psi_{1\varkappa})$ is a chart on $A \times X$ at (a_0, x_0) and $(V_\varkappa, \Psi_{2\varkappa})$ is a chart on X at x_0. Let $h_1 : I_1 \times I_3 \to R^{n-1}$ be the local representation of H_1 with respect to $(U_\varkappa, \Psi_{1\varkappa})$, $(V_\varkappa, \Psi_{2\varkappa})$. Then $h_1 = id$. Let $U(H_1) = \{F \in C^r(\overline{W}_1 \times (\overline{W}_2 \cap \Sigma), \Sigma) \mid \mid \hat{F} \in M\}$, where $\hat{F}$ is the local representation of $\tilde{H}_1$ with respect to $(U_\varkappa, \Psi_{1\varkappa})$, $(V_\varkappa, \Psi_{2\varkappa})$. Then $\tilde{h}_1 \in M$ and $g_1 = \mathscr{F}^{-1}(\tilde{h}_1)$ is such that $\varphi_{\tau_1(\mu,z),g_1}(\mu, z) = z + \int_0^{\tau_1(\mu,z)} g_1(\mu, v, \varphi_{\gamma,g_1}(\mu, z))\,dv = h_1(\mu, z)$ for $(\mu, z) \in I_1 \times I_3$, where $g_1(\mu, t, z) = = \Psi(\mu, t, z)\, h_1(\mu, z)$. Since $\Psi \equiv 0$ outside R_1 (R_1 is defined on the p. 75), so $g_1 \equiv 0$ outside R_1. Let $g \in C^r(I_1 \times I_2 \times I_3, R^n)$ be defined by $g(\mu, t, z) = (1, g_1(\mu, t, z))$ for $(\mu, t, z) \in I_1 \times I_2 \times I_3$. We can define a parametrized vectorfield $\tilde{\xi}$ such that g is the local representation of $\tilde{\xi}$ with respect to the chart $(T_1(\varkappa_1, \varkappa_2), \Phi^{-1}_{\varkappa_1,\varkappa_2})$ and $\tilde{\xi} = \xi$ outside $T_1(\varkappa_1, \varkappa_2)$. From the properties of g it follows that $\tilde{\xi} \in G^r(A, X)$. The construction of $\tilde{\xi}$ yields: (1) $\tilde{\varphi}(a, x, \tau(a, x)) = \tilde{H}_1(a, x)$ for $(a, x) \in \overline{W}_1 \times (\overline{W}_2 \cap \Sigma)$, where $\tilde{\varphi}$ is the parametrized flow of $\tilde{\xi}$. (2) For every neighborhood $V(\xi)$ of ξ, there is a neighborhood $\tilde{U}(H_1) \subset U(H_1)$ of the mapping H_1 in $C^r(\overline{W}_1 \times (\overline{W}_2 \cap \Sigma), \Sigma)$ such that if $\tilde{H}_1 \in \tilde{U}(H_1)$, then there is a $\tilde{\xi} \in U(\xi)$ such that $\tilde{\varphi}(a, x, \tau(a, x)) = \tilde{H}_1(a, x)$ for $(a, x) \in \overline{W}_1 \times (\overline{W}_2 \cap \Sigma)$ and $\tilde{\xi}$ depends continuously on $\tilde{H}_1$.

Remark. Let $H : V_1 \times (V_2 \cap \Sigma) \to \Sigma$ be the Poincaré mapping and let $\hat{H} : V_1 \times (V_2 \cap \Sigma) \to \Sigma \times \Sigma$ be the mapping given by $\hat{H}(a, x) = (x, H(a, x))$. Let $\Delta(\Sigma)$ be the diagonal in $\Sigma \times \Sigma$. Denote $Z = \hat{H}^{-1}(\Delta(\Sigma))$, $W(Z, \xi) = \{(\mu, t, z) \mid (\alpha^{-1}(\mu), \beta^{-1}(0, z)) \in Z, 0 \leqq t \leqq \tau_1(\mu, z)$. We can choose the function Ψ from the proof of Lemma 5 such that $\Psi = 0$ on $W(Z, \xi)$. Then for every $a \in A$, the vectorfield $\tilde{\xi}_a$ has the same closed orbits as the vectorfield ξ_a.

Let $\xi \in G^r(A, X)$ and let γ be a closed orbit of the vectorfield ξ_{a_0} through x_0 of a prime period τ_0. Let $H = H[\xi, a_0, x_0, \gamma, V_1 \times (V_2 \cap \Sigma)]$ be the Poincaré mapping. For $a \in V_1$, define the mapping $H_a : V_2 \cap \Sigma \to \Sigma$, $H_a(x) = H(a, x)$ for $x \in V_2 \cap \Sigma$. Denote by $G^r_S(A, X)$ the set of all $\xi \in G^r_\Delta(A, X)$ such that the mapping $T_{x_0}H_{a_0} : : T_{x_0}(V_2 \cap \Sigma) \to T_{x_0}\Sigma$ has the following properties:

(1) It has no eigenvalue on $S = \{\lambda \in C \mid |\lambda| = 1\}$ of multiplicity $\geqq 2$.
(2) All eigenvalues of this mapping meet S transversally at (a_0, x_0).
(3) If a complex eigenvalue of this mapping lies on S, then there is no other eigenvalue on S except of its complex conjugate.
(4) It has no complex eigenvalue λ such that $\lambda^m = 1$ for a natural number $m > 1$.

Remark. The condition (2) means the following: If λ_0 is an eigenvalue of $T_{x_0}H_{a_0}$, $\lambda_0 \in S$, then there is an open neighborhood of (a_0, x_0) in Z ($Z = \hat{H}^{-1}(\Delta(\Sigma))$) and a unique C^r mapping $\hat{\lambda} : N \to R^2$ such that $\hat{\lambda} = (\lambda_1, \lambda_2)$, $\lambda(a, x) = \lambda_1(a, x) + i\lambda_2(a, x)$ is an eigenvalue of the mapping T_xH_a for $(a, x) \in N$, $\lambda(a_0, x_0) = \lambda_0$ and $\hat{\lambda} \pitchfork \{(\mu_1, \mu_2) \in R^2 \mid \mu_1^2 + \mu_2^2 = 1\}$.

Denote by $G^r_{ijqm}(S)$ the set of all $\xi \in G^r_{ijq}$ such that if for $(a_0, x_0) \in \overline{N(\bigcup_{k=0}^{j} Y_k(\xi), 2q^{-1})}$ there is a closed orbit of the vectorfield ξ_{a_0} through x_0 of a prime period $\tau_0 \in [jb_i - \delta_i, (j+1)\, b_i - \delta_i]$, then the mapping $T_{x_0}H_{a_0}$ has the properties (1)–(3) from the definition of the set $G^r_S(A, X)$ and has no complex eigenvalue such that $\lambda^m = 1$ (m being a natural number).

Lemma 7. *The set $G^r_{ijqm}(A, X)$ is open and dense in G^r_{ijq}.*

Proof. Openness. Let $\xi_0 \in G^r_{ijqm}(S)$. From [5, Theorem 3] it follows that there is a $\delta_1 > 0$ and an open neighborhood $N_{ijq}(\xi_0)$ of ξ_0 in $G^r_{L_i}(A, X)$ such that for $\xi \in N_{ijq}(\xi_0)$, $N(\bigcup_{k=0}^{j} Y_k(\xi), 2q^{-1}) \subset N(\bigcup_{k=0}^{j} Y_k(\xi_0), q^{-1} - \delta)$ where $\delta > \delta_1$ and $N_{ijq}(\xi_0) \subset G^r_{ijq}$. Now, define the mapping $\Psi : N_{ijq}(\xi_0) \to C^{r-1}(N(\bigcup_{k=0}^{j} Y_k(\xi_0), q^{-1} - \delta) \times (0, (j+1)\, b_i) \times L(T(X), T(X))$ ($L(T(X), T(X))$ is defined in [4, §9]), $\Psi(\xi) = \Psi_\xi$, where $\Psi_\xi(a, x, t) = T_x\tilde{\varphi}^\xi_{(t,a)}$, $\tilde{\varphi}^\xi = \varphi^\xi \mid N(\bigcup_{k=0}^{j} Y_k(\xi_0), q^{-1} - \delta) \times (0, (j+1)\, b_i)$, $\tilde{\varphi}^\xi_{(t,a)} y = \tilde{\varphi}^\xi(a, y, t)$ for $y \in N(\bigcup_{k=0}^{j} Y_k(\xi_0), q^{-1} - \delta)$.

Let $\hat{W} \subset L(T(X), T(X))$ be the set of all $B \in L(T(X), T(X))$ such that

(1) $B \in L(T_xX, T_xX)$ for some $x \in X$;
(2) B has eigenvalues on S (different from 1) of multiplicity $\geqq 2$.

The set $\hat{W}$ is a closed subset of $L(T(X), T(X))$. By [4, Theorem 18.1] the set $K_{ijq} = \{\xi \in N_{ijq}(\xi_0) \mid \{\Psi(\xi)\,(\overline{N(\bigcup_{k=0}^{j} Y_k(\xi_0), q^{-1} - \delta)} \times [jb_i - \delta_i, (j+1)\, b_i - \delta_i]\} \cap \hat{W} = \emptyset\}$ is open in $N_{ijq}(\xi_0)$. Therefore, there exists an open neighborhood $\hat{N}_{ijq}(\xi_0)$ of ξ_0 in $G^r_{L_i}(A, X)$ such that for $\xi \in \hat{N}_{ijq}(\xi_0)$, $\{\Psi(\xi)\,[\overline{N(\bigcup_{k=0}^{j} Y_k(\xi), 2q^{-1})} \times [jb_i - \delta_i, (j+1)\, b_i - \delta_i]]\} \cap \hat{W} = \emptyset$ and this proves the openness of (1). The openness of (4) can be proved similarly. The openness of (2) follows from [4, Theorem 18.2] and the openness of (3) is clear.

Density. Let $\xi \in G^r_{ijq}$, $(a_0, x_0, \tau_0) \in \overline{N(\bigcup_{k=0}^{j} Y_k(\xi), 2q^{-1})} \times [jb_i - \delta_i, (j+1)\, b_i - \delta_i]$ and let γ be a closed orbit of the vectorfield ξ_{a_0} through x_0 of a prime period τ_0. Let $H = H[\xi, a_0, x_0, \gamma, V_1 \times (V_2 \cap \Sigma)]$ be the Poincaré mapping such that $Z = \hat{H}^{-1}(\Delta(\Sigma))$ is an open 1-dimensional submanifold of $V_1 \times (V_2 \cap \Sigma)$. Let $W_1 \times W_2$ be an open neighborhood of (a_0, x_0) such that $\overline{W}_1 \times \overline{W}_2 \subset V_1 \times V_2$. By [3, Theorem 2] there is an $F \in C^r(\overline{W}_1 \times (\overline{W}_2 \cap \Sigma), \Sigma)$ arbitrary close to $H/\overline{W}_1 \times \overline{W}_2$ such that for $(a, x) \in W_1 \times (W_2 \cap \Sigma)$ the mapping $T_xF_a\,(F_a(y) = F(a, y))$ for $y \in W_2 \cap \Sigma$) has the properties (1)–(4). By Lemma 6 there is a $\xi \in G^r(A, X)$ such that $H[\tilde{\xi}, a_0, x_0, \tilde{\gamma}, W_1 \times (W_2 \cap \Sigma)] = F/W_1 \times (W_2 \cap \Sigma)$, where $\tilde{\gamma}$ is a closed orbit of $\tilde{\xi}_{a_0}$ close to γ which can be constructed arbitrarily close to ξ if F is close enough to $H/\overline{W}_1 \times (\overline{W}_2 \cap \Sigma)$. Since the set $\overline{N(\bigcup_{k=0}^{j} Y_k(\xi), 2q^{-1})} \times [jb_i - \delta_i, (j+1)\, b_i - \delta_i]$ is compact, the proof of Lemma 7 is complete.

Proposition 2. *The set* $G_S^r(A, X)$ $(r \geqq 1)$ *is residual in* $G^r(A, X)$.

The proof of this proposition follows from Lemma 7 analogously as Proposition 1 from Lemma 4.

For $\xi \in G^r(A, X)$ denote by $P_1(\xi)$ the set of $(a, x) \in A \times X$ such that the vectorfield ξ_a has a closed orbit through x of a prime period τ and $\lambda = 1$ is the eigenvalue of the mapping $T_x\varphi_{(\tau,a)}$ of multiplicity 2. Let $P_2(\xi)$ be the set of $(a, x) \in A \times X$ such that $\lambda = -1$ is an eigenvalue of the mapping $T_x\varphi_{(\tau,a)}$.

Let $\xi \in G^r_{ijqm}(S)$, $(a_0, x_0) \in P_1(\xi)$. Then there is a chart $(U \times V, \alpha \times \beta)$ on $A \times X$ at (a_0, x_0) such that $\alpha(a_0) = 0$, $\beta(x_0) = 0$ and the local representation of the mapping $H = H[\xi, a_0, x_0, \gamma, V_1 \times (V_2 \cap \Sigma)]$ with respect to this chart has the form

$$y_2 = y_1 + \alpha_1\mu + \alpha_2 y_1^2 + \omega(\mu, y_1, z_1)\,, \quad z_2 = Bz_1 + X(\mu, y_1, z_1)\,,$$

where $\dim y_1 = 1$, $\dim z_1 = n - 2$, $\omega, X \in C^r$, $X(0, 0, 0) = 0$, $\omega(\mu, y_1, 0)$ contains only μ^2, μy_1 and terms of higher order than 2 and B is a matrix which has the following properties:

(i) B has no eigenvalue on S of multiplicity $\geqq 2$.
(ii) If a complex eigenvalue of B lies on S, then there is no other complex eigenvalue on S except of its complex conjugate and $\lambda = 1$.
(iii) B has no complex eigenvalue λ such that $\lambda^m = 1$ for a natural number $m \geqq 2$.

Let D^r_{ijqm} be the subset of $G^r_{ijqm}(S)$ such that for all $\xi \in D^r_{ijqm}$ the matrix B from the expression of the local representation of H has no complex eigenvalue on S and $\lambda = -1$ is not an eigenvalue of B. This set is open and dense in G^r_{ijqm}. The openness is obvious. To prove density we assume $\xi \in D^r_{ijqm}$. We change H into $\tilde{H}$ by changing the term Bz_1 in the local representation of H into $(B + \Psi(\mu, y_1, z_1)\,\delta E)\, z_1$, where E is the unit matrix, Ψ is a C^r bump function vanishing outside $(\alpha \times \beta)(U \times V)$ and equal to 1 at a neighborhood of $(0, 0, 0)$, $0 < \delta$ is a real number such that $B + \delta E$ has no complex eigenvalue on S and $\lambda = -1$ is not an eigenvalue of $B + \delta E$. By Lemma 6 there is a $\tilde{\xi}$ such that for every $a \in A$ the vectorfield $\tilde{\xi}_a$ has the same closed orbits as ξ_a, $H[\tilde{\xi}, a_0, x_0, \gamma, V_1 \times (V_2 \cap \Sigma)] = \tilde{H}$ and $\tilde{\xi}$ can be constructed arbitrarily close to ξ if δ is sufficiently small.

Denote by L^r_{ijqm} the set of all $\xi \in D^r_{ijqm}$ such that if $(a_0, x_0, \tau_0) \in \overline{N(\bigcup_{k=0}^{j} Y_k(\xi), 2q^{-1})} \times [jb_i - \delta_i, (j+1)\, b_i - \delta_i]$ and γ is a closed orbit of ξ_{a_0}, then there is a chart $(U \times V, \alpha \times \beta)$ as before such that $\alpha_2 \neq 0$.

Lemma 8. *The set* L^r_{ijqm} $(r \geqq 2)$ *is open and dense in* $G^r(A, X)$.

The proof of this lemma is analogous to the proof of Lemma 7.

Define the set $G_2^r(A, X) = \bigcap_{j,q,m=1}^{\infty} \bigcap_{i=1}^{\infty} L_{ijqm}^r$. For $\xi \in G^r(A,X)$, $H = H[\xi, a_0, x_0, \gamma, V_1 \times (V_2 \cap \Sigma)]$ define the sets $Z_k(H) = \{(a, x) \in V_1 \times (V_2 \cap \Sigma) \mid H_a^k(x) = x, H_a^j(x) \neq x$ for $0 < j < k\}$, $k = 1, 2, \ldots$, where $H_a^1(x) = H_a(x) = H(a, x)$, $H_a^k(x) = H_a(H_a^{k-1}(x))$.

Theorem 1. *There is a residual set $G_2^r(A, X)$ $(r \geqq 2)$ in $G^r(A, X)$ such that the following is true: If $\xi \in G_2^r(A, X)$, then*

(1) *the set $P_1(\xi)$ consists of isolated points.*

(2) *If $(a_0, x_0) \in A \times X$, γ is a closed orbit of the vectorfield ξ_{a_0} through x_0, then there is a chart $(V_1 \times V_2, h_1 \times h_2)$ on $A \times X$ at (a_0, x_0), $h_1(a_0) = 0$, $h_2(x_0) = 0$ such that*

(a) *the Poincaré mapping $H = H[\xi, a_0, x_0, \gamma, V_1 \times (V_2 \cap \Sigma)]$ is defined and $Z_1 = Z_1(H)$ is a 1-dimensional submanifold of $A \times X$.*

(b) *If $(a_0, x_0) \in P_1(\xi)$, then $(h_1 \times h_2)(Z_1(H)) = \{(\mu, y_1, y_2, \ldots, y_n) \mid \mu = \varphi_0(y_1)$, $y_i = \varphi_i(y_1)$, $i = 1, \ldots, n$, $y_1 \in J\}$, where J is an open interval, $0 \in J$, $\varphi_i \in C^r$, $i = 0, 1, \ldots, n$,*

$$\varphi_0(0) = 0\,, \quad \frac{\mathrm{d}}{\mathrm{d}y_1}\varphi_0(0) = 0\,, \quad \frac{\mathrm{d}^2}{\mathrm{d}y_1^2}\varphi_0(0) > 0\,.$$

(c) *If $\mu > 0$, then there are exactly two numbers $y_1 > 0$, $z_1 < 0$ such that $(a_1, x_1) = (h_1 \times h_2)^{-1}(\mu, y_1, 0) \in Z_1(H)$, $(a_1, x_2) = (h_1 \times h_2)^{-1}(\mu, z_1, 0) \in Z_1(H)$ and the following is true: If s is the number of eigenvalues of the mapping $T_{x_2}H_{a_1}$ with moduli >1, then the number of eigenvalues of the mapping $T_{x_2}H_{a_1}$ with moduli >1 is $s - 1$.*

(3) *If $(a, x) \in P_1(\xi)$, then the mapping T_xH_a has exactly one eigenvalue equal to 1.*

(4) *$V_1 \times (V_2 \cap \Sigma) - Z_1(H)$ contains no invariant set.*

Proof. It is possible to prove this theorem by virtue of Lemma 6 and using the results of P. Brunovský [3], who has proved a similar theorem for one-parameter families of diffeomorphisms.

Let $\xi \in G_{ijqm}^r(S)$, $(a_0, x_0) \in P_2(\xi)$. Then there is a chart $(U \times V, \alpha \times \beta)$ on $A \times X$ at (a_0, x_0) such that $\alpha(a_0) = 0$, $\beta(x_0) = 0$ and the local representation of the mapping $H = H[\xi, a_0, x_0, \gamma, V_1 \times (V_2 \cap \Sigma)]$ with respect to this chart has the form

$$y_2 = -y_1 + \alpha_1\mu y_1 + \alpha_2 y_1^2 + \gamma_1 y_1^3 + \omega(\mu, y_1, z_1)\,,$$

$$z_2 = Cz_1 + X(\mu, y_1, z_1)\,,$$

where $\dim y_1 = 1$, $\dim z_1 = n - 2$, $\omega, X \in C^r$, $X(0, 0, 0) = 0$, $\omega(\mu, y_1, 0)$ contains only μ^2, μy_1 and terms of higher order than 2 and C is a matrix which has the properties (i)–(iii) as the matrix B above (see the case $(a_0, x_0) \in P_1(\xi)$).

Denote by M_{ijqm}^r the set of all $\xi \in G_{ijqm}^r(S)$ such that the matrix C from the expression of the local representation of H has no complex eigenvalue on S and $\lambda = 1$ is not an eigenvalue of C. By the same argument as in the case $(a_0, x_0) \in Y_1(\xi)$ the set M^r_{ijqm} is open and dense in $G^r_{ijqm}(S)$. Denote by N^r_{ijqm} the set of all $\xi \in M^r_{ijqm}$ such that $\alpha_2^2 + \gamma_1 \neq 0$. This set is open and dense in $G^r(A, X)$. Therefore the set $G^r_3(A, X) = \bigcap_{j,q,m=1}^{\infty} \bigcup_{i=1}^{\infty} N^r_{ijqm}$ is residual.

Using [3, Theorem 4] and using our method of construction of vectorfields to the Poincaré mapping, it is possible to prove the following theorem.

Theorem 2. *There is a residual set $G^r_3(A, X)$ $(r \geqq 3)$ in $G^r(A, X)$ such that the following is true: For $\xi \in G^r_3(A, X)$,*

(1) *the set $P_2(\xi)$ consists of isolated points.*

(2) *If $(a_0, x_0) \in P_2(\xi)$ and $H = H[\xi, a_0, x_0, \gamma, V_1 \times (V_2 \cap \Sigma)]$ is the Poincaré mapping, then $\bar{Z}_2 = \overline{Z_2(H)}$ is a 1-dimensional C^{r-1} submanifold of $A \times X$.*

(3) *$V_1 \times (V_2 \cap \Sigma) - (Z_1 \cup Z_2)$ contains no invariant set.*

Let T be a positive real number and let $G^r(A, X, T)$ be the set of $\xi \in G^r(A, X)$ with the following properties: If γ is a closed orbit of the vectorfield ξ_a $(a \in A)$ through x of a prime period $\tau \leqq T$ and $H = H[\xi, a, x, \gamma, V_1 \times (V_2 \cap \Sigma)]$ is the Poincaré mapping, then

(1) *γ is Δ-transversal,*

(2) the mapping $T_x H_a$ $(H_a(x) = H(a, x)$ for $x \in V_1 \times (V_2 \cap \Sigma))$ has the properties (1)–(4) from the definition of the set $G^r_5(A, X)$.

(3) a) If $(a, x) \in P_1(\xi)$, then $T_x H_a$ has no complex eigenvalue on S and has not the eigenvalue $\lambda = -1$.

b) The Poincaré mapping $H = H[\xi, a, \gamma, V_1 \times (V_2 \cap \Sigma)]$ has the local representation as on p. 79, where $\alpha_2 \neq 0$.

(4) a) If $(a, x) \in P_2(\xi)$, then $T_x H_a$ has no complex eigenvalue on S and has not the eigenvalue $\lambda = 1$.

b) The Poincaré mapping $H = H[\xi, a, x, \gamma, V_1 \times (V_2 \cap \Sigma)]$ has the local representation as on p. 80, where $\alpha_2^2 + \gamma_1 \neq 0$.

(5) The mapping $T_x H_a$ has no complex eigenvalue λ such that $\lambda^m = 1$ for a natural number $m < [T/\tau]$, where $[z]$ denotes the greatest integer strictly less than z.

For $\xi \in G^r(A, X)$ denote by $P_1(\xi, T)$ $(P_2(\xi, T))$ the set of $(a, x) \in P_1(\xi)$ $((a, x) \in P_2(\xi))$ such that the closed orbit of the vectorfield ξ_a through x has a prime period $\tau \leqq T$.

Let $Y_0(\xi) = \{(a, x) \in A \times X \mid \xi(a, x) = 0_x\}$ for $\xi \in G^r(A, X)$, where 0_x denotes the zero of the space $T_x X$. For $(a, x) \in Y_0(\xi)$ denote by $\dot{\xi}_a(x) : T_x X \to T_x X$ the Hessian of the vectorfield ξ_a at x ([4, § 22]).

Let $G_4^r(A, X)$ be the set of all $\xi \in G^r(A, X)$ with the following properties: If $(a, x) \in Y_0(\xi)$, then

(1) if the mapping $\dot{\xi}_a(x)$ has an eigenvalue 0, then it has multiplicity 1,
(2) if $\dot{\xi}_a(x)$ has a complex eigenvalue with zero real part, then it has multiplicity 1,
(3) if $\dot{\xi}_a(x)$ has an eigenvalue 0, then it has no complex eigenvalue with zero real part.

By [1, Theorem 1, Theorem 2] the set $G_4^r(A, X)$ is open and dense in $G^r(A, X)$. Let $G_1^r(A, X, T) = G^r(A, X, T) \cap G_4^r(A, X)$. We shall prove the following lemma.

Lemma 8. *The set $G_1^r(A, X, T)$ $(r \geqq 3)$ is open and dense in $G^r(A, X)$.*

Proof. Density follows from $G_1^r(A, X, T) \supset G_2^r(A, X) \cap G_3^r(A, X) \cap G_4^r(A, X)$. Now, we shall prove the openness. It suffices to prove it for the set $G_L^r(A, X, T) = G_L^r(A, X) \cap G_1^r(A, X, T)$, because $G^r(A, X, T) = \bigcup_{i=1}^{\infty} G_{L_i}^r(A, X, T)$, where $\{L_i\}_{i=1}^{\infty}$ is an increasing sequence of positive numbers such that $\lim_{i \to \infty} L_i = +\infty$. If $\xi \in G_L^r(A, X, T)$, then by Lemma 1 for $a \in A$ every closed orbit of the vectorfield ξ_a has a prime period $\geqq b$, where $b = 4/L$.

Let $\Phi : G^r(A, X) \to C^r(A \times X \times R^+, X \times R^+ \times X)$ be the mapping defined on p. 71. The properties (1)–(5) of the set $G^r(A, X, T)$ together with the properties (1)–(3) of the set $G_4^r(A, X)$ imply that if $\xi_0 \in G_1^r(A, X, T)$, then $\Phi(\xi_0) \mathrel{\bar{\pitchfork}} \Delta$ on $A \times X \times [b, T]$. By [4, Theorem 18.2] there is an open neighborhood $N(\xi_0)$ of ξ_0 in $G_1^r(A, X, T)$ such that $\Phi(\xi) \mathrel{\bar{\pitchfork}} \Delta$ on $A \times X \times [b, T]$ for $\xi \in N(\xi_0)$ and this yields the openness of the property (1).

Let $L(\tau_X) : L(T(X), T(X)) \to X \times X$ be the linear map bundle defined in [4, § 9], whose fiber over a ponit $(x, y) \in X \times X$ is the Banach space $L(T_xX, T_yX)$ of continuous linear maps from T_xX into T_yX, i.e. $L(T(X), T(X)) = \bigcup_{(x,y)} L(T_xX, T_yX)$.

Let W_i $(i = 1, 2, 3)$ be the set of all $A \in L(T(X), T(X))$ such that

(H) $A \in L(T_xX, T_xX)$ for some $x \in X$,
(H1) $A \in W_1$ has the eigenvalue $\lambda = -1$ of multiplicity >1,
(H2) $A \in W_2$ has a complex eigenvalue on S of multiplicity >1,
(H3) $A \in W_3$ has a complex eigenvalue λ such that $\lambda^k = 1$ for a natural number $k < [T/b]$.

By an argument similar to [4, Theorem 30.2], $W_i = \bigcup_{j=1}^{k_i} W_{ij}$ $(i = 1, 2, 3)$, where W_{ij} are submanifolds of $L(T(X), T(X))$ and W_i $(i = 1, 2, 3)$ are closed.

Define the following mapping:

$\Phi' : G^r(A, X) \to C^{r-1}(A \times X \times R^+, L(T(X), T(X))$ for $\xi \in G^r(A, X)$, $\Phi'(\xi) = \Phi'_\xi$ for $\xi \in G^r(A, X)$, where $\Phi'_\xi(a, x, t) = T_x\varphi^\xi_{(t,a)}$, $(a, x, t) \in A \times X \times R^+$, $\varphi^\xi_{(t,a)}(x) = \varphi^\xi(a, x, t)$, φ^ξ is the parametrized flow of ξ. The mapping Φ' is a C^{r-1} representation.

Let $\xi_0 \in G_1^r(A, X, T)$. From the properties (1)–(4) of the set $G^r(A, X, T)$ and from the properties (1)–(3) of the set $G_4^r(A, X)$ we obtain that $\Phi'(\xi_0)(A \times X \times [b, T]) \cap \cap W_i = \emptyset$ for $i = 1, 2, 3$. Since $A \times X \times [b, T]$ is compact and W_i $(i = 1, 2, 3)$ are closed, [4, Theorem 18.2] implies that there is an open neighborhood $N_1(\xi_0)$ in $G_1^r(A, X. T)$ such that $\Phi'(\xi)(A \times X \times [b, T]) \cap W_i = \emptyset$ for $i = 1, 2, 3$, $\xi \in \in N_1(\xi_0)$. This establishes the openness of the properties (2)–(5) except of the openness of the property that there are not two eigenvalues of $T_x H_a$ on S and that $\alpha_2 \neq 0$ $(\alpha_2^2 + \gamma_1 \neq 0)$. It is clear that if $(a, x) \in P_1(\xi_0, T)$ $((a, x) \in P_2(\xi_0, T))$, then there is a neighborhood $U \times V$ of (a, x) in $A \times X$ and a neighborhood $N_2(\xi_0)$ such that for all $\xi \in N_2(\xi_0)$ the sets $P_1(\xi, T) \subset U \times V(P_2(\xi, T) \subset U \times V)$. Let $(\bar{a}, \bar{x}) \in P_1(\xi, T) \cap \cap (U \times V)$ and let $\bar{\gamma}$ be the closed orbit of $\xi_{\bar{a}}$ through $\bar{x}$. Since $\xi_0 \in G_1^r(A, X, T)$, so for $N_2(\xi_0)$ sufficiently small, the Poincaré mapping $H = H[\xi, \bar{a}, \bar{x}, \bar{\gamma}, U \times \times (V \cap \Sigma)]$ has the form as on p. 79 (p. 80) such that $\alpha_2 \neq 0$ $(\alpha_2^2 + \gamma_1 \neq 0)$ and $T_x H_a$ has no two eigenvalues on S. Since $A \times X \times [b, T]$ is compact, the sets $P_1(\xi_0, T)$, $P_2(\xi_0, T)$ are finite and the proof of Lemma 8 is complete.

The following theorem is a consequence of Lemma 8:

Theorem 3. *There is an open, dense set $G_2^r(A, X, T)$ in $G^r(A, X)$ $(r \geqq 3)$ such that if $\xi \in G_2^r(A, X, T)$, then*

(I) *$P_1(\xi, T)$ and $P_2(\xi, T)$ are finite.*

(II) *If $(a_0, x_0) \in A \times X$ and γ is a closed orbit of the vectorfield ξ_{a_0} through x_0 of a prime period $\tau \leqq T$, then the properties (2)–(4) of Theorem 1 and the properties (2)–(3) of Theorem 2 are fulfilled.*

References

[1] *M. Medveď:* Generic properties of parametrized vectorfields I, Czechoslovak Math. J., *25 (100)* 1975, 376–388.

[2] *P. Brunovský:* On one-parameter families of diffeomorphisms, Commentationnes mathematicae Universita Carolinae *11* (1970), 559–582.

[3] *P. Brunovský:* On one-parameter families of diffeomorphisms, Commentationes mathematicae Universita Carolinae *12* (1971), 765–784.

[4] *R. Abraham, J. Robbin:* Transversal Mappings and Flows, Benjamin, New York, 1967.

[5] *А. А. Андробов, Е. А. Леонтович, И. И. Гордон, А. Г. Майер:* Теория бифуркаций динамических систем на плоскости, Наука, Москва 1967.

[6] *H. Whitney:* Elementary structure of real algebraic varietes, Annals of Mathematics *66* (1957), 545–556.

[7] *J. Dieudonné:* Foundations of modern analysis, Russian translation, Mir, Moscow 1964.

[8] *Л. А. Люстерник, В. Й. Соболев:* Элементы функционального анализа, Наука, Москва 1965.

[9] *A. Lasota, A. James, Yorke:* Bounds for periodic solutions of differential equations in Banach spaces, J. Diff. Equations *10* (1971), 83–91.

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