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ON SYMMETRIC DERIVATIVES AND ON PROPERTIES OF ZAHORSKI

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Let f be a real function defined on the real line R. The symmetric derivative $f^{(1)}(x)$ of f at $x \in R$ is defined to be

$$\lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}.$$

The Lebesgue measure of a measurable set E will be denoted by $m(E)$.

For convenience we shall write

$$E_\alpha(f) = \{t : f(t) > \alpha\}, \quad E^\alpha(f) = \{t : f(t) < \alpha\}.$$

We introduce the following definitions, which are abstract formulation of those considered by ZAHORSKI [14].

Definitions. For any given $x \in R$

I. $M_0[x]$ is the family of all non empty F_σ-sets E such that x is a bilateral point of accumulation of E,

II. $M_1[x]$ is the family of all non empty F_σ-sets E such that x is a bilateral point of condensation of E,

III. $M_2[x]$ is the family of all non empty F_σ-sets E such that every unilateral neighbourhood of x intersects E in a set of positive measure,

IV. $M_3[x]$ is the family of all non empty F_σ-sets E satisfying the following property:

for every $c > 0$ there is $\varepsilon(x, c) > 0$ such that for any two reals h, h_1 satisfying $hh_1 > 0$, $h/h_1 < c$, $|h + h_1| < \varepsilon(x, c)$, the relation

$$m(E \cap J) > 0$$

is true, where J is the interval with end points $x + h$ and $x + h + h_1$.

It is clear from the above definitions that for any $x \in R$,

$$M_3[x] \subset M_2[x] \subset M_1[x] \subset M_0[x].$$

For any set $F \subset R$, we write

$$M_i[F] = \bigcap_{x \in F} M_i[x], \quad i = 0, 1, 2, 3.$$

V. For any given set $F \subset R$, $M_4[F]$ is the family of all non empty F_σ-sets E satisfying the following property:

there is a sequence of closed sets $\{E_n\}$ and a sequence of numbers $\{\eta_n\}$, $0 < \eta_n < 1$, such that $E = \bigcup_{n=1}^{\infty} E_n$ and that for every $c > 0$ and any $x \in F \cap E_n$ there is $\varepsilon(x, c) > 0$ such that for any two reals h, h_1 satisfying $hh_1 > 0$, $h/h_1 < c$, $|h + h_1| < \varepsilon(x, c)$ the relation

$$\frac{m(E \cap J)}{m(J)} > \eta_n$$

is true, where J is the interval with end points $x + h$ and $x + h + h_1$.

It is clear from the above definitions that for any set $F \subset R$

$$M_4[F] \subset M_3[F] \subset M_2[F] \subset M_1[F] \subset M_0[F].$$

Finally, M_i is the family of all non empty F_σ-sets E such that

$$E \in M_i[E], \quad 0 \leqq i \leqq 4.$$

VI. For any given $x \in R$, $\mathcal{M}_i[x]$ is the family of all functions f such that for arbitrary real $\alpha < f(x)$ the set $E_\alpha(f) \in M_i[x]$ and that for arbitrary real $\alpha > f(x)$ the set $E^\alpha(f) \in M_i[x]$, $0 \leqq i \leqq 3$.

For any given set $F \subset R$, $\mathcal{M}_i[F]$ is the family of all functions f such that for arbitrary real α, $E_\alpha(f) \in M_i[F \cap E_\alpha]$ and $E^\alpha(f) \in M_i[F \cap E^\alpha]$, $0 \leqq i \leqq 4$.

It is clear that

$$\mathcal{M}_i[F] = \bigcap_{x \in F} \mathcal{M}_i[x], \quad 0 \leqq i \leqq 3$$

and that

$$\mathcal{M}_4[F] \subset \mathcal{M}_3[F].$$

Finally, $\mathcal{M}_i$ is the family of all functions f such that for arbitrary real α, $E_\alpha(f) \in M_i$, $E^\alpha(f) \in M_i$, $0 \leqq i \leqq 4$. $\mathcal{D}$ will denote the family of all functions f satisfy Denjoy property i.e., for any two reals α, β $(\alpha < \beta)$ the set

$$\{x : \alpha < f(x) < \beta\}$$

is either void or is of positive measure [2].

Zahorski [14] proved that $\mathscr{M}_0 = \mathscr{M}_1$, each of these classes being equal to the class of Darboux function of Baire type 1. He also proved that the ordinary derivative f' (possibly infinite) of a continuous function f belongs to the class $\mathscr{M}_2$ and f' belongs to the class $\mathscr{M}_3$ when f' remains finite and f' belongs to $\mathscr{M}_4$ when f' remains bounded. Also it was known that the derivative f' (possibly infinite) of a continuous function f belongs to the class $\mathscr{D}$ [2,1]. It is found latter that $\mathscr{M}_2 = \mathscr{M}_0 \cap \mathscr{D}$ [8, 10], (for details see also [6, 12, 13]). In the present work we study the symmetric derivative along this directions. Since the symmetric derivative lacks some of the interesting properties of the ordinary derivatives, these are not obvious from the known properties of the ordinary derivatives. It may be mentioned that the Dini derivatives of real functions also enjoy under certain conditions, the Zahorski properties [11] and Denjoy property [7].

Theorem 1. *Let f be continuous, $f^{(1)}$ exist (possibly infinite) and possess Darboux property and let $f^{(1)}$ be finite except enumerable set. Then $f^{(1)} \in \mathscr{M}_2$.*

Proof. Let α be arbitrary. We show that $E_\alpha(f^{(1)}) \in M_2$. Since f is continuous, $f^{(1)}$ is of Baire type 1 and hence $E_\alpha(f^{(1)})$ is an F_σ-set. Let $x_0 \in E_\alpha(f^{(1)})$ and let $\delta > 0$ be arbitrary. Suppose that $[x_0, x_0 + \delta) \cap E_\alpha(f^{(1)})$ is of measure zero. Then $f^{(1)}(x) \leqq \leqq \alpha$ for almost all $x \in [x_0, x_0 + \delta)$ and hence $f(x) - \alpha x$ is non-increasing in $(x_0, x_0 + \delta)$ [4, 5] and so $f^{(1)}(x) \leqq \alpha$ for $x \in (x_0, x_0 + \delta)$. Since $x_0 \in E_\alpha(f^{(1)})$, $f^{(1)}(x_0) > \alpha$ and since $f^{(1)}$ has Darboux property, this is a contradiction. So $[x_0, x_0 + \delta) \cap E_\alpha(f^{(1)})$ is of positive measure and hence $E_\alpha(f^{(1)}) \in M_2[x_0]$. Since $x_0 \in E_\alpha(f^{(1)})$ is arbitrary, $E_\alpha(f^{(1)}) \in M_2$. Similarly it can be shown that $E^\alpha(f^{(1)}) \in M_2$. Hence $f^{(1)} \in \mathscr{M}_2$. This completes the proof.

Since for Darboux functions of Baire type 1, the property $\mathscr{M}_2$ and $\mathscr{D}$ are equivalent [8, 10], we get

Corollary. *If f is continuous, $f^{(1)}$ exists and possesses Darboux property and if $f^{(1)}$ is finite except enumerable set, then $f^{(1)} \in \mathscr{D}$.*

This is an improvement of a result of [9].

Lemma 1. *Let f be continuous, $f^{(1)}$ exist and be finite and possess Darboux property. Then $f^{(1)}$ possesses mean value property i.e., for any open interval (a, b) there is $\xi \in (a, b)$ such that*

$$f(b) - f(a) = (b - a) f^{(1)}(\xi)\,.$$

This is proved in [5].

Theorem 2. *Let f be continuous, $f^{(1)}$ exist and be finite and possess Darboux property. Then $f^{(1)} \in \mathscr{M}_3[F]$, where F is the set of points x where $f'(x)$ exists.*

Proof. Since $f^{(1)}$ exists and is finite, f' exists almost everywhere [3]. We have to show that $f^{(1)} \in \mathscr{M}_3[x]$ for every $x \in F$. Let $x_0 \in F$. Let α be such that $\alpha < f^{(1)}(x_0)$. If $\alpha = -\infty$, then $E_\alpha(f^{(1)}) = R$, the whole real line and hence $E_\alpha(f^{(1)}) \in M_3[x_0]$. So, we suppose $-\infty < \alpha < f^{(1)}(x_0)$. Since $x_0 \in F$, $f'(x_0)$ exists and $f'(x_0) = f^{(1)}(x_0) = \lambda$ say. Choose β such that $\lambda < \beta < \infty$. Since $\lambda = f'(x_0)$, we have

$$f(x_0 + h) = f(x_0) + \lambda h + \varepsilon_1 h \tag{1}$$

$$f(x_0 + h + h_1) = f(x_0) + \lambda(h + h_1) + \varepsilon_2(h + h_1) \tag{2}$$

where $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$ as $h \to 0$ and $h + h_1 \to 0$, respectively. Let $c > 0$ be arbitrary. Suppose $hh_1 > 0$, $h/h_1 < c$. Then $h \to 0$ as $h + h_1 \to 0$ and hence there is $\varepsilon(x_0, c) > 0$ such that

$$c|\varepsilon_2 - \varepsilon_1| + |\varepsilon_2| < \min\left(\frac{\lambda - \alpha}{2}, \frac{\beta - \lambda}{2}\right), \tag{3}$$

whenever $|h + h_1| < \varepsilon(x_0, c)$. So, if $hh_1 > 0$, $h/h_1 < c$, $|h + h_1| < \varepsilon(x_0, c)$, we have from (1), (2) and (3) that

$$\left|\frac{f(x_0 + h + h_1) - f(x_0 + h)}{h_1} - \lambda\right| \leqq \frac{h}{h_1}|\varepsilon_2 - \varepsilon_1| + |\varepsilon_2| <$$

$$< c|\varepsilon_2 - \varepsilon_1| + |\varepsilon_2| < \min\left(\frac{\lambda - \alpha}{2}, \frac{\beta - \lambda}{2}\right)$$

i.e.,

$$\alpha < \frac{\alpha + \lambda}{2} < \frac{f(x_0 + h + h_1) - f(x_0 + h)}{h_1} < \frac{\beta + \lambda}{2} < \beta\,. \tag{4}$$

Let J be the open interval with end points $x_0 + h$ and $x_0 + h + h_1$. Then from the above lemma there is $\eta \in J$ such that

$$f^{(1)}(\eta) = \frac{f(x_0 + h + h_1) - f(x_0 + h)}{h_1}$$

and hence from (4), $\alpha < f^{(1)}(\eta)$; i.e., $\eta \in J \cap E_\alpha(f^{(1)})$. By Theorem 1, $f^{(1)} \in \mathscr{M}_2$ and so, since $\eta \in E_\alpha(f^{(1)})$,

$$m(J \cap E_\alpha(f^{(1)})) > 0\,.$$

Since $f^{(1)}$ is of Baire type 1, $E_\alpha(f^{(1)})$ is an F_σ-set and hence $E_\alpha(f^{(1)}) \in M_3[x_0]$. It can similarly be shown that $E^\alpha(f^{(1)}) \in M_3[x_0]$ by interchanging α and β in the above argument. Thus $f^{(1)} \in \mathscr{M}_3[x_0]$. Since $x_0 \in F$ arbitrary, $f^{(1)} \in \mathscr{M}_3[F]$.

Lemma 2. *Let f be continuous, $f^{(1)}$ exist and be bounded and possess Darboux property. Let $|f^{(1)}(x)| \leqq K$, $K > 0$ for all x and let $f'(x_0)$ exist and be finite. Then for every $\alpha < f'(x_0)$, the set $E_\alpha(f^{(1)})$ satisfies the following property:*

for every $c > 0$ there is $\varepsilon(x_0, c) > 0$ such that for all h, h_1 satisfying $hh_1 > 0$, $h/h_1 < c$ and $|h + h_1| < \varepsilon(x_0, c)$, we have

$$\text{(i)} \qquad \frac{m(J \cap E_\alpha(f^{(1)}))}{m(J)} > \min\left(\frac{1}{2}, \frac{f'(x_0) - \alpha}{2(K + |\alpha|)}\right).$$

And for every $\alpha > f'(x_0)$ then set $E^\alpha(f^{(1)})$ satisfies (under same condition)

$$\text{(ii)} \qquad \frac{m(J \cap E^\alpha(f^{(1)})}{m(J)} > \min\left(\frac{1}{2}, \frac{\alpha - f'(x_0)}{2(K + |\alpha|)}\right)$$

where J is the interval with end points $x_0 + h$ and $x_0 + h + h_1$.

Proof. We shall prove the first part of the theorem. The proof of the second part is similar. If $-\infty \leqq \alpha < -k$, then $E_\alpha(f^{(1)}) \cap J = J$ and (i) is clear. So we suppose that α is finite. We may assume $\alpha = 0$. For, if $\alpha \neq 0$ we are to consider the function $f(x) - \alpha x$ instead of $f(x)$. Put $\lambda = f'(x_0)$. Then $\lambda > 0$. Let $c > 0$ be arbitrary. Then as in Theorem 2, there exists $\varepsilon(x_0, c) > 0$ such that for all h, h_1 satifying $hh_1 > 0$, $h/h_1 < c$, and $|h + h_1| < \varepsilon(x_0, c)$, we have

$$\text{(1)} \qquad 0 < \frac{\lambda}{2} < \frac{f(x_0 + h + h_1) - f(x_0 + h)}{h_1}.$$

Let J denote the interval with end points $x_0 + h$, $x_0 + h + h_1$. Under the hypotheses f' exists almost every in J [3]. Let $J_0 = \{x : x \in J; f'(x) \text{ exists}\}$. Then $m(J) = m(J_0)$. If

$$\varphi_n(x) = n\left\{f\left(x + \frac{1}{n}\right) - f(x)\right\}$$

then the sequence $\{\varphi_n(x)\}$ converges everywhere in J_0 to $f'(x)$. So by Lemma 1, $f^{(1)}$ satisfies mean value property

$$\varphi_n(x) = f^{(1)}(\xi), \quad \xi \in \left(x, x + \frac{1}{n}\right)$$

and hence

$$|\varphi_n(x)| = |f^{(1)}(\xi)| \leqq K \quad \text{for all } n \text{ and all } x.$$

So,

$$\text{(2)} \qquad \lim_{n \to \infty} \int_{J_0} \varphi_n(x)\, dx = \int_{J_0} f'(x)\, dx.$$

Since f is continuous

$$\text{(3)} \qquad \lim_{n\to\infty} \int_J \varphi_n(x)\, \mathrm{d}x = \begin{cases} f(x_0 + h + h_1) - f(x_0 + h) & \text{if} \quad h_1 > 0 \\ f(x_0 + h) - f(x_0 + h + h_1) & \text{if} \quad h_1 < 0 \end{cases}.$$

Hence from (2) and (3)

$$\text{(4)} \qquad \frac{|h_1|}{h_1} \{f(x_0 + h + h_1) - f(x_0 + h)\} =$$

$$= \int_{J_0} f'(x)\, \mathrm{d}x = \int_J f^{(1)}(x)\, \mathrm{d}x \leqq Km(J \cap E_0(f^{(1)}))\,.$$

Hence from (1) and (4)

$$0 < \frac{\lambda}{2K} < \frac{m(J \cap E_0(f^{(1)}))}{m(J)}\,.$$

This completes the proof.

Theorem 3. *Let f be continuous, $f^{(1)}$ exist and be bounded and possess Darboux property. Then $f^{(1)} \in \mathscr{M}_4[F]$ where F is the set of points x where $f'(x)$ exists.*

Proof. Let α be arbitrary. We shall show that $E_\alpha(f^{(1)}) \in M_4[F \cap E_\alpha(f^{(1)})]$. If $\alpha = \pm\infty$ this follows immediately. So, we suppose $-\infty < \alpha < +\infty$. Let K be a positive number such that $K > \frac{1}{2} - |\alpha|$ and $|f^{(1)}(x)| \leqq K$ for all x. Now

$$E_\alpha(f^{(1)}) = \{x : f^{(1)}(x) > \alpha\} = \bigcup_{n=1}^{\infty} \left\{x : f^{(1)}(x) > \alpha + \frac{1}{n}\right\}.$$

Since $f^{(1)}$ is of Baire type 1, the set $\{x : f^{(1)}(x) > \alpha + 1/n\}$ is an F_σ-set for each n and so there is a sequence of closed sets $\{E_{mn}\}$ such that

$$\left\{x : f^{(1)}(x) > \alpha + \frac{1}{n}\right\} = \bigcup_{m=1}^{\infty} E_{mn}\,.$$

Hence

$$E_\alpha(f^{(1)}) = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} E_{mn}\,.$$

Set $\eta_{mn} = 1/2n(K + |\alpha|)$ for all m, n. Then if $x_0 \in F \cap E_{mn}$, we have $f'(x_0) > \alpha + 1/n$ and hence from Lemma 2, we see that for every $c > 0$ there is $\varepsilon(x_0, c) > 0$ such that for all h, h_1 satisfying $hh_1 > 0$, $h/h_1 < c$, and $|h + h_1| < \varepsilon(x_0, c)$, we have

$$\text{(1)} \qquad \frac{m(J \cap E_\alpha(f^{(1)}))}{m(J)} > \frac{f'(x_0) - \alpha}{2(K + |\alpha|)} > \frac{1}{2n(K + |\alpha|)} = \eta_{mn}$$

where J is the interval with end points $x_0 + h$ and $x_0 + h + h_1$. Writing the double sequences $\{E_{mn}\}$ and $\{\eta_{mn}\}$ in terms of simple sequences, we obtain a sequence of closed sets $\{E_n\}$ and a sequence of numbers $\{\eta_n\}$, $0 < \eta_n < 1$, and these, together with relation (1) show that $E_\alpha(f^{(1)}) \in M_4[F \cap E_\alpha(f^{(1)})]$. Similarly $E^\alpha(f^{(1)}) \in \in M_4[F \cap E^\alpha(f^{(1)})]$. Hence $f^{(1)} \in \mathscr{M}_4[F]$.

From Theorems 2 and 3 it is seen that if the set F is the whole real line R, i.e., when f' exists everywhere then $f' \in \mathscr{M}_3[R]$ and if, more over, f' is bounded then $f' \in \mathscr{M}_4[R]$. But from the definition it follows that $\mathscr{M}_3[R] = \mathscr{M}_3$ and $\mathscr{M}_4[R] = = \mathscr{M}_4$. Hence the following result of Zahorski [14] is clear from Theorems 2 and 3.

Corollary. *A finite derivative* $f' \in \mathscr{M}_3$ *and a bounded derivative* $f' \in \mathscr{M}_4$.

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