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POSITIVE FUNCTIONS FROM $\mathscr{S}$-INDECOMPOSABLE SEMIGROUPS INTO PARTIALLY ORDERED SETS

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INTRODUCTION

Throughout S we will denote a semigroup, Z^+ the set of positive integers, Z the set of all integers, R^+ the set of positive reals and R the set of all reals. If $a \in S$, then $\langle a \rangle = \{a^i \mid i \in Z^+\}$ is the cyclic semigroup generated by a; the relation $\omega = S \times S$ is called the universal relation on S. For notions of semilattice decompositions and $\mathscr{S}$-indecomposable semigroups, see for example TAMURA [8, 9, 10, 11], PETRICH [1, 2] and the author [3, 4]. Positive quasi-orders on semigroups have been studied from different points of view by SCHEIN [14], Tamura [10, 12, 13], the author [6, 7] and others. Positive quasi-orders and positive mappings are naturally related [6, 7]. In this paper we are primarily interested in positive mappings and as such repeat the definition.

Definition. Let S be a semigroup.

(1) Let $a, b \in S$. Then $a \mid b$ if $b \in S^1 a S^1$.

(2) By a *positive mapping* on S we mean a mapping $\varphi : S \to (P, \leqq)$ where $(P, \leqq)$ is a partially ordered set such that for all $u, v \in S$, $\varphi(uv) \geqq \varphi(u)$ and $\varphi(uv) \geqq \varphi(v)$. Then clearly for all $a, b \in S$, $a \mid b$ implies $\varphi(a) \leqq \varphi(b)$.

(3) Let φ be a positive mapping on S. Let $\sim$ on S be given by: $a \sim b$ if and only if $\varphi(a^i) = \varphi(b^j)$ for some $i, j \in Z^+$. Then we say S is *φ-connected* if the transitive closure of $\sim$ is the universal relation on S.

1. φ-CONNECTEDNESS

Since by the Tamura semilattice decomposition theorem, every semigroup is a semilattice of $\mathscr{S}$-indecomposable semigroups, we restrict our attention mostly to $\mathscr{S}$-indecomposable semigroups.

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Proposition 1.1. *Suppose S is an $\mathscr{S}$-indecomposable semigroup and φ a positive mapping on S. Suppose further that for all $u \in S$ and $x \in S^1$ there exists $N \in Z^+$ such that for all $m \geqq N$, $\varphi(xu^N) = \varphi(xu^m)$. Assume also that for all $u, v \in S$ there exists $N \in Z^+$ such that for all $m \geqq N$, $\varphi(u^N v^N) = \varphi(u^N v^m)$ and $\varphi(v^N u^N) = \varphi(v^N u^m)$. Then S is φ-connected.*

Proof. Define $\sim$ on S as: $a \sim b$ if and only if $\varphi(a^i) = \varphi(b^j)$ for some $i, j \in Z^+$. Let $\equiv$ be the transitive closure of $\sim$. Clearly $\equiv$ is an equivalence relation. We must show that $\equiv$ is the universal relation. Since S is $\mathscr{S}$-indecomposable, by [4; Theorem 1.1] we just have to show that for all $a \in S$, $b \in S^1$, $ab \equiv aba \equiv ba$.

First let $u, v \in S$. There exists $n \in Z^+$ such that $\varphi((uv)^n) = \varphi((uv)^m)$ and $\varphi((vu)^n) = \varphi((vu)^m)$ for all $m \geqq n$. Since $(uv)^n \mid (vu)^{n+1}$ and $(vu)^n \mid (uv)^{n+1}$, we have

$$\begin{aligned}\varphi((uv)^n) &\leqq \varphi((vu)^{n+1}) = \varphi((vu)^n) \\ &\leqq \varphi((uv)^{n+1}) = \varphi((uv)^n).\end{aligned}$$

Thus $\varphi((uv)^n) = \varphi((vu)^n)$. Hence $uv \sim vu$. We use this fact without further comment.

Clearly for all $a \in S$, $a \sim a^2$. Thus we are left with showing that for all $a, b \in S$, $ab \equiv aba$, i.e., $ab \equiv a^2b$. Let $u = ab$ and $v = a^2b$. There exists $N \in Z^+$ such that for all $m \geqq N$, $\varphi(u^N v^N) = \varphi(u^N v^m)$ and $\varphi(v^N u^N) = \varphi(v^N u^m)$. Next let $A = \{u^i \mid i = 1, \ldots, N\} \cup \{uv^i \mid i = 1, \ldots, N\} \cup \{v^i \mid i = 1, \ldots, N\}$. Now A is a finite set. Thus there exists $M \in Z^+$, such that

$$M \geqq N; \quad \text{for all} \quad x \in A^1, \quad n \geqq M, \quad \varphi(xu^M) = \varphi(xu^n) \quad \text{and} \quad \varphi(xv^M) = \varphi(xv^n).$$

Clearly $u \sim u^M$. So there exists a largest non-negative integer k such that $k \leqq N$ and $u \equiv v^k u^M$. Thus

$$(1) \qquad u \equiv v^k u^M, \quad 0 \leqq k \leqq N \quad (k \text{ maximal}).$$

Our claim is that $k = N$. So we assume $k < N$ and obtain a contradiction. Since $v^k \in A^1$, $\varphi(v^k u^M) = \varphi(v^k u^{M+1})$. Thus $v^k u^M \equiv v^k u^{M+1} \equiv uv^k u^M$. Since $k < N$, $uv^k \in A$. Therefore we have

$$\varphi(uv^k u^M) \leqq \varphi(uv^k u^M a) \leqq \varphi(uv^k u^{M+1}) = \varphi(uv^k u^M).$$

Consequently $\varphi(uv^k u^M) = \varphi(uv^k u^M a)$. We therefore have,

$$u \equiv v^k u^M \equiv uv^k u^M \equiv uv^k u^M a \equiv auv^k u^M = v^{k+1} u^M.$$

But this contradicts the maximality of k in (1). This contradiction shows that $u \equiv v^N u^M$. Since $M \geqq N$, $\varphi(v^N u^M) = \varphi(v^N u^N)$ and $v^N u^M \equiv v^N u^N$. So we have

$$(2) \qquad u \equiv v^N u^N \equiv u^N v^N.$$

Next we notice that $v \sim v^M$. Thus there exists a largest non-negative integer k, $k \leqq N$ such that $v \equiv u^k v^M$. Thus

$$v \equiv u^k v^M, \quad 0 \leqq k \leqq N \quad (k \text{ maximal}). \tag{3}$$

Our claim is that $k = N$. So we assume $k < N$ and obtain a contradiction. Since $u^k \in A^1$,

$$u^k v^M \equiv u^k v^{M+1} = u^k v^M a u \equiv u^{k+1} v^M a .$$

Since $k < N$, $u^{k+1} \in A$ and so

$$\varphi(u^{k+1}v^M a) \geqq \varphi(u^{k+1}v^M) = \varphi(u^{k+1}v^{M+1}) \geqq \varphi(u^{k+1}v^M a).$$

Thus $\varphi(u^{k+1}v^M a) = \varphi(u^{k+1}v^M)$ and therefore

$$v \equiv u^k v^M \equiv u^{k+1} v^M a \equiv u^{k+1} v^M, \quad k + 1 \leqq N .$$

This however contradicts the maximality of k in (3). This contradiction shows that $v \equiv u^N v^M$. Since $M \geqq N$, $\varphi(u^N v^M) = \varphi(u^N v^N)$ and $u^N v^M \equiv u^N v^N$. So we have

$$v \equiv u^N v^N . \tag{4}$$

Combining (2) and (4) we obtain $u \equiv v$. Thus $ab \equiv a^2 b$, proving the theorem.

Theorem 1.2. *Suppose S is an $\mathcal{S}$-indecomposable semigroup and φ a positive mapping on S. Suppose further that for all $u, v \in S$, the sets $\{\varphi(uv^n) \mid n \in Z^+\}$, $\{\varphi(u^n v^n) \mid n \in Z^+\}$ are both finite. Then S is φ-connected.*

Proof. We have $\{\varphi(uu^n) \mid n \in Z^+\}$ is finite, whence $\{\varphi(u^n) \mid n \in Z^+\}$ is finite. Thus for any $x \in S^1$, the set $\{\varphi(xu^n) \mid n \in Z^+\}$ is finite. So for each $x \in S^1$, there exists $N \in Z^+$ such that for each $m \geqq N$, there exists $i \leqq N$ such that $\varphi(xu^m) = \varphi(xu^i)$. By positivity,

$$\varphi(xu^m) = \varphi(xu^i) \leqq \varphi(xu^N) \leqq \varphi(xu^m).$$

Hence $\varphi(xu^N) = \varphi(xu^m)$ for all $m \geqq N$.

Next let $u, v \in S$. Then $\varphi\{u^n v^n \mid n \in Z^+\}$ is finite. So there exists $M \in Z^+$ such that for each $n \geqq M$, there exists $i \leqq M$ such that $\varphi(u^n v^n) = \varphi(u^i v^i)$. By positivity,

$$\varphi(u^n v^n) = \varphi(u^i v^i) \leqq \varphi(u^M v^M) \leqq \varphi(u^n v^n).$$

So $\varphi(u^M v^M) = \varphi(u^n v^n)$ for all $n \geqq M$. Similarly there exists $N \in Z^+$ such that for all $n \geqq N$, $\varphi(v^N u^N) = \varphi(v^n u^n)$. Let $K = N + M$. Then for all $n \geqq K$, $\varphi(u^K v^K) = \varphi(u^n v^n)$ and $\varphi(v^K u^K) = \varphi(v^n u^n)$. By positivity,

$$\varphi(u^K v^K) \leqq \varphi(u^K v^n) \leqq \varphi(u^n v^n) = \varphi(u^K v^K).$$

So for all $n \geqq K$, $\varphi(u^K v^K) = \varphi(u^K v^n)$. Similarly, for all $n \geqq K$, $\varphi(v^K u^K) = \varphi(v^K u^n)$. Consequently, the hypothesis of Proposition 1 is satisfied and S is φ-connected.

Following is now an immediate consequence.

Theorem 1.3. *Let S be an $\mathcal{S}$-indecomposable semigroup and φ a positive mapping on S such that $\varphi(S)$ is finite. Then S is φ-connected.*

Theorem 1.4. *Let S be an $\mathcal{S}$-indecomposable semigroup and φ a positive mapping on S. Suppose that for all $u \in S$, there exists $N \in Z^+$ such that for all $x \in S^1$, $\varphi(xu^N) = \varphi(xu^{N+1})$. Then S is φ-connected.*

Proof. Let $u \in S$. Then there exists $N \in Z^+$ such that for all $x \in S^1$, $\varphi(xu^N) = \varphi(xu^{N+1})$. In particular, $\varphi(xuu^N) = \varphi(xuu^{N+1})$ so that $\varphi(xu^{N+1}) = \varphi(xu^{N+2})$. By induction $\varphi(xu^K) = \varphi(xu^N)$ for all $K \geqq N$. Next let $u, v \in S$. Then by the above, there exist $M, N \in Z^+$ such that for all $x \in S^1$ and $k \geqq M$, $l \geqq N$, $\varphi(xu^k) = \varphi(xu^M)$ and $\varphi(xv^l) = \varphi(xv^N)$. It follows that for all $n \geqq M + N$, $x \in S^1$, $\varphi(xu^{M+N}) = \varphi(xu^n)$ and $\varphi(xv^{M+N}) = \varphi(xv^n)$. In particular $\varphi(v^{M+N}u^{M+N}) = \varphi(v^{M+N}u^n)$ and $\varphi(u^{M+N}v^{M+N}) = \varphi(u^{M+N}v^n)$ for all $n \geqq M + N$. Consequently the hypothesis of Proposition 1.1 is satisfied and S is φ-connected.

Corollary 1.5. *Let S be an $\mathcal{S}$-indecomposable semigroup such that a power of each element in S lies in a right simple subsemigroup of S. Then for every positive mapping φ on S, S is φ-connected.*

Proof. Let $u \in S$. Then there exists $N \in Z^+$ such that u^N lies in a right simple subsemigroup T of S. Then $u^{2N} \in T$. So there exists $y \in T$ such that $u^{2N}y = u^N$. Let $z = u^{N-1}y$. Then for all $x \in S^1$, $xu^{N+1}z = xu^{2N}y = xu^N$. Hence $xu^{N+1} | xu^N | xu^{N+1}$. By positivity, $\varphi(xu^N) = \varphi(xu^{N+1})$ for all $x \in S^1$. By Theorem 1.4. S is φ-connected.

Remark. In case that S has the property that a power of each element lies in a subgroup, Corollary 1.5 yields an equivalent formulation of the author [5; Corollary 2].

Problem. Let S be an $\mathcal{S}$-indecomposable semigroup and φ a positive mapping on S. Suppose that for all cyclic subsemigroups $\langle a \rangle$ of S, $\varphi(\langle a \rangle)$ is a finite set. Then is S necessarily φ-connected?

2. REAL VALUED POSITIVE FUNCTIONS

Theorem 2.1. *Let S be an $\mathcal{S}$-indecomposable semigroup and φ a real valued positive mapping on S such that for all $u, v \in S$, $\varphi(uv) = \varphi(vu)$. Then for all $a, b \in S$, $\lim_{n\to\infty} \varphi(a^n) = \lim_{n\to\infty} \varphi(b^n)$ in the extended real line.*

Proof. Let $a \in S$. By positivity, $\langle\varphi(a^n)\rangle_{n\in Z^+}$ is a non-decreasing sequence. So $\sup_{n\in Z^+} \varphi(a^n) = \lim_{n\to\infty} \varphi(a^n)$ exists in the extended real line. Let $\Psi(a) = \lim_{n\to\infty} \varphi(a^n) = \sup_{n\in Z^+} \varphi(a^n)$. For $a, b \in S$, define $a \equiv b$ if and only if $\Psi(a) = \Psi(b)$. Clearly $\equiv$ is an equivalence relation. We will be done once we show that $\equiv$ is the universal relation on S. Since S is $\mathcal{S}$-indecomposable, by [4; Theorem 1.1], we just have to show that for all $a \in S$, $b \in S^1$, $ab \equiv aba \equiv ba$. Now for each $n \in Z^+$, $(ab)^n \mid (ba)^{n+1}$ whence $\varphi((ab)^n) \leqq \varphi((ba)^{n+1}) \leqq \Psi(ba)$. Thus $\Psi(ab) \leqq \Psi(ba)$. Similarly $\Psi(ba) \leqq \Psi(ab)$ and $\Psi(ab) = \Psi(ba)$. Hence $ab \equiv ba$. So we are left with showing that $ab \equiv aba$, i.e., $ab \equiv a^2b$. Now for each $n \in Z^+$,

$$\varphi((ab)^n) \leqq \varphi((a^2b)(ab)^{n-1}) = \varphi((ab)^{n-1}a^2b) \leqq \varphi((a^2b)(ab)^{n-2}(a^2b)) = \\ = \varphi((ab)^{n-2}(a^2b)^2) \leqq \ldots \leqq \varphi((a^2b)^n) \leqq \Psi(a^2b)\,.$$

Thus $\Psi(ab) \leqq \Psi(a^2b)$. Also for each $n \in Z^+$,

$$\varphi((a^2b)^n) = \varphi((ab)(a^2b)^{n-1}a) \leqq \varphi((ab)(a^2b)^{n-1}ab) = \varphi((a^2b)^{n-1}(ab)^2) \leqq \\ \leqq \varphi((ab)(a^2b)^{n-2}(ab)^2a) \leqq \ldots \leqq \varphi((ab)^{2n}) \leqq \Psi(ab)\,.$$

Hence $\Psi(a^2b) \leqq \Psi(ab)$. Consequently $\Psi(ab) = \Psi(a^2b)$ and $ab \equiv a^2b$. This proves the theorem.

Remark. Theorem 2.1 can be proved in an alternate way as follows: Let $a, b \in S$ and $a \mid b$. Then $xay = b$ for some $x, y \in S^1$. Therefore

$$\begin{aligned} \varphi(a) &\leqq \varphi(b)\,; \\ \varphi(a^2) &\leqq \varphi(a^2yx) = \varphi(ayxa) \leqq \varphi(xayxay) = \varphi(b^2)\,; \\ \varphi(a^3) &\leqq \varphi(a^3yx) = \varphi(a^2yxa) \leqq \varphi(a^2yxayx) = \\ &= \varphi(ayxayxa) \leqq \varphi(xayxayxay) = \varphi(b^3)\,. \end{aligned}$$

This argument can easily be generalized to show that for all $i \in Z^+$, $\varphi(a^i) \leqq \varphi(b^i)$. Thus for any $a, b \in S$, $a \mid b$ implies $\varphi(a^i) \leqq \varphi(b^i)$ for all $i \in Z^+$. This result in conjunction with Tamura [11] easily yields that for any $a, b \in S$, $\varphi(a) \leqq \varphi(b^j)$ for some $j \in Z^+$. Hence for any $a, b \in S$, $n \in Z^+$ there exists $m \in Z^+$ such that $\varphi(a^n) \leqq \varphi(b^m) \leqq \Psi(b)$. So $\Psi(a) \leqq \Psi(b)$. Similarly $\Psi(b) \leqq \Psi(a)$ and $\Psi(a) = \Psi(b)$.

Theorem 2.2. *Let S be an $\mathcal{S}$-indecomposable semigroup and φ a positive mapping on S such that for all $a, b \in S$, $\varphi(a) \leqq \varphi(b)$ implies $\varphi(a^2) \leqq \varphi(b^i)$ for some $i \in Z^+$. Then for any $a, b \in S$, $\lim_{n\to\infty} \varphi(a^n) = \lim_{n\to\infty} \varphi(b^n)$.*

Proof. By the author [7] the hypothesis implies that for any $a, b \in S$, there exists $n \in Z^+$ such that $\varphi(a) \leqq \varphi(b^n)$. By the argument given in the remark after Theorem 2.1, the result follows.

Next we study boundedness of real valued positive functions on semigroups.

Definition. Let S be a semigroup and φ a positive mapping into the positive reals R^+.

(1) φ is *locally bounded* if for all $r \in R^+ \cup \{0\}$, there exists $\varepsilon > 0$ and $N \in Z^+$ such that for all $a \in S$ with $|\varphi(a) - r| < \varepsilon$, $\varphi(\langle a \rangle) \subseteq [0, N]$.

(2) φ is *bounded* if there exists $N \in Z^+$ such that $\varphi(S) \subseteq [0, N]$.

(3) $\mathfrak{B}(R^+)$ is *the class of all semigroups* T such that every locally bounded positive mapping of T into the positive reals is bounded. $\mathfrak{B}(Z^+)$ is *the class of all semigroups* T such that every locally bounded positive mapping of S into the positive integers is bounded. Clearly $\mathfrak{B}(R^+) \subseteq \mathfrak{B}(Z^+)$.

Remark. (1) Let φ be a positive mapping into Z^+. Then φ is locally bounded if and only if for all $r \in Z^+$ there exists $N \in Z^+$ such that for all $a \in S$ and $\varphi(a) = r$, $\varphi(\langle a \rangle) \subseteq [0, N]$. Also φ is bounded if and only if $\varphi(S)$ is finite.

(2) A homomorphic image of a semigroup in $\mathfrak{B}(R^+)$ (or $\mathfrak{B}(Z^+)$) is again in $\mathfrak{B}(R^+)$ (or $\mathfrak{B}(Z^+)$).

Lemma 2.3. *Let S be a semigroup and $\varphi : S \to R^+$ a positive mapping. Then the following are equivalent*:

(1) *φ is locally bounded.*

(2) *For each $r \in R^+$, there exists $N \in Z^+$ such that for all $a \in S$ and $\varphi(a) < r$, $\varphi(\langle a \rangle) \subseteq [0, N]$.*

Proof. (1) $\Rightarrow$ (2). The proof is by contradiction. So suppose there exists $r \in R^+$ such that for each $i \in Z^+$ there exists $a_i \in S$ such that $\varphi(a_i) < r$ but $\varphi(\langle a_i \rangle) \nsubseteq [0, i]$. Now $\{\varphi(a_i) \mid i \in Z^+\} \subseteq [0, r]$. Thus the sequence $\langle \varphi(a_i) \rangle_{i=1}^{\infty}$ must have an accumulation point $r_0 \in [0, r]$. Since φ is locally bounded, there exists $\varepsilon > 0$ and $N \in Z^+$ such that for all $a \in S$, $|\varphi(a) - r_0| < \varepsilon$ implies $\varphi(\langle a \rangle) \subseteq [0, N]$. Now there exists $i > N$ such that $|\varphi(a_i) - r_0| < \varepsilon$. Hence $\varphi(\langle a_i \rangle) \subseteq [0, N] \subseteq [0, i]$, a contradiction.

(2) $\Rightarrow$ (1). Let $r \in R^+ \cup \{0\}$. Set $r_0 = r + 1$. There exists $N \in Z^+$ such that for all $a \in S$, $\varphi(a) < r_0$ implies $\varphi(\langle a \rangle) \subseteq [0, N]$. Let $\varepsilon = 1$. Then for each $a \in S$, $|\varphi(a) - r| < \varepsilon$ implies $\varphi(a) < r_0$ and therefore $\varphi(\langle a \rangle) \subseteq [0, N]$. Consequently φ is locally bounded.

We assume familiarity with results of [11], [4] and use the notation of [4] without further comment.

Definition. (1) Let S be a semigroup and $a, b \in S$. If there is no sequence from a to b we set $d(a, b) = \infty$. If $a \to b$ we set $d(a, b) = 0$. Otherwise we let $d(a, b)$ be the length of a minimal sequence from a to b. If in need of clarification, we use d_S for d.

(2) If $u \in S$, then $\Phi(u) = \sup_{a \in S} d(a, u)$. If in need of clarification, we use Φ_S for Φ.

We now characterize $\mathscr{S}$-indecomposable semigroups in $\mathfrak{B}(R^+)$.

Theorem 2.4. *Let S be an $\mathscr{S}$-indecomposable semigroup. Then the following are equivalent*:

(1) *There exist $u \in S$ such that $\Phi(u) < \infty$.*

(2) *For each $a \in S$, $\Phi(a) < \infty$.*

(3) $S \in \mathfrak{B}(R^+)$.

(4) $S \in \mathfrak{B}(Z^+)$.

Proof. (1) $\Rightarrow$ (2). Suppose for some $u \in S$, $\Phi(u) < \infty$. Let $a \in S$. Since S is $\mathscr{S}$-indecomposable, $d(u, a) < \infty$. Thus for any $x \in S$, $d(x, a) \leqq d(x, u) + d(u, a) + 1 \leqq \Phi(u) + d(u, a) + 1$. Hence $\Phi(a) < \infty$.

(2) $\Rightarrow$ (3). Let $\varphi : S \to R^+$ be a locally bounded positive mapping. By Lemma 2.3, for each $r \in R^+$, there exists $\alpha(r) \in Z^+$ such that for each $a \in S$, $\varphi(a) < r$ implies $\varphi(\langle a \rangle) \subseteq [0, \alpha(r)]$. Next we note that for $a, b \in S$, $a \to b$ implies that $\varphi(a) \leqq \varphi(b^i)$ for some $i \in Z^+$. Now choose $u \in S$. Let $A_0 = \{x \mid x \in S,\ x \to u\} = \{x \mid x \in S, d(x, u) = 0\}$. In general $A_{n+1} = \{x \mid x \in S,\ x \to a \text{ for some } a \in A_n\} = \{x \mid x \in S, d(x, u) \leqq n + 1\}$. Evidently for each $x \in A_0$, $\varphi(x) \leqq \alpha(u)$. Hence $\varphi(\langle x \rangle) \subseteq \alpha(\alpha(u))$ for each $x \in A_0$.
It follows that for each $x \in A_1$, $\varphi(x) \leqq \alpha(\alpha(u))$. In general for each $i \in Z^+$ there exists $N_i \in Z^+$ such that $\varphi(A_i) \subseteq [0, N_i]$. Now $\varphi(u) < \infty$. Let $K = \Phi(u)$. Then $A_K = S$. Consequently, $\varphi(S) = \varphi(A_K) \subseteq [0, N_K]$.

(3) $\Rightarrow$ (4). Obvious.

(4) $\Rightarrow$ (1). Let $u \in S$. Then since S is $\mathscr{S}$-indecomposable $d(a, u) < \infty$ for each $a \in S$. Define $\varphi : S \to Z^+$ as $\varphi(a) = d(a, u)$. By [4; Lemma 1.5] φ is positive. If $\langle x_1, \ldots, x_n \rangle$ is a sequence from a to u, then for any $k \in Z^+$, $\langle a, x_1, \ldots, x_n \rangle$ is a sequence from a^k to u. So $d(a^k, u) \leqq d(a, u) + 1$. Consequently $\varphi(\langle a \rangle) \subseteq [0, \varphi(a) + 1]$ and φ is locally bounded and positive. Hence φ is bounded. Thus $\Phi(u) < \infty$.

Next we take up the task of studying semigroups in $\mathfrak{B}(R^+)$ which are not necessarily $\mathscr{S}$-indecomposable.

Lemma 2.5. *Let Ω be a countable semilattice. Then the following are equivalent.*

(1) $\Omega \in \mathfrak{B}(R^+)$.

(2) $\Omega \in \mathfrak{B}(Z^+)$.

(3) *Ω has a zero.*

Proof. (1) $\Rightarrow$ (2). Obvious.

(2) $\Rightarrow$ (3). Clearly we may assume $|\Omega| > 1$. As is well known, Ω is a subdirect product of copies of the semilattice $I = \{0, 1\}$. Since Ω is countable, we easily obtain

that Ω is a subdirect product of I_i $(i \in Z^+)$ where each $I_i = \{0_i, 1_i\} \cong I$. Let σ_i $(i \in Z^+)$ be the projection maps. We assume Ω does not have zero and obtain a contradiction. For each $a \in \Omega$, there exists a smallest $i \in Z^+$ such that $\sigma_i(a) \neq 0_i$. Let $\varphi(a)$ denote this integer i. Then $\varphi : S \to Z^+$ is clearly positive and locally bounded. Let $a \in S$. Set $j = \varphi(a)$. Then there exists $b \in S$ such that $\sigma_j(b) = 0_j$. Hence $\varphi(ab) > \varphi(a)$. Consequently $\varphi(S)$ is infinite and hence unbounded. This contradiction shows that Ω has a zero.

(3) $\Rightarrow$ (1). Clearly any positive mapping on Ω attains a maximum at the zero.

Theorem 2.6. *Let S be a semigroup and Ω its maximal semilattice homomorphic image. Suppose that either Ω is countable or has a zero. Then the following are equivalent.*

(1) *There exists $u \in S$ such that $\Phi(u) < \infty$.*

(2) *Ω has a zero 0 and the corresponding $\mathcal{S}$-indecomposable component S_0 of S is in $\mathfrak{B}(R^+)$.*

(3) *There exists an ideal I of S such that $I \in \mathfrak{B}(R^+)$.*

(4) *$S \in \mathfrak{B}(R^+)$.*

(5) *$S \in \mathfrak{B}(Z^+)$.*

Proof. (1) $\Rightarrow$ (2). Let $\Psi : S \to \Omega$ be the natural homomorphism. Since $\Phi(u) < \infty$, for any $a \in S$, there exists a sequence from a to u. It follows by [4; Lemma 2.2] that $\Psi(u) = 0$ is the zero of Ω. Let $S_0 = \Psi^{-1}(\{0\})$. Then S_0 is an $\mathcal{S}$-indecomposable semigroup, an ideal of S and contains u. Let $a \in S_0$. By [4; Lemma 2.2], $d_S(a, u) = d_{S_0}(a, u)$. Hence

$$d_{S_0}(a, u) = d_S(a, u) \leqq \Phi_S(u)\,.$$

So $\Phi_{S_0}(u) \leqq \Phi_S(u) < \infty$. By Theorem 2.4, $S_0 \in \mathfrak{B}(R^+)$.

(2) $\Rightarrow$ (3). Clearly S_0 is an ideal of S.

(3) $\Rightarrow$ (4). Let $\varphi : S \to R^+$ be a locally bounded positive mapping. Then φ is a locally bounded positive mapping on I. Since $I \in \mathfrak{B}(R^+)$ there exists $M \in Z^+$ such that $\varphi(I) \subseteq [0, M]$. Choose $u \in I$. Then for any $a \in S$, $au \in I$. Hence $\varphi(a) \leqq \varphi(au) \leqq M$. Therefore $\varphi(S) \subseteq [0, M]$ and φ is bounded. Consequently $S \in \mathfrak{B}(R^+)$.

(4) $\Rightarrow$ (5). Obvious.

(5) $\Rightarrow$ (1). Since $S \in \mathfrak{B}(Z^+)$, the homomorphic image $\Omega \in \mathfrak{B}(Z^+)$. By Lemma 2.5, Ω has a zero 0. Let S_0 be the corresponding $\mathcal{S}$-indecomposable component of S. Fix $u \in S_0$. Let $a \in S$. Then $au \in S_0$. By [4; Lemma 1.5], $d_S(a, u) \leqq d_S(au, u)$. Since S_0 is $\mathcal{S}$-indecomposable $d_{S_0}(au, u) < \infty$. Clearly $d_S(au, u) \leqq d_{S_0}(au, u) < \infty$. It follows that $d_S(a, u) < \infty$ for all $a \in S$. Let $\varphi : S \to Z^+$, be defined by $\varphi(a) = d_S(a, u)$. Then as in Theorem 2.4, we see that φ is bounded. Hence $\Phi_S(u) < \infty$.

Definition. Let S be a semigroup and $\varphi : S \to R$ a positive mapping.

(1) φ is *locally bounded* if for all $r \in R$ there exists $\varepsilon > 0$ and $N \in Z^+$ such that for all $a \in S$, $|\varphi(a) - r| < \varepsilon$ implies $\varphi(\langle a \rangle) \subseteq (-\infty, N]$. φ is *bounded* below if $\varphi(S) \subseteq (M, \infty)$ for some $M \in R$. φ is bounded above if $\varphi(S) \subseteq (-\infty, M)$ for some $M \in R$. φ is bounded if it is bounded above and below. (Clearly we could take M to be in Z.)

(2) $\mathfrak{B}(R)$ is *the class of all semigroups* T such that every locally bounded positive mapping $\varphi : T \to R$ is bounded. Clearly $\mathfrak{B}(R) \subseteq \mathfrak{B}(R^+)$.

Remark. Let $\varphi : S \to R$ be locally bounded and positive. Then it is easy to check that for any $M \in R$, $\varphi + M$ is also locally bounded and positive.

Lemma 2.7. *Let $S \in \mathfrak{B}(R^+)$ and $\varphi : S \to R$ a locally bounded positive mapping which is bounded below. Then φ is bounded.*

Proof. There exists $M \in R$ such that $\varphi(S) \subseteq (M, \infty)$. If $M \geqq 0$, we are clearly done since $S \in \mathfrak{B}(R^+)$. Otherwise $M < 0$ and $\varphi - M : S \to R^+$ is positive and locally bounded. Since $S \in \mathfrak{B}(R^+)$, $\varphi - M$ is bounded. Then clearly φ is bounded.

Theorem 2.8. *Let S be a finitely generated semigroup. Then the following are equivalent.*

(1) *There exists $u \in S$ such that $\Phi(u) < \infty$.*

(2) $S \in \mathfrak{B}(R^+)$.

(3) $S \in \mathfrak{B}(R)$.

Proof. Since S is finitely generated, it is countable. By Theorem 2.6, (1) $\Leftrightarrow$ (2). Evidently (3) $\Rightarrow$ (2). So we are left with showing (2) $\Rightarrow$ (3). So let $S \in \mathfrak{B}(R^+)$. Let $\varphi : S \to R$ be a positive, locally bounded mapping. Since S is finitely generated, $S = \langle u_1, \ldots, u_n \rangle$ for some $u_1, \ldots, u_n \in S$. For each $a \in S$, $u_i \mid a$ for some $i \in \{1, \ldots, n\}$. Hence $\varphi(u_i) \leqq \varphi(a)$. Consequently φ is bounded below by $\min \{\varphi(u_1), \ldots, \varphi(u_n)\}$. By Lemma 2.7, φ is bounded. Consequently, $S \in \mathfrak{B}(R)$.

Example. Let X be an infinite set and $\mathfrak{J}_X$ the full transformation semigroup on X. If $\sigma \in \mathfrak{J}_X$, let $\varphi(\sigma) = |\text{range of } \sigma|$. Let $S = \{\sigma \mid \sigma \in \mathfrak{J}_X \text{ and } \varphi(\sigma) < \infty\}$. Then S is subsemigroup of $\mathfrak{J}_X$. Define $\varphi_1 : S \to R$ as $\varphi_1(\sigma) = -\varphi(\sigma)$.

Then $\varphi_1(S) \subseteq (-\infty, 0)$ and is positive. Being bounded above, φ_1 is locally bounded. On the other hand φ_1 is unbounded. Thus $S \notin \mathfrak{B}(R)$. However, it is routine to verify that S is an $\mathscr{S}$-indecomposable semigroup of rank 1 (see [4] for definition of semirank and rank of a semigroup). In contrast, by Theorem 2.4, every $\mathscr{S}$-indecomposable semigroup of finite semirank must be in $\mathfrak{B}(R^+)$.

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