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# ON DECOMPOSITIONS OF COMPLETE $k$-UNIFORM HYPERGRAPHS

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## INTRODUCTION

The problem of decomposing graphs or hypergraphs into factors with given properties occurs in many modifications. One of these is to determine some conditions for the existence of a decomposition of a complete $k$-uniform hypergraph into $m$ factors with given diameters $d_1, d_2, \ldots, d_m$. Our aim is to determine the smallest cardinal number $F^k(d_1, d_2, \ldots, d_m) = t$ (if such a cardinal number exists) such that a complete $k$-uniform hypergraph $K_t^k$ on $t$ vertices can be decomposed into $m$ factors with diameters $d_1, d_2, \ldots, d_m$ and (if it is possible) to find the decomposition required by a constructive method.

In [1] it was proved that $F^2(d_1, d_2, \ldots, d_m) \leqq md - m$ for $d \geqq 3$ and $m \geqq 3$, where $d = \max d_i$. Relatively more complicated it was to find a decomposition into factors with diameters equal to two. J. BOSÁK, P. ERDÖS, A. ROSA in [2] proved that

$$4m - 1 \leqq F_m^2(2) \leqq \binom{6m-2}{2m-2}.$$

Later on, J. Bosák proved in [3] an inequality which is the best one known till now:

$$6m - 52 \leqq F_m^2(2) \leqq 6m .$$

The papers [9], [11], [10] investigate the problem of decomposing a complete digraph into factors with given diameters.

D. PALUMBÍNY in [5], [6] studied the problem of decomposing a complete graph into factors with equal diameters. He proved in [5] that $F_m^2(d) = 2m$ for $m \geqq 2$ and $3 \leqq d \leqq 2m - 1$. Even though his aim was not to find a decomposition into isomorphic factors, $m$ factors with diameter equal to $d$ of his decomposition of the complete graph with $2m$ vertices are isomorphic for $d$ odd. The study of the decompositions of complete graphs into isomorphic factors with given diameters was initiated in [4], where the decomposition of a complete graph into three isomorphic factors with a given diameter $d \geqq 2$ is considered.

In [7] the problem of decomposing a complete $k$-uniform hypergraph ($k \geqq 2$) into isomorphic factors with a given diameter was systematically studied. It seems that this is the shortest way to solve the problem of the existence of the cardinal number $F^k(d_1, d_2, \ldots, d_m)$.

We shall give in this paper an upper bound for the number $F^k(d_1, d_2, \ldots, d_m)$ where $3 \leqq k < m$ and $2 \leqq d_1 \leqq d_2 \leqq \ldots \leqq d_m$. The case when at least one of the diameters is equal to one was solved in [8].

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First we give some notations and definitions. A hypergraph is an ordered pair of sets $G = (V, H)$ where $H \subset P(V)$ (the potency of $V$). Let $k$ be a positive integer. A hypergraph $G$ is said to be a *$k$-uniform hypergraph* if for each $h \in H$ we have $|h| = k$. For $k = 2$ we obtain graphs. If the set $H$ contains all the $k$-element subsets of $V$ then $G$ is said to be a *complete $k$-uniform hypergraph* and we denote $G$ by $K_n^k$ where $n = |V|$. The distance $d(x, y)$ of two vertices $x$ and $y$ is the length of the shortest path joining them. The diameter of a hypergraph is defined by $d = \sup_{x,y \in V} d(x, y)$.

A factor of $G$ is such a subhypergraph of $G$ which contains all vertices of $G$. Let $F^k(d_1, d_2, \ldots, d_m) = t$ be the smallest integer (if it exists) such that the hypergraph $K_t^k$ is decomposable into $m$ factors with diameters $d_1, d_2, \ldots, d_m$. If $d_1 = \ldots = d_m = d$ we put $F_m^k(d) = F^k(d_1, d_2, \ldots, d_m)$. We shall say that $G_1 = (V_1, H_1)$ and $G_2 = (V_2, H_2)$ are isomorphic if there exists a bijection $f : V_1 \to V_2$ such that $h \in H_1$ if and only if $f(h) \in H_2$.

Let $G_m^k(d) = t$ be the smallest cardinal number such that $K_t^k$ can be decomposed into $m$ isomorphic factors with the diameter $d$.

**Definition 1.** Let $G$ be an arbitrary group of automorphisms of the hypergraph $K_n^k$ and let there exist a surjection $h : G \to R$, where $R$ is a decomposition of the hypergraph $K_n^k$ into isomorphic factors, with the following property:

$$x(h(y)) = h(xy) \quad \text{for every} \quad x, y \in G .$$

Then we shall say that $R$ is a *decomposition of $K_n^k$* by the group $G$. If the mapping $h$ is a bijection then we shall say that $R$ is a *simple decomposition* of $K_n^k$ by $G$. The factor $h(x)$ will be denoted by $G_x$.

In our further considerations we shall need the following propositions.

**Proposition 1.** *Let $H$ be an Abelian group of a finite order $m > 1$ and let $k \geqq 3$ be a natural number such that $(m, k!) = 1$. Then the following two statements are equivalent:*

1. *There exists a group $H_1 \cong H$ such that the hypergraph $K_n^k$ has a simple decomposition by the group $H_1$.*

2. *m divides* $\binom{n}{k}$ *and divides just one of the numbers* $n, n-1, \ldots, n-k+1$.

This statement was proved in [7]. It was also remarked there that a weaker condition is sufficient for the existence of a simple decomposition of $K_n^k$, namely $(m, k) = 1$. We use this fact in the sequel.

**Proposition 2.** *Let* $F^k(d_1, d_2, \ldots, d_m)$ *exist. Then the complete hypergraph* $K_k^N$ *can be decomposed into m factors with diameters* $d_1, d_2, \ldots, d_m$ *if and only if* $N \geqq F^k(d_1, d_2, \ldots, d_m)$.

**Proposition 3.** *Let* $3 \leqq k < m$, $(m, k) = 1$, $d \geqq 2$ *be integers. Then* $G_m^k(d)$ *exists and*

$$G_m^k(d) \leqq m[(d-2)(k-1)+] \quad \text{if} \quad d \geqq 3\,, \qquad G_m^k(2) \leqq 2m\,.$$

Proposition 2 was proved in [8] and Proposition 3 in [7]. Now we are able to prove

**Theorem 1.** *Let* $3 \leqq k < m$, $(m, k) = 1$, $2 \leqq d_1 \leqq d_2 \leqq \ldots \leqq d_m$ *be integers. Then* $F^k(d_1, d_2, \ldots, d_m)$ *exists and*

$$F^k(d_1, d_2, \ldots, d_m) \leqq m[(d_m - 2)(k-1)+1] \quad \text{if} \quad d_m \geqq 3\,,$$
$$F^k(d_1, d_2, \ldots, d_m) \leqq 2m \quad \text{if} \quad d_m = 2\,.$$

Proof. I. Assume $d_m \geqq 3$ and put $n = mt$ where $t = (d_m - 2)(k-1) + 1$. Denote the vertices of $K_n^k$ by $i_j$, $1 \leqq i \leqq m$, $1 \leqq j \leqq t$. Obviously $m$ divides $\binom{n}{k}$. Moreover, $m$ divides $n$. Because $(m, k) = 1$, the sufficient condition for the existence of a simple decomposition of $K_n^k$ by a cyclic group $H$ of order $m$ generated by a permutation $\beta = (1_1, 2_1, \ldots, m_1)(1_2, 2_2, \ldots, m_2) \ldots (1_t, 2_t, \ldots, m_t)$ is satisfied. In the proof of Proposition 3 a special simple decomposition $R = \{G_\alpha / \alpha \in H\}$ of $K_n^k$ by the group $H$ was constructed, the factors of which have the diameters equal to $d$.

The factors $G_\alpha$ are constructed as follows:

1. The factor $G_0$ corresponding to zero of $H$ contains a path of length $d - 2$ formed by the edges

$$h_i = \{1_{t_i}, 1_{t_i+1}, \ldots, 1_{t_{i+1}}\} \quad \text{where} \quad t_i = 1 + (i-1)(k-1)\,, \quad 1 \leqq i \leqq d-2$$

and the edge $f = \{1_1, 2_1, \ldots, k_1\}$.

2. $\{B \cup \{2_j\}\} \in G_0$ for any $(k-1)$-tuple $B \subset A_i = \{2_i, 3_i, \ldots, m_i\}$, $i \neq j$, $i, j = 1, 2, \ldots, t$.

3. $G_\alpha = \alpha(G_0)$ for every $\alpha \in H$.

Obviously the diameter of $G_\alpha$ is equal to $d$.

4. All the remaining edges are divided into factors preserving the diameters.

Now we shall modify this decomposition $R$ of the hypergraph $K_n^k$ in order to obtain the required diameters of the factors. Put $S = \{f_i = \{1_i, 2_i, \dots, k_i\} / i = 2, 3, \dots, t\}$. Choose the edges $\alpha(S)$ for every $\alpha \in H$ from the factors of the decomposition $R$. It is evident that the diameters of the factors remain the same as before. Let $p(d_i)$ denote the number of the members of the set $D = \{d_1, d_2, \dots, d_m\}$ which are equal to $d_i$.

(a) Assume $d_1 = 2$. Then insert the edges $f_i$, $i = 2, 3, \dots, t$ into the factor $G_0$ and the edges $\beta^{j-1}(f_i)$ into the factor $G_{\beta^{j-1}}$ where $2 \leqq j \leqq p(2)$.

(b) Assume $d_1 > 2$. If $d_1 = d_m$ then all the diameters are the same and equal to $d_m$. However, by Proposition 3 the number $G_m^k(d_m)$ exists. Since $F_m^k(d_m) \leqq G_m^k(d_m)$ the proof is completed. So we can assume $d_1 < d_m$. Then there exists the greatest index $x$ such that $d_{G_0}(1_x, 1_t) = d_1 - 2$ (the index $G_0$ denotes the distance taken in $G_0$). Insert the edges $f_i$, $i = 2, 3, \dots, x$ into the factor $G_0$ and the edges $\beta^{j-1}(f_i)$ into the factors $G_{\beta^{j-1}}$, $2 \leqq j \leqq p(d_1)$, $2 \leqq i \leqq x$.

Now consider an arbitrary $d_r \in D$, $d_1 < d_r < d_m$. If such $d_r$ does not exist the required decomposition is achieved, because the factors $G_{\beta s}$ where $p(d_1) \leqq s < m$ have the diameter equal to $d_m$ and the remaining edges of the sets $\alpha(S)$ can be put back into the same factors from which they were chosen.

Let such a $d_r$ exist. Denote by $r^0$ the smallest number such that $d_{r^0} = d_r$. We can assume without loss of generality that $j \leqq r$ for every $d_j = d_r$. Since $d_r < d_m$, there exists the greatest index $x$ such that

$$d_{G\beta^{r^0-1}}((r^0)_x, (r^0)_t) = d_r - 2 \, .$$

Then let $\beta^s(f_i) \in G_{\beta^s}$ where $r^0 - 1 \leqq s < r$. Insert all the other edges from the sets $\beta^s(S)$, $r^0 - 1 \leqq s < r$ into $G_{\beta^{r^0-1}}$. It is evident that the diameter of the factors $G_{\beta^s}$ constructed in this way is equal to $d_r$.

This construction can be applied to every diameter $d_q < d_m$. Denote by $m^0$ the smallest number such that $d_{m^0} = d_m$ and insert the edges $\beta^s(S)$, $m^0 - 1 \leqq s < m$ into the same factors from which they were chosen.

It can be easily verified that $\{G_\alpha / \alpha \in H\}$ is the required decomposition of the hypergraph $K_n^k$ into factors with diameters $d_1, d_2, \dots, d_m$.

II. Assume $d_m = 2$. Then all the diameters are equal to 2. By Proposition 3 the number $G_m^k(2)$ exists and since $F_m^k(2) \leqq G_m^k(2) \leqq 2m$ the proof is complete.

In the following theorem the existence of the number $F^k(d_1, d_2, \dots, d_m)$ in the case $(m, k) > 1$ will be investigated. The next proposition proved in [7] will be very useful for the purpose.

**Proposition 4.** *Let* $3 \leqq k < m$, $(m, k) > 1$, $d \geqq 2$ *be integers. Then* $G_m^k(d)$ *exists and*

$$G_m^k(d) \leqq km[(d-2)(k-1)+1] \quad \text{if} \quad d \geqq 3 \, ,$$

$$G_m^k(2) \leqq 2mk \, .$$

**Theorem 2.** *Let* $3 \leqq k < m$, $(m, k) > 1$, $2 \leqq d_1 \leqq d_2 \leqq \ldots \leqq d_m$ *be integers. Then* $F^k(d_1, d_2, \ldots, d_m)$ *exists and*

$$F^k(d_1, d_2, \ldots, d_m) \leqq mk[(d_m - 2)(k - 1) + 1] \quad \text{if} \quad d_m \geqq 3\,,$$

$$F^k(d_1, d_2, \ldots, d_m) \leqq 2mk \quad \text{if} \quad d_m = 2\,.$$

Proof. I. Assume $d_m \geqq 3$. Put $n = kmt$ where $t = (d_m - 2)(k - 1) + 1$. Denote the vertices of the hypergraph $K_n^k$ by $i_j$, $1 \leqq i \leqq km$, $1 \leqq j \leqq t$. Let $H$ be the group generated by the permutation $\beta = (1_1, 2_1, \ldots, (km)_1)(1_2, 2_2, \ldots, (km)_2) \ldots \ldots (1_t, 2_t, \ldots, (km)_t)$. In the proof of Proposition 4, a decomposition $R$ of $K_n^k$ by the cyclic group $H$ of order $km$ into $m$ isomorphic factors with diameter $d_m$ was found.

The factors $G_\alpha$ were constructed as follows:

1. $h_j = \{m_j, (2m)_j, \ldots, (km)_j\} \in G_0$ for $1 \leqq j \leqq t$, where $G_0$ is the factor corresponding to the zero element of $H$.

2. The factor $G_0$ contains a path of length $d_m - 2$:
Put $f_s = \{m_{t_s}, (2m)_{t_s+1}, \ldots, (km)_{t_{s+1}}\}$, $t_s = 1 + s(k - 1)$, $0 \leqq s < d_m - 2$. Then $\beta^{mr}(f_s) \in G_0$ for every $1 \leqq r \leqq k$, $0 \leqq s < d_m - 2$.

3. Put $A_j = \{i_j/1 \leqq i \leqq km\} - h_j$ and take an arbitrary $(k - 1)$-tuple $B \subset A_j$. Then $\beta^{mr}(B \cup (m + 1)_i) \in G_0$ for every $i \neq j$, $i, j = 1, 2, \ldots, t$, $1 \leqq r \leqq k$. The edge $g \subset \{1_1, 2_1, \ldots, (km)_1\}$ which contains the vertices $m_1$, $(m + 1)_1$ and no other vertices of $h_1$ is also inserted into the factor $G_0$, together with its images $\beta^{mr}(g)$, $1 \leqq r \leqq k$.

Obviously the diameter of $G_0$ is equal to $d_m$. If we put $G_{\beta^i} = \beta^i(G_0)$ for every $0 \leqq i < m$, then the factors $G_{\beta^i}$ have the diameters equal to $d_m$.

All the remaining edges are divided into factors preserving the diameters.

Now we shall modify this decomposition $R$ with the aim to obtain the required diameters of the factors. Put

$$S = \{e_i = \{m_i, (m + 1)_i, \ldots, (m + k - 1)_i\}/2 \leqq i \leqq t\}\,.$$

Choose the edges $\alpha(S)$ for every $\alpha \in H$ from the factors of the decomposition $R$. It is evident that the diameters of the factors remain the same as before. Let $p(d_i)$ denote the number of members of the set $D = \{d_1, d_2, \ldots, d_m\}$ which are equal to $d_i$.

(a) Suppose $d_1 = 1$. Then insert the edges $\beta^{mr}(e_i)$ into the factor $G_0$ and the edges $\beta^{r(j-1)}(e_i)$ into the factor $G_{\beta^{j-1}}$, where $1 \leqq i \leqq t$, $1 \leqq r \leqq k$, $2 \leqq j \leqq p(2)$.

(b) Suppose $d_1 > 2$. If $d_1 = d_m$ then all the diameters are the same and equal to $d_m$. However, by Proposition 4 the number $G_m^k(d_m)$ exists. Since $F_m^k(d_m) \leqq G_m^k(d_m)$ the proof is completed. So it can be supposed $d_1 < d_m$. Then there exists the greatest index $x$ such that $d_{G_0}(m_x, m_t) = d_1 - 2$. Insert the edges $\beta^{mr}(e_i)$ into the factor $G_0$ and the edges $\beta^{r(j-1)}(e_i)$ into the factors $G_{\beta^{j-1}}$ where $2 \leqq j \leqq p(d_1)$, $2 \leqq i \leqq x$, $1 \leqq r \leqq k$.

Now consider an arbitrary $d_r \in D$, $d_1 < d_r < d_m$. If such $d_r$ does not exist the required decomposition is achieved, because the factors $G_{\beta^s}$, $p(d_1) \leqq s < m$ have the diameters equal to $d_m$ and the remaining edges of the sets $\alpha(S)$ can be put back into the same factors from which they were chosen.

Let such a $d_r$ exist. It can be supposed without loss of generality that $j \leqq r$ for every $d_j = d_r$. Denote by $r^0$ the smallest number such that $d_{r^0} = d_r$. Since $d_r < d_m$ there exists the greatest index $x$ such that

$$d_{G\beta^{r^0-1}}((m + r^0)_x, (m + r^0)_t) = d_r - 2 .$$

Then let $\beta^{ms}(e_i) \in G_{\beta^s}$ where $r^0 - 1 \leqq s < r$, $2 \leqq i \leqq x$. Insert all the other edges from the sets $\beta^s(S)$, $r^0 - 1 \leqq s < r$ into the factor, say, $G_{\beta^{r^0-1}}$. It is evident that the diameters of the factors $G_{\beta^s}$, $r^0 - 1 \leqq s < r$ constructed in this way are equal to $d_r$.

This construction can be applied to every $d_q < d_m$. Denote by $m^0$ the smallest number such that $d_{m^0} = d_m$ and insert the edges $\beta^s(S)$, $m^0 - 1 \leqq s < m$ into the same factors from which they were chosen.

It can be easily verified that $\{G_\alpha / \alpha \in H\}$ is the required decomposition of the hypergraph $K_n^k$ into factors with diameters $d_1, d_2, \ldots, d_m$.

II. Suppose $d_m = 2$. Then all the diameters are equal to 2. By Proposition 4 the number $G_m^k(2)$ exists and since $F_m^k(2) \leqq G_m^k(2) \leqq 2km$ the proof is complete.

The preceding results yield the following theorem.

**Theorem 3.** *Let* $3 \leqq k < m$, $2 \leqq d_1 \leqq d_2 \leqq \ldots \leqq d_m$, $d_m \geqq 3$, *be integers.*

*If* $(m, k) = 1$ *and* $n \geqq m[(d_m - 2)(k - 1) + 1]$ *then* $K_n^k$ *can be decomposed into m factors with diameters* $d_1, d_2, \ldots, d_m$.

*If* $(m, k) > 1$ *and* $n \geqq km[(d_m - 2)(k - 1) + 1]$ *then* $K_n^k$ *can be also decomposed into m factors with diameters* $d_1, d_2, \ldots, d_m$.

Proof. Applying Proposition 2 to Theorems 1 and 2 we obtain the above statement.

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