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THE STRUCTURE OF TANGENTORS AND THEIR MORPHISMS

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1. PREFACE

In his early works, CH. EHRESMANN [3–8] established the jet theory to use it in higher-order differential geometry, especially to construct higher-order tangent spaces of manifolds and jet prolongations of vector bundles. Subsidiary constructions for manifold and vector-bundle morphisms were performed, too. In this way, tangent functors and those of jet prolongations of vector bundles were obtained; see for example [1, 10, 11].

Certain common features of the functors mentioned above have led us to consider them as particular cases of the so-called tangentors.

A functor T from the category of $(r + n)$-times continuously differentiable manifolds into the category of r-times continuously differentiable vector bundles is said to be *a manifold tangentor of order n* iff T satisfies the conditions given below.

a) If A is a manifold then the basis of $T(A)$ coincides with A. If α is a manifold morphism then the basis of $T(\alpha)$ coincides with α.

b) Let $\alpha_1, \alpha_2 : A \to A'$ be manifold morphisms, $p \in A$, $p' \in A'$. If α_1 and α_2 belong to the same n-jet with the source p and the target p' then $T_p(\alpha_1), T_p(\alpha_2) : T_p(A) \to T_{p'}(A')$ coincide.

c) Let A, A' be manifolds, A an open submanifold of A', ι the natural embedding from A into A'. Then $T(A)$ is an open subbundle of $T(A')$ and $T(\iota)$ is the natural embedding from $T(A)$ into $T(A')$.

d) If A is a local manifold then $T(A)$ is a local bundle.

A functor T from the category of $(r + n)$-times continuously differentiable vector bundles into the category of r-times continuously differentiable vector bundles is said to be *a bundle tangentor of order n* iff T satisfies conditions analogous to the foregoing ones.

An attempt is made here to survey the structure of tangentors and their morphisms. To achieve the goal, we proceed in the following way.

a) Bundles are treated as families of their fibers rather than projections because non-vector bundles cannot be omitted here.

b) Differential calculus in Banach spaces is used to avoid the abundance of indices [2]. For the sake of clarity, induction is preferred whenever possible.

c) To include various cases (real, complex and others) without separating them, manifolds are supposed to be modelled on normable topological right R-modules; R is a normable topological commutative and associative ring with unit, containing the field of reals as a subring. This is made possible by keeping the rules of the Banach-space calculus in this more general case as well.

The exposition of the present paper is in the spirit of [10]. For a monograph related to the subject, see [12]. Finally, the author expresses his gratitude to Professor I. KOLÁŘ for valuable consulation and kind help.

2. PRELIMINARIES

Basic agreements, definitions and notation are listed here. The reader can omit this part and proceed immediately to the next sections, consulting the list given below if necessary.

2.1. Notation. Let R be a normable topological commutative and associative ring with unit, containing the field of reals as a subring. The symbol Mod denotes the category of normable topological right R-modules and continuous R-linear mappings.

2.2. Agreement. The symbol r denotes a non-negative integer or ∞.

2.3. Notation. The symbol Lom^r denotes the category of open sets of Mod-objects and r-times continuously differentiable mappings.

2.4. Notation. The symbol Man^r denotes the category of r-times continuously differentiable manifolds and r-times continuously differentiable manifold morphisms. Manifolds are supposed to be modelled on Mod-objects.

Remark that Lom^r can be regarded as a subcategory of Man^r.

2.5. Definition. A category K is said to be *a* Lom^r-*category* iff: (1) $\mathrm{Hom}(A, A')$ is a Lom^r-object for any K-objects A, A', (2) the composition mapping from $\mathrm{Hom}(A, A') \times \mathrm{Hom}(A', A'')$ into $\mathrm{Hom}(A, A'')$ is a Lom^r-morphism for any K-objects A, A', A''.

2.6. Definition. A category K is said to be *a* Man^r-*category* iff: (1) $\mathrm{Hom}(A, A')$ is a Man^r-object for any K-objects A, A', (2) the composition mapping from

Hom $(A, A') \times$ Hom (A', A'') into Hom (A, A'') is a Man^r-morphism for any K-objects A, A', A''.

Remark that any Lom^r-category can be regarded as a Man^r-category.

2.7. Definition. A functor $T: K \to K'$ is said to be *a* Lom^r*-functor* iff: (1) K and K' are Lom^r-categories, (2) T: Hom $(A, A') \to$ Hom $(T(A), T(A'))$ is a Lom^r-morphism for any K-objects A, A'.

2.8. Definition. A functor $T: K \to K'$ is said to be *a* Man^r*-functor* iff: (1) K and K' are Man^r-categories, (2) T: Hom $(A, A') \to$ Hom $(T(A), T(A'))$ is a Man^r-morphism for any K-objects A, A'.

Remark that any Lom^r-functor can be regarded as a Man^r-functor.

2.9. Agreement. Let F be a finite sequence. The symbol LF denotes the sequence obtained from F by deleting the last member. The symbol RT denotes the sequence obtained from F by deleting the first member.

2.10. Agreement. Let F be a pair of finite sequences. The symbol $\tilde{L}F$ denotes the pair $(L(LF), L(RF))$. The symbol $\tilde{R}F$ denotes the pair $(R(LF), R(RF))$.

2.11. Definition. *The local r-times continuously differentiable bundlification of a* Man^r*-category* K is the category Lob$^r(K)$ of local r-times continuously differentiable K-bundles and r-times continuously differentiable K-bundle morphisms, defined as follows.

a) A is a Lob$^r(K)$-object iff: (1) A is a pair, (2) LA is a Lom^r-object (called the basis of A), (3) RA is a family of K-objects (called fibers of A) indexed by LA, (4) if $p_1, p_2 \in \in LA$ then $p_1RA = p_2RA$.

b) Let A, A' be Lob$^r(K)$-objects. α is a Lob$^r(K)$-morphism from A into A' iff: (1) α is a pair, (2) $L\alpha$ is a Lom^r-morphism from LA into LA' (called the basis of α), (3) $R\alpha$ is a family of K-morphisms (called fiber morphisms of α) indexed by LA, (4) if $p \in LA$ then $pR\alpha$ is a morphism from pRA into $(pL\alpha)\, RA'$.

c) If $\alpha : A \to A'$, $\alpha' : A' \to A''$ are Lob$^r(K)$-morphisms then $L(\alpha \circ \alpha') = L\alpha \circ L\alpha'$ and $pR(\alpha \circ \alpha') = pR\alpha \circ (pL\alpha)\, R\alpha'$ for each $p \in LA$.

2.12. Definition. *The r-times continuously differentiable bundlification of a* Man^r*-category* K is the category Bun$^r(K)$ of r-times continuously differentiable K-bundles and r-times continuously differentiable K-bundle morphisms, defined as follows.

a) A is a Bun$^r(K)$-object iff: (1) A is a pair, (2) LA is a topological space (called the basis of A), (3) RA is a family of K-objects (called fibers of A) indexed by LA, (4) A is equipped with an r-times continuously differentiable atlas.

Here an atlas of A means any set with the properties: (1) each element of the set is a pair χ such that $L\chi$ is a homeomorphism from an open set of LA onto a Lom^r-

object and $R\chi$ is a family of K-isomorphisms indexed by the domain of $L\chi$, provided that the domain of $pR\chi$ coincides with pRA for each p from the domain of $L\chi$ and the codomains of $p_1R\chi$ and $p_2R\chi$ coincide for each p_1, p_2 from the domain of $L\chi$, (2) if χ_1 and χ_2 are elements of the set then $(L\chi_1)^{-1} \circ (L\chi_2)$ and $p \mapsto ((p(L\chi_1)^{-1}) . R\chi_1)^{-1} \circ ((p(L\chi_1)^{-1}) R\chi_2)$ are r-times continuously differentiable mappings, (3) if $p \in LA$ then p is from the domain of $L\chi$ for some chart χ from the set.

b) Let A, A' be $\mathrm{Bun}^r(K)$-objects. α is a $\mathrm{Bun}^r(K)$-morphism from A into A' iff α satisfies the conditions: (1) α is a pair, (2) $L\alpha$ is a continuous mapping from LA into LA', (3) $R\alpha$ is a family of K-morphisms (called fiber morphisms of α) indexed by LA, (4) if $p \in LA$ then $pR\alpha$ is a morphism from pRA into $(pL\alpha)\,RA'$, (5) if χ and χ' are charts from the atlases of A and A' respectively then $(L\chi)^{-1} \circ L\alpha \circ (L\chi')$ and $p \mapsto ((p(L\chi)^{-1} R\chi)^{-1} \circ (p(L\chi)^{-1})\, R\alpha \circ (((p(L\chi)^{-1})\, L\alpha)\, R\chi')$ are r-times continuously differentiable mappings.

c) If $\alpha : A \to A'$, $\alpha' : A' \to A''$ are $\mathrm{Bun}^r(K)$-morphisms then $L(\alpha \circ \alpha') = L\alpha \circ L\alpha'$ and $pR(\alpha \circ \alpha') = pR\alpha \circ (pL\alpha)\, R\alpha'$ for each $p \in LA$.

Remark that $\mathrm{Lob}^r(K)$ can be regarded as a subcategory of $\mathrm{Bun}^r(K)$.

2.13. Definition. Let A, A' be $\mathrm{Lob}^r(K)$-objects.

a) A is said to be *an open subbundle of A* iff: (1) LA and LA' are open sets of the same Mod-object, (2) LA is a subset of LA', (3) if $p \in LA$ then $pRA = pRA'$.

b) If A is an open subbundle of A then the natural embedding from A into A' is the $\mathrm{Lob}^r(K)$-morphism $\iota : A \to A'$ defined as follows: (1) if $p \in LA$ then $pL\iota = p$, (2) if $p \in LA$ then $pR\iota$ is the identical K-morphism of pRA.

2.14. Definition. Let A, A' be $\mathrm{Bun}^r(K)$-objects.

a) A is said to be *an open subbundle of* A' iff the atlas of A is a subset of the atlas of A'.

b) If A is an open subbundle of A' then the natural embedding from A into A' is the $\mathrm{Bun}^r(K)$-morphism $\iota : A \to A'$ defined as follows: (1) if $p \in LA$ then $pL\iota = p$, (2) if $p \in LA$ then $pR\iota$ is the identical K-morphism of pRA.

2.15. Definition. *The local r-times continuously differentiable bundlification of a* Man^r*-functor* $T: K \to K'$ is the functor $\mathrm{Lob}^r(T) : \mathrm{Lob}^r(K) \to \mathrm{Lob}^r(K')$ defined as follows.

a) If A is a $\mathrm{Lob}^r(K)$-object then $L(\mathrm{Lob}^r(T)(A)) = LA$ and $pR(\mathrm{Lob}^r(T)(A)) = T(pRA)$ for each $p \in LA$.

b) If $\alpha : A \to A'$ is a $\mathrm{Lob}^r(K)$-morphism then $L(\mathrm{Lob}^r(T)(\alpha)) = L\alpha$ and $pR(\mathrm{Lob}^r(T)(\alpha)) = T(pRA)$ for each $p \in LA$.

2.16. Definition. Let T, $T' : H \to K$ be Man^r-functors. *The local r-times continuously differentiable bundlification of a morphism* $M : T \to T'$ is the morphism $\mathrm{Lob}^r(M)$: $\mathrm{Lob}^r(T) \to \mathrm{Lob}^r(T')$ defined as follows.

If A is a $\mathrm{Lob}^r(H)$-object then $L(\mathrm{Lob}^r(M)(A))$ is the identical mapping of LA and $pR(\mathrm{Lob}^r(M)(A)) = M(pRA)$ for each $p \in LA$.

2.17. Agreement. Let $T, T' : H \to K$ be functors and let $M : T \to T'$ be a morphism.

a) If $P : G \to H$ is a functor then the symbol $M \circ P$ denotes the morphism from $T \circ P$ into $T' \circ P$ defined as follows: if A is a G-object then $(M \circ P)(A) = M(P(A))$.

b) If $Q : H \to K$ is a functor then the symbol $Q \circ M$ denotes the morphism from $Q \circ T$ into $Q \circ T'$ defined as follows: if A is an H-object then $(Q \circ M)(A) = Q(M(A))$.

2.18. Agreement. Let A be a Mod-object. If U is an open set in A then $[U]$ denotes A.

2.19. Agreement. $pD^i\alpha$ denotes the i-th derivative of α at p.

2.20. Agreement. $a\alpha$ will be written instead of $(a\alpha_1, \ldots, a\alpha_n)$.

2.21. Agreement. The symbol n denotes a positive integer.

3. THE STRUCTURE OF MANIFOLD TANGENTORS AND THEIR MORPHISMS

To define manifold tangentors, the Taylor manifold functors will be introduced first. In the local case, one might say that the functors are universal local manifold tangentors. The Taylor manifold functor of order n maps Lom^{r+n} into $\mathrm{Lob}^r(\mathrm{Tac}^n)$, where Tac^n is the n-th Taylor category.

3.1. Definition. *The n-th Leibniz category* is the Lom^∞-category Lec^n defined as follows.

a) A is a Lec^n-object iff A is a Mod-object.

b) If A, A' are Lec^n-objects then

$$\mathrm{Hom}^n(A, A') = \mathrm{Hom}(A, A') \times \ldots \times \mathrm{Hom}(\underbrace{A, \ldots, \mathrm{Hom}(A}_{n\text{-times}}, A') \ldots) .$$

c) If $\alpha : A \to A'$, $\alpha' : A' \to A''$ are Lec^1-morphisms then $\alpha \circ \alpha'$ is their composition as Mod-morphisms. If $\alpha : A \to A'$, $\alpha' : A' \to A''$ are Lec^{n+1}-morphisms then $(\alpha \circ \alpha')_1 = \alpha_1 \circ \alpha'_1$ and

$$aR(\alpha \circ \alpha') = (aR\alpha)((L\alpha)\, D(\xi \vdash \xi \circ L\alpha')) +$$
$$+ ((a\alpha_1)\, R\alpha')((L\alpha')\, D(\xi' \vdash L\alpha \circ \xi')) , \quad a \in A ,$$

where ξ and ξ' are running $\mathrm{Hom}^n(A, A')$ and $\mathrm{Hom}^n(A', A'')$, respectively.

The correctness of the above definition can be established easily by induction, together with the following propositions.

a) Let A be a Lec^n-object. Denote by ι the identical Mod-morphism of A. Then the n-tuple $(\iota, \emptyset, \ldots, \emptyset)$ is the identical Lec^n-morphism of A.

b) If $\alpha : A \to A'$, $\alpha' : A' \to A''$ are Lec^{n+1}-morphisms then $L(\alpha \circ \alpha') = L\alpha \circ L\alpha'$.

3.2. Definition. *The n-th Taylor manifold functor* is the functor $\mathrm{Tmf}^n : \mathrm{Lom}^{r+n} \to \to \mathrm{Lob}^r(\mathrm{Lec}^n)$ defined as follows.

a) If A is a Lom^{r+n}-object then $L(\mathrm{Tmf}^n(A)) = A$ and $pR(\mathrm{Tmf}^n(A)) = [A]$ for each $p \in A$.

b) If $\alpha : A \to A'$ is a Lom^{r+n}-morphism then $L(\mathrm{Tmf}^n(\alpha)) = \alpha$ and $pR(\mathrm{Tmf}^n(\alpha)) = = (pD^1\alpha, \ldots, pD^n\alpha)$ for each $p \in A$.

3.3. Agreement. Let $\alpha : A \to A'$ be a Lec^n-morphism. The symbol $[\alpha]$ denotes the Lom^∞-morphism from A into A' defined by the equality

$$p[\alpha] = p\alpha_1(1/1!) + \ldots + p \ldots p\alpha_n(1/n!)\,, \quad p \in A\,.$$

One can see at once that if $\alpha : A \to A'$ is a Lec^n-morphism and $\alpha_1, \ldots, \alpha_n$ are symmetrical then $\emptyset[\alpha] = \emptyset'$ and $\emptyset R(\mathrm{Tmf}^n[\alpha]) = \alpha$. This enables us to prove the correctness of the definition given below. Namely, if $\alpha : A \to A'$, $\alpha' : A' \to A''$ are Tac^n-morphisms then

$$\alpha \circ \alpha' = \emptyset R(\mathrm{Tmf}^n[\alpha]) \circ \emptyset' R(\mathrm{Tmf}^n[\alpha']) =$$
$$= \emptyset R(\mathrm{Tmf}^n[\alpha] \circ \mathrm{Tmf}^n[\alpha']) = \emptyset R(\mathrm{Tmf}^n([\alpha] \circ [\alpha']))\,;$$

thus, $\alpha \circ \alpha'$ is a Tac^n-morphism as well.

3.4. Definition. *The n-th Taylor category* is the Lom^∞-subcategory Tac^n of Lec^n defined as follows.

a) Every Lec^n-object is a Tac^n-object.

b) A Lec^n-morphism $\alpha : A \to A'$ is a Tac^n-morphism iff $\alpha_1, \ldots, \alpha_n$ are symmetrical.

Evidently, Tmf^n maps Lom^{r+n} into $\mathrm{Lob}^r(\mathrm{Tac}^n)$. Note also that I. Kolář observed Lec^n and Tac^n to be closely related to the semi-holonomic and holonomic jets, respectively.

Now, structural theorems concerning manifold tangentors will be given for the local case. The proofs can be carried out easily by using elementary properties of Tmf^n, although certain tricks are needed.

3.5. Definition. A functor $T : \mathrm{Lom}^{r+n} \to \mathrm{Lob}^r(K)$ is said to be *a manifold tangentor of order n* iff T satisfies the conditions given below.

a) If A is a Lom^{r+n}-object then the basis of $T(A)$ coincides with A. If α is a Lom^{r+n}-morphism then the basis of $T(\alpha)$ coincides with α.

b) Let $\alpha_1, \alpha_2 : A \to A'$ be Lom^{r+n}-morphisms, $p \in A$. If $p\alpha_1 = p\alpha_2$ and $pR(\mathrm{Tmf}^n(\alpha_1)) = pR(\mathrm{Tmf}^n(\alpha_2))$ then $pR(T(\alpha_1)) = pR(T(\alpha_2))$.

c) Let A, A' be Lom^{r+n}-objects, A an open submanifold of A', ι the natural embedding from A into A'. Then $T(A)$ is an open subbundle of $T(A')$ and $T(\iota)$ is the natural embedding from $T(A)$ into $T(A')$.

In the local case, manifold tangentors can be constructed by the following method.

3.6. Theorem. *If* $T : \mathrm{Tac}^n \to K$ *is a* Man^r*-functor then* $\mathrm{Lob}^r(T) \circ \mathrm{Tmf}^n$ *is a manifold tangentor of order* n.

It will be shown that in the local case any manifold tangentor of order n is isomorphic to $\mathrm{Lob}^r(T) \circ \mathrm{Tmf}^n$ for some T, namely, for the rectification of the tangentor.

3.7. Definition. Let $T : \mathrm{Lom}^{r+n} \to \mathrm{Lob}^r(K)$ be a manifold tangentor of order n. *The rectification of* T is the Man^r-functor $\mathrm{Rtf}(T) : \mathrm{Tac}^n \to K$ defined as follows.

a) If A is a Tac^n-object then $\mathrm{Rtf}(T)(A) = \emptyset R(T(A))$.

b) If $\alpha : A \to A'$ is a Tac^n-morphism then $\mathrm{Rtf}(T)(\alpha) = \emptyset R(T[\alpha])$.

3.8. Theorem. *If* $T : \mathrm{Tac}^n \to K$ *is a* Man^r*-functor then the rectification of* $\mathrm{Lob}^r(T) \circ \mathrm{Tmf}^n$ *coincides with* T.

The isomorphism mentioned above is the torsion of the tangentor.

3.9. Definition. Let $T : \mathrm{Lom}^{r+n} \to \mathrm{Lob}^r(K)$ be a manifold tangentor of order n. *The torsion of* T is the isomorphism $\mathrm{Tsn}(T) : \mathrm{Lob}^r(\mathrm{Rtf}(T)) \circ \mathrm{Tmf}^n \to T$ defined as follows.

If A is a Lom^{r+n}-object then $L(\mathrm{Tsn}(T)(A))$ is the identical mapping of A and $pR(\mathrm{Tsn}(T)(A)) = \emptyset R(T(a \mapsto a + p))$ for each $p \in A$; a is running $[A]$.

3.10. Theorem. *If* $T : \mathrm{Tac}^n \to K$ *is a* Man^r*-functor then the torsion of* $\mathrm{Lob}^r(T) \circ$ $\circ\, \mathrm{Tmf}^n$ *coincides with its identical morphism.*

Torsion-free manifold tangentors or certain fragments of them are used more frequently than the others. Nevertheless, manifold tangentors with any torsions are closely related to higher-order connections. For example, a usual connection of order n on a local manifold A is essentially an automorphism of $\mathrm{Tmf}^n(A)$, whose basis is the identical mapping of A; however, the automorphism is a fragment of the torsion of a certain manifold tangentor $T : \mathrm{Lom}^{r+n} \to \mathrm{Lob}^r(\mathrm{Tac}^n)$ or order n.

Now, the structure of manifold-tangentor morphisms will be described. Remember that the local case is being discussed.

3.11. Definition. Let $T, T' : \mathrm{Lom}^{r+n} \to \mathrm{Lob}^r(K)$ be manifold tangentors of order n and let $M : T \to T'$ be a morphism. *The rectification of* M is the morphism $\mathrm{Rtf}(M) :$ $: \mathrm{Rtf}(T) \to \mathrm{Rtf}(T')$ defined as follows.

If A is a Tac^n-object then $\mathrm{Rtf}(M)(A) = \emptyset R(M(A))$.

3.12. Theorem. *Let* $T, T' : \mathrm{Tac}^n \to K$ *be* Man^r*-functors and let* $M : T \to T'$ *be a morphism. Then the rectification of* $\mathrm{Lob}^r(M) \circ \mathrm{Tmf}^n$ *coincides with* M.

3.13. Theorem. *Let* $T, T' : \mathrm{Lom}^{r+n} \to \mathrm{Lob}^r(K)$ *be manifold tangentors of order* n *and let* $M : T \to T'$ *be a morphism. Then the diagram*

$$\begin{array}{ccc} \mathrm{Lob}^r(\mathrm{Rtf}(T)) \circ \mathrm{Tmf}^n & \xrightarrow{\mathrm{Lob}^r(\mathrm{Rtf}(M)) \circ \mathrm{Tmf}^n} & \mathrm{Lob}^r(\mathrm{Rtf}(T')) \circ \mathrm{Tmf}^n \\ {\scriptstyle \mathrm{Tsn}(T)}\downarrow & & \downarrow{\scriptstyle \mathrm{Tsn}(T')} \\ T & \xrightarrow[M]{} & T' \end{array}$$

is commuting.

Finally, the global case is to be discussed. According to ideas of Ch. Ehresmann [9], I. Kolář (his lecture, Summer school in modern geometry, Poprad, CSSR, July 1975) and others, we define manifold tangentors to be extensions of those from Definition 3.5. Hence, the manifold tangentors of order n map Man^{r+n} into $\mathrm{Bun}^r(K)$. Recall that any such extension is determined uniquely up to an isomorphism. If $M : T \to T'$ is a morphism of local manifold tangentors and P, P' are extensions of T, T' respectively then there is an extension $N : P \to P'$ of M, determined uniquely by M.

The survey concerning the manifold tangentors is accomplished.

4. THE STRUCTURE OF BUNDLE TANGENTORS AND THEIR MORPHISMS

The approach to the subject of the present section does not differ from that of the preceding one.

To define bundle tangentors, the Taylor bundle functors will be introduced first. In the local case, one might say that the functors are universal local bundle tangentors. The Taylor bundle functor of order n maps $\mathrm{Lob}^{r+n}(H)$ into $\mathrm{Lob}^r(\mathrm{Tex}^n(H))$, where H is a Lom^{r+n}-category and $\mathrm{Tex}^n(H)$ is the n-th Taylor extension of H.

4.1. Definition. *The n-th Leibniz extension of a* Lom^{r+n}*-category* K is the Lom^r-category $\mathrm{Lex}^n(K)$ defined as follows.

a) A is a $\mathrm{Lex}^n(K)$-object iff: (1) A is a pair, (2) LA is a Lec^n-object, (3) RA is a K-object.

b) If A, A' are $\mathrm{Lex}^n(K)$-objects then

$$\mathrm{Hom}^n(A, A') = \mathrm{Hom}^n(LA, LA') \times (\mathrm{Hom}(RA, RA) \times \dots$$
$$\dots \times \mathrm{Hom}(\underbrace{LA, \dots, \mathrm{Hom}(LA}_{n\text{-times}}, [\mathrm{Hom}(RA, RA')]) \cdots)) \,.$$

c) If $\alpha : A \to A'$, $\alpha' : A' \to A''$ are $\mathrm{Lex}^1(K)$-morphisms then

$$L(\alpha \circ \alpha') = L\alpha \circ L\alpha' , \quad (R(\alpha \circ \alpha'))_\emptyset = (R\alpha)_\emptyset \circ (R\alpha')_\emptyset ,$$

$$a(R(\alpha \circ \alpha'))_1 = (a(R\alpha)_1)\left((R\alpha)_\emptyset \, D(\xi \mapsto \xi \circ (R\alpha')_\emptyset)\right) +$$

$$+ \left((aL\alpha)(R\alpha')_1\right)\left((R\alpha')_\emptyset \, D(\xi' \mapsto (R\alpha)_\emptyset \circ \xi')\right), \quad a \in LA ,$$

where ξ and ξ' are running $\mathrm{Hom}(RA, RA')$ and $\mathrm{Hom}(RA', RA'')$, respectively. If $\alpha : A \to A'$, $\alpha' : A' \to A''$ are $\mathrm{Lex}^{n+1}(K)$-morphisms then

$$(L(\alpha \circ \alpha'))_1 = (L\alpha)_1 \circ (L\alpha')_1 , \quad (R(\alpha \circ \alpha'))_\emptyset = (R\alpha)_\emptyset \circ (R\alpha')_\emptyset ,$$

$$a\tilde{R}(\alpha \circ \alpha') = (a\tilde{R}\alpha)\left((\tilde{L}\alpha) \, D(\xi \mapsto \xi \circ \tilde{R}\alpha')\right) +$$

$$+ \left((a(\tilde{L}\alpha)_1)\, \tilde{R}\alpha'\right)\left((\tilde{L}\alpha') \, D(\xi' \mapsto \tilde{R}\alpha \circ \xi')\right), \quad a \in LA ,$$

where ξ and ξ' are running $\mathrm{Hom}^n(A, A')$ and $\mathrm{Hom}^n(A', A'')$ respectively.

The correctness of the above definition can be established easily by induction together with the following propositions.

a) Let A be a $\mathrm{Lex}^n(K)$-object. Denote by ι the identical Lec^n-morphism of LA and by ε the identical K-morphism of RA. Then $(\iota, (\varepsilon, \emptyset, \ldots, \emptyset))$ is the identical $\mathrm{Lex}^n(K)$-morphism of A.

b) If $\alpha : A \to A'$, $\alpha' : A' \to A''$ are $\mathrm{Lex}^{n+1}(K)$-morphisms then $L(\alpha \circ \alpha') = L\alpha \circ L\alpha'$.

4.2. Definition. *The n-th Leibniz extension of a* Lom^{r+n}*-functor* $T : K \to K'$ is the Lom^r-functor $\mathrm{Lex}^n(T) : \mathrm{Lex}^n(K) \to \mathrm{Lex}^n(K)$ defined as follows.

a) If A is a $\mathrm{Lex}^n(K)$-object then $\mathrm{Lex}^n(T)(A) = (LA, T(RA))$.

b) If $\alpha : A \to A'$ is a $\mathrm{Lex}^1(K)$-morphism then

$$L(\mathrm{Lex}^1(T)(\alpha)) = L\alpha , \quad (R(\mathrm{Lex}^1(T)(\alpha)))_\emptyset = T(R\alpha)_\emptyset ,$$

$$a(R(\mathrm{Lex}^1(T)(\alpha)))_1 = (a(R\alpha)_1)\left((R\alpha)_\emptyset \, D(\xi \mapsto T(\xi))\right), \quad a \in LA ,$$

where ξ is running $\mathrm{Hom}(RA, RA')$. If $\alpha : A \to A'$, $\alpha' : A' \to A''$ are $\mathrm{Lex}^{n+1}(K)$-morphisms then

$$(L(\mathrm{Lex}^{n+1}(T)(\alpha)))_1 = (L\alpha)_1 , \quad (R(\mathrm{Lex}^{n+1}(T)(\alpha)))_\emptyset = T(R\alpha)_\emptyset ,$$

$$a\tilde{R}(\mathrm{Lex}^{n+1}(T)(\alpha)) = (a\tilde{R}\alpha)\left((\tilde{L}\alpha) \, D(\xi \mapsto \mathrm{Lex}^n(T)(\xi))\right), \quad a \in LA ,$$

where ξ is running $\mathrm{Hom}^n(A, A')$.

The correctness of the above definition can be established easily by induction together with the following proposition.

If $\alpha : A \to A'$ is a $\mathrm{Lex}^{n+1}(K)$-morphism then $\tilde{L}(\mathrm{Lex}^{n+1}(T)(\alpha)) = \mathrm{Lex}^n(T)(\tilde{L}\alpha)$.

4.3. Theorem. a) *If I is the identical functor of a* Lom^{r+n}*-category K then* $\mathrm{Lex}^n(I)$ *is the identical functor of* $\mathrm{Lex}^n(K)$.

b) *If $T : K \to K'$, $T' : K' \to K''$ are* Lom^{r+n}*-functors then* $\mathrm{Lex}^n(T' \circ T) = \mathrm{Lex}^n(T') \circ \mathrm{Lex}^n(T)$.

4.4. Definition. Let $T, T' : H \to K$ be Lom^{r+n}-functors. *The n-th Leibniz extension of a morphism $M : T \to T'$* is the morphism $\mathrm{Lex}^n(M) : \mathrm{Lex}^n(T) \to \mathrm{Lex}^n(T')$ defined as follows.

Let A be a $\mathrm{Lex}^n(H)$-object. Denote by ι the identical Lec^n-morphism of LA. Then $\mathrm{Lex}^n(M)(A) = (\iota, (M(RA), \emptyset, \ldots, \emptyset))$.

4.5. Theorem. a) *If I is the identical morphism of a* Lom^{r+n}*-functor T then* $\mathrm{Lex}^n(I)$ *is the identical morphism of* $\mathrm{Lex}^n(T)$.

b) *Let $T, T', T'' : H \to K$ be* Lom^{r+n}*-functors. If $M : T \to T'$, $M' : T' \to T''$ are morphisms then* $\mathrm{Lex}^n(M \circ M') = \mathrm{Lex}^n(M) \circ \mathrm{Lex}^n(M')$.

4.6. Definition. *The n-th Taylor bundle functor is the functor* $\mathrm{Tbf}^n : \mathrm{Lob}^{r+n}(K) \to \mathrm{Lob}^r(\mathrm{Lex}^n(K))$, where K is a Lom^{r+n}-category, defined as follows.

a) If A is a $\mathrm{Lob}^{r+n}(K)$-object then $L(\mathrm{Tbf}^n(A)) = LA$ and $pR(\mathrm{Tbf}^n(A)) = ([LA], pRA)$ for each $p \in LA$.

b) If $\alpha : A \to A'$ is a $\mathrm{Lob}^{r+n}(K)$-morphism then $L(\mathrm{Tbf}^n(\alpha)) = L\alpha$ and

$$pR(\mathrm{Tbf}^n(\alpha)) = (pR(\mathrm{Tmf}^n(L\alpha)), \ (pD^{\emptyset}(R\alpha), \ldots, pD^n(R\alpha))), \quad p \in LA .$$

4.7. Theorem. *If $T : K \to K'$ is a* Lom^{r+n}*-functor then the diagram*

$$\begin{array}{ccc}
\mathrm{Lob}^{r+n}(K) & \xrightarrow{\ \mathrm{Lob}^{r+n}(T)\ } & \mathrm{Lob}^{r+n}(K') \\
\Big\downarrow{\scriptstyle \mathrm{Tbf}^n} & & \Big\downarrow{\scriptstyle \mathrm{Tbf}^n} \\
\mathrm{Lob}^r(\mathrm{Lex}^n(K)) & \xrightarrow[\ \mathrm{Lob}^r(\mathrm{Lex}^n(T))\ ]{} & \mathrm{Lob}^r(\mathrm{Lex}^n(K'))
\end{array}$$

is commuting.

4.8. Theorem. *Let $T, T' : H \to K$ be* Lom^{r+n}*-functors. If $M : T \to T'$ is a morphism then* $\mathrm{Tbf}^n \circ \mathrm{Lob}^{r+n}(M) = \mathrm{Lob}^r(\mathrm{Lex}^n(M)) \circ \mathrm{Tbf}^n$.

4.9. Agreement. Let $\alpha : A \to A'$ be a $\mathrm{Lex}^n(K)$-morphism, where K is a Lom^{r+n}-category. The symbol $[\alpha]$ denotes the $\mathrm{Lob}^{r+n}(K)$-morphism from $(LA, p \mapsto RA)$ into $(LA', p' \mapsto RA')$ defined by the equalities $L[\alpha] = [L\alpha]$ and

$$pR[\alpha] = (R\alpha)_{\emptyset}(1/\emptyset!) + \ldots + p \ldots p(R\alpha)_n (1/n!), \quad p \in LA .$$

One can see at once that if $\alpha : A \to A'$ is a $\mathrm{Lex}^n(K)$-morphism, $L\alpha$ is a Tac^n-morphism and $(R\alpha)_1, \ldots, (R\alpha)_n$ are symmetrical then $\emptyset L[\alpha] = \emptyset'$ and $\emptyset R(\mathrm{Tbf}^n[\alpha]) = \alpha$. This enables us to prove the correctness of the definition given below. Namely, if $\alpha : A \to A'$, $\alpha' : A' \to A''$ are $\mathrm{Tex}^n(K)$-morphisms then

$$\alpha \circ \alpha' = \emptyset R(\mathrm{Tbf}^n[\alpha]) \circ \emptyset' R(\mathrm{Tbf}^n[\alpha']) =$$
$$= \emptyset R(\mathrm{Tbf}^n[\alpha] \circ \mathrm{Tbf}^n[\alpha']) = \emptyset R(\mathrm{Tbf}^n([\alpha] \circ [\alpha'])) ;$$

thus $\alpha \circ \alpha'$ is a $\mathrm{Tex}^n(K)$-morphism as well.

4.10. Definition. *The n-th Taylor extension of a* Lom^{r+n}*-category* K is the Lom^r-subcategory $\mathrm{Tex}^n(K)$ of $\mathrm{Lex}^n(K)$ defined as follows.

a) Every $\mathrm{Lex}^n(K)$-object is a $\mathrm{Tex}^n(K)$-object.

b) A $\mathrm{Lex}^n(K)$-morphism $\alpha : A \to A'$ is a $\mathrm{Tex}^n(K)$-morphism iff $L\alpha$ is a Tac^n-morphism and $(R\alpha)_1, \ldots, (R\alpha)_n$ are symmetrical.

Evidently, Tbf^n maps $\mathrm{Lob}^{r+n}(K)$ into $\mathrm{Lob}^r(\mathrm{Tex}^n(K))$. Note also that, in general, Leibniz and Taylor extensions of Man^r-categories cannot be introduced canonically.

The correctness of the definition given below results from Theorem 4.7. Indeed, if $\alpha : A \to A'$ is a $\mathrm{Tex}^n(K)$-morphism then

$$\mathrm{Tex}^n(T)(\alpha) = \mathrm{Lex}^n(T)(\emptyset R(\mathrm{Tbf}^n[\alpha])) =$$
$$= \emptyset R(\mathrm{Lob}^r(\mathrm{Lex}^n(T))(\mathrm{Tbf}^n[\alpha])) = \emptyset R(\mathrm{Tbf}^n(\mathrm{Lob}^{r+n}(T)[\alpha])) ;$$

thus $\mathrm{Tex}^n(T)(\alpha)$ is a $\mathrm{Tex}^n(K')$-morphism.

4.11. Definition. *The n-th Taylor extension of a* Lom^{r+n}*-functor* $T: K \to K'$ is the Lom^r-functor $\mathrm{Tex}^n(T) : \mathrm{Tex}^n(K) \to \mathrm{Tex}^n(K')$ defined as follows.

a) If A is a $\mathrm{Tex}^n(K)$-object then $\mathrm{Tex}^n(T)(A) = \mathrm{Lex}^n(T)(A)$.

b) If α is a $\mathrm{Tex}^n(K)$-morphism then $\mathrm{Tex}^n(T)(\alpha) = \mathrm{Lex}^n(T)(\alpha)$.

4.12. Definition. Let $T, T' : H \to K$ be Lom^{r+n}-functors. *The n-th Taylor extension of a morphism* $M : T \to T'$ is the morphism $\mathrm{Tex}^n(M) : \mathrm{Tex}^n(T) \to \mathrm{Tex}^n(T')$ defined as follows.

If A is a $\mathrm{Tex}^n(H)$-object then $\mathrm{Tex}^n(M)(A) = \mathrm{Lex}^n(M)(A)$.

Now, structural theorems concerning bundle tangentors will be given for the local case. The proofs can be carried out easily by using elementary properties of Tbf^n, although certain tricks are needed.

4.13. Definition. A functor $T: \mathrm{Lob}^{r+n}(H) \to \mathrm{Lob}^r(K)$, where H is a Lom^{r+n}-category, is said to be *a bundle tangentor of order n* iff T satisfies the conditions given below.

a) If A is a $\mathrm{Lob}^{r+n}(H)$-object then the basis of $T(A)$ coincides with the basis of A. If α is a $\mathrm{Lob}^{r+n}(H)$-morphism then the basis of $T(\alpha)$ coincides with the basis of α.

b) Let $\alpha_1, \alpha_2 : A \to A'$ be $\mathrm{Lob}^{r+n}(H)$-morphisms, $p \in LA$. If $pL\alpha_1 = pL\alpha_2$ and $pR(\mathrm{Tbf}^n(\alpha_1)) = pR(\mathrm{Tbf}^n(\alpha_2))$ then $pR(T(\alpha_1)) = pR(T(\alpha_2))$.

c) Let A, A' be $\mathrm{Lob}^{r+n}(H)$-objects, A an open subbundle of A', ι the natural embedding from A into A'. Then $T(A)$ is an open subbundle of $T(A')$ and $T(\iota)$ is the natural embedding from $T(A)$ into $T(A')$.

In the local case, bundle tangentors can be constructed by the following method.

4.14. Theorem. *If* $T: \mathrm{Tex}^n(H) \to K$ *is a* Man^r*-functor then* $\mathrm{Lob}^r(T) \circ \mathrm{Tbf}^n$ *is a bundle tangentor of order* n.

It will be shown that in the local case any bundle tangentor of order n is isomorphic to $\mathrm{Lob}^r(T) \circ \mathrm{Tbf}^n$ for some T, namely, for the rectification of the tangentor.

4.15. Definition. Let $T: \mathrm{Lob}^{r+n}(H) \to \mathrm{Lob}^r(K)$ be a bundle tangentor of order n. The rectification of T is the Man^r-functor $\mathrm{Rrf}(T) : \mathrm{Tex}^n(H) \to K$ defined as follows.

a) If A is a $\mathrm{Tex}^n(H)$-object then $\mathrm{Rtf}(T)(A) = \emptyset R(T(LA, p \mapsto RA))$.

b) If $\alpha : A \to A'$ is a $\mathrm{Tex}^n(H)$-morphism then $\mathrm{Rtf}(T)(\alpha) = \emptyset \mathrm{R}(T[\alpha])$.

4.16. Theorem. *If* $T: \mathrm{Tex}^n(H) \to K$ *is a* Man^r*-functor then the rectification of* $\mathrm{Lob}^r(T) \circ \mathrm{Tbf}^n$ *coincides with* T.

The isomorphism mentioned above is the torsion of the tangentor.

4.17. Definition. Let $T: \mathrm{Lob}^{r+n}(H) \to \mathrm{Lob}^r(K)$ be a bundle tangentor of order n. *The torsion of* T is the isomorphism $\mathrm{Tsn}(T): \mathrm{Lob}^r(\mathrm{Rtf}^n(T)) \circ \mathrm{Tbf}^n \to T$ defined as follows.

If A is a $\mathrm{Lob}^{r+n}(H)$-object then $L(\mathrm{Tsn}(T)(A))$ is the identical mapping of LA and $pR(\mathrm{Tsn}(T)(A)) = \emptyset R(T(a \mapsto a + p,\ a \mapsto \varepsilon))$ for each $p \in LA$; a is running $[LA]$ and ε denotes the identical H-morphism of pRA.

4.18. Theorem. *If* $T: \mathrm{Tex}^n(H) \to K$ *is a* Man^r*-functor then the torsion of* $\mathrm{Lob}^r(T) \circ \mathrm{Tbf}^n$ *coincides with its identical morphism.*

Torsion-free bundle tangentors or certain fragments of them are used more frequently than the others. Nevertheless, bundle tangentors with any torsions are closely related to higher-order connections. For example, a usual connection of order n on a local vector bundle A is essentially an automorphism of $\mathrm{Tbf}^n(A)$, whose basis is the identical mapping of LA; however, the automorphism is a fragment of the torsion of a certain bundle tangentor $T: \mathrm{Lob}^{r+n}(\mathrm{Mod}) \to \mathrm{Lob}^r(\mathrm{Tex}^n(\mathrm{Mod}))$ of order n.

Now, the structure of bundle-tangentor morphisms will be described. Remember that the local case is being discussed.

4.19. Definition. Let $T, T' : \mathrm{Lob}^{r+n}(H) \to \mathrm{Lob}^r(K)$ be bundle tangentors of order n and let $M : T \to T'$ be a morphism. *The rectification of* M is the morphism $\mathrm{Rtf}(M) : \mathrm{Rtf}(T) \to \mathrm{Rtf}(T')$ defined as follows.

If A is a $\mathrm{Tex}^n(H)$-object then $\mathrm{Rtf}(M)(A) = \emptyset R(M(LA, p \mapsto RA))$.

4.20. Theorem. *Let* $T, T' : \mathrm{Tex}^n(H) \to K$ *be* Man^r*-functors and let* $M : T \to T'$ *be a morphism. Then the rectification of* $\mathrm{Lob}^r(M) \circ \mathrm{Tbf}^n$ *coincides with* M.

4.21. Theorem. *Let* $T, T' : \mathrm{Lob}^{r+n}(H) \to \mathrm{Lob}^r(K)$ *be bundle tangentors of order n and let* $M : T \to T'$ *be a morphism. Then the diagram*

$$\begin{array}{ccc}
\mathrm{Lob}^r(\mathrm{Rtf}(T)) \circ \mathrm{Tbf}^n & \xrightarrow{\mathrm{Lob}^r(\mathrm{Rtf}(M))\,\mathrm{Tbf}^n} & \mathrm{Lob}^r(\mathrm{Rtf}(T')) \circ \mathrm{Tbf}^n \\
{\scriptstyle \mathrm{Tsn}(T)}\downarrow & & \downarrow{\scriptstyle \mathrm{Tsn}(T')} \\
T & \xrightarrow[M]{} & T'
\end{array}$$

is commuting.

Finally, the global case is to be discussed. According to ideas of Ch. Ehresmann [9], I. Kolář (his lecture, Summer school in modern geometry, Poprad, CSSR, July 1975) and others, we define bundle tangentors to be extensions of those from Definition 4.13. Hence, the bundle tangentors of order n map $\mathrm{Bun}^{r+n}(H)$ into $\mathrm{Bun}^r(K)$, where H is a Lom^{r+n}-category. Recall that any such extension is determined uniquely up to an isomorphism. If $M : T \to T'$ is a morphism of local bundle tangentors and P, P' are extensions of T, T' respectively then there is an extension $N : P \to P'$ of M, determined uniquely by M.

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