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ON q-IDEALS IN COMPACT SEMIGROUPS

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A *semigroup* is a non-empty Hausdorff space together with a continuous associative multiplication, denoted by juxtaposition, $(x, y) \to xy$. In what follow S will denote a semigroup. If S contains a *zero*. i.e. an element 0 such that $x0 = 0 = 0x$ for all $x \in S$, S is said to be a *semigroup with* 0. In this paper, to make the explanation more concise, we treat only semigroups with 0, so we shall use the term semigroup to denote semigroup with 0. However, the results obtained in this paper are valid for semigroups without zero elements.

Our main objective of this paper is to generalize a number of results of HOO and SHUM [1] and SHUM [9] dealing with radicals and compressed ideals in a compact semigroup. After elementary preparations in § 1, we shall give, in § 2, the definition of *q-ideal* in S (Definition 2.1). This concept is a generalization of the concepts of radical and open semiprime ideal of a compact semigroup. And we shall give a necessary and sufficient condition that an ideal in a compact semigroup be a q-ideal (Theorem 2.6). We shall also introduce, in this section, the notion of *Q-primitive idempotent*, where Q is a q-ideal of S (Definition 2.8). And we shall extend a result of KOCH [2] concerning primitive idempotents in a compact semigroup (Theorem 2.9). In § 3, we shall discuss about Q-divisors in S. If Q is a q-ideal in S, then for any element a of S, the set $\{x \in S : xSa \subset Q\}$ is called the *Q-divisor* of a, and we denote it by $(Q : a)$. This concept is an analogy of the topological B-divisor defined by Shum [9]. Some properties of the set $(Q : a)$ will be treated and some results in [1] and [9] concerning topological B-divisors in a compact semigroup will be generalized. The last section § 4, is devoted to examples of q-ideals in compact semigroups.

For most of the terminology used in this paper we refer to A. B. PAALMAN-DE MIRANDA [8]. Unless otherwise stated, the word "ideal" shall mean two-sided ideal of S.

I. PRELIMINARIES

Throughout this paper we shall adhere to the following notation.

$\bar{A}$ = the topological closure of a subset A of S.

$X - Y$ = the set-theoretical complement of Y in X, X and Y being any sets.

E = the set of idempotents of S.

$J(A)$ = $A \cup AS \cup SA \cup SAS$, i.e. the smallest ideal containing A, A being a non-empty subset of S.

$R(A) = A \cup AS$, i.e. the smallest right ideal containing A, A being a non-empty subset of S.

$L(A) = A \cup SA$, i.e. the smallest left ideal containing A, A being a non-empty subset of S.

It is clear that if e is an idempotent of S, then $J(e) = SeS$, $R(e) = eS$ and $L(e) = Se$ hold. Furthermore it is obvious that if S and A are compact, then $J(A)$, $R(A)$ and $L(A)$ are compact.

$J_0(A)$ = the union of all ideals contained in A, i.e. the largest ideal contained in A if $0 \in A$.

It has been shown in [3] that if A is open and S is compact, then $J_0(A)$ is open.

We define T^ω and $\Gamma(a)$ as follows:

$$T^\omega = \bigcap_{n=1}^{\infty} T^n, \quad \text{where } T \text{ is a subsemigroup of } S.$$

$$\Gamma(a) = \overline{\{a^n : n = 1, 2, \ldots\}}, \quad a \text{ being any element of } S.$$

Generally we shall not distinguish between x and $\{x\}$ if confusion of meaning is unlikely, so that we write xA in place of $\{x\}A$, $x \cup A$ in place of $\{x\} \cup A$ and $A - x$ in place of $A - \{x\}$.

The following two propositions, Propositions 1.1 and 1.3, are well-known results in the ring theory (refer to Theorems 4.3 and 4.12 in McCoy [4]).

Proposition 1.1. *If P is an ideal of S, then all of the following conditions are equivalent:*

(i) *If A and B are ideals of S such that $AB \subset P$, then $A \subset P$ or $B \subset P$.*

(ii) *If $a, b \in S$ such that $aSb \subset P$, then $a \in P$ or $b \in P$.*

(iii) *If R_1 and R_2 are right ideals of S such that $R_1R_2 \subset P$, then $R_1 \subset P$ or $R_2 \subset P$.*

(iv) *If L_1 and L_2 are left ideals of S such that $L_1L_2 \subset P$, then $L_1 \subset P$ or $L_2 \subset P$.*

Definition 1.2. An ideal P of S is said to be a *prime ideal* if it satisfies one (hence all) of the conditions of Proposition 1.1.

Proposition 1.3. *If Q_0 is an ideal of S, then all of the following conditions are equivalent:*

(i) *If A is an ideal of S such that $A^2 \subset Q_0$, then $A \subset Q_0$.*

(ii) *If $a \in S$ such that $aSa \subset Q_0$, then $a \in Q_0$.*

(iii) *If R is a right ideal of S such that $R^2 \subset Q_0$, then $R \subset Q_0$.*

(iv) *If L is a left ideal of S such that $L^2 \subset Q_0$, then $L \subset Q_0$.*

Definition 1.4. An ideal Q_0 of S is said to be a *semiprime ideal* if it satisfies one (hence all) of the conditions of Propostion 1.3.

It is obvious that a prime ideal is semiprime. Furthemore it follows easily by induction that if Q_0 is a semiprime ideal and A a one-sided ideal such that $A^n \subset Q_0$ for some positive integer n, then $A \subset Q_0$.

The next propostion is McCoy's Corollary 4.16 in [4].

Proposition 1.5. *An ideal Q_0 of S is a semiprime ideal if and only if it is an intersection of prime ideals of S.*

We conclude this section by presenting known results concerning the radical and open prime ideals of S, which will be used later.

An element b of S is said to be *nilpotent* if $b^n \to 0$, that is, if for every neighborhood U of 0 there exists a positive integer n_0 such that $b_n \in U$ for all $n \geqq n_0$. We denote here the set of all nilpotent elements in S by N_0. Clearly, N_0 is not empty, because $0 \in N_0$.

Definition 1.6. (Numakura [5]). The largest ideal contained in N_0 is called the *radical* of S and is denoted by N, namely $N = J_0(N_0)$.

The next proposition is the author's Theorem 1 in [7].

Proposition 1.7. *If S is compact, then N is the intersection of all open prime ideals of S.*

The following proposition is also the author's Theorem 2 in [7].

Proposition 1.8. *If S is compact, then each open prime ideal $P(\neq S)$ has the form $P = J_0(S - e)$ for some non-zero idempotent e, and conversely for each non-zero idempotent e, $J_0(S - e)$ is an open prime ideal.*

II. q-IDEALS AND Q-PRIMITIVE IDEMPOTENTS

In this section we first present the concept of q-ideal in the following definition.

Definition 2.1. An ideal Q of S is said to be a *q-ideal* provided Q can be expressed as an intersection of open prime ideals.

In view of Proposition 1.5, it is evident that a q-ideal is a semiprime ideal. Moreover, from Proposition 1.7, it can easily be seen that the following theorem holds.

Theorem 2.2. *The radical of a compact semigroup is a q-ideal.*

We shall next prove the following result.

Proposition 2.3. *If I is an ideal of S, then all of the following conditions are equivalent:*

(i) *Every ideal of S which is not contained in I has an idempotent e such that $e \notin I$.*

(ii) *Every right ideal of S which is not contained in I has an idempotent e such that $e \notin I$.*

(iii) *Every left ideal of S which is not contained in I has an idempotent e such that $e \notin I$.*

Proof. It is obvious that (ii) implies (i) and (iii) implies (i).

We shall show that (i) implies (ii). Suppose that R is a right ideal of S not contained in I. Certainly $L(R) = R \cup SR$ is an ideal of S not contained in I, and so it has an idempotent e such that $e \notin I$. If $e \in R$, there is nothing to prove. Otherwise, $e \in SR$ and so e can be written in the form $e = ab$, where we may assume that $a = ea \in eS$ and $b = be \in Re$. Let $f = ba$. Then we have

$$f^2 = b(ab)\,a = bea = ba = f$$

and

$$f = ba \in (Re)\,(eS) \subset R\,.$$

That is, f is an idempotent contained in R. Moreover we have $f \notin I$. For, otherwise we obtain

$$e = e^2 = a(ba)\,b = afb \in aIb \subset I\,,$$

and this contradicts to the assumption. Hence R has an idempotent f such that $f \notin I$.

Analogously it can be shown that (i) implies (iii).

It will be convenient here to give the following definition.

Definition 2.4. (Property $\mathscr{E}$). We say that an ideal I of S *has the property* $\mathscr{E}$ if I satisfies one (hence all) of the conditions of Proposition 2.3.

The next lemma gives a class of ideals in a compact semigroup having the property $\mathscr{E}$.

Lemma 2.5. *If Q_1 is an open semiprime ideal of a compact semigroup S, then Q_1 has the property $\mathscr{E}$.*

Proof. Let M be an arbitrary ideal of S which is not contained in Q_1. Take an element x from $M - Q_1$, and consider the ideal $J(x)$. Clearly $J(x)$ is a compact ideal contained in M but not contained in Q_1. We shall show that $J(x)$ has an idempotent e which is not contained in Q_1. Assume now, by the way of contradiction, that $E \cap J(x) \subset Q_1$. By Lemma 7 in [7], we have $J(x)^\omega = J(x)\,(E \cap J(x))\,J(x) \subset Q_1$. Since Q_1 is open and $J(x)$ is compact, there exists a positive integer n such that $J(x)^n \subset Q_1$. Using the fact that Q_1 is a semiprime ideal, we obtain $J(x) \subset Q_1$. This gives the desired contradiction, which completes the proof.

We can now easily prove the following theorem which characterizes a q-ideal in a compact semigroup.

Theorem 2.6. *Let I be an ideal of a compact semigroup S. Then I is a q-ideal if and only if I has the property $\mathscr{E}$.*

Proof. Only if part: Suppose that I is a q-ideal of S. Let M be an arbitrary ideal of S which is not contained in I. We shall show that M contains an idempotent e such that $e \notin I$. Since I is an intersection of open prime ideals and M is not contained in I, there exists an open prime ideal P which contains I but does not contain M. By Lemma 2.5, there exists an idempotent e in M such that $e \notin P$. It is clear that e is a desired idempotent. Therefore I has the property $\mathscr{E}$.

If part: Suppose that I is an ideal of S having the property $\mathscr{E}$. Let I_1 denote the intersection of all open prime ideals containing I. We shall show that $I = I_1$. Certainly $I \subset I_1$, so let us assume that $I \neq I_1$ and seek a contradiction. Since I_1 is not contained in I, there exists an idempotent $e \in I_1$ such that $e \notin I$. Let $P_1 = J_0(S - e)$. Then P_1 is an open prime ideal containing I but not containing I_1. This gives the desired contradiction. Hence we obtain $I = I_1$, so that I is a q-ideal.

The next theorem is an immediate consequence of Lemma 2.5 and Theorem 2.6.

Theorem 2.7. *An open semiprime ideal of a compact semigroup is a q-ideal.*

We now define another concept whose significance will be indicated in the next section.

Definition 2.8. (Shum [9]). Let I be an ideal of S. An idempotent e of S is said to be an *I-primitive idempotent* if $e \notin I$ and e is the only idempotent in $eSe - I$.

We remark here that (0)-primitive or N-primitive idempotents are non-zero primitive idempotents in an ordinary sense.

The following result gives a generalization of Koch's Theorem 1 in [2].

Theorem 2.9. *Let S be a compact semigroup and Q a q-ideal of S. If e is an idempotent of S, then all of the following conditions are equivalent*:

(i) *e is a Q-primitive idempotent.*

(ii) *SeS is a minimal ideal not contained in Q.*

(iii) *Se (eS) is a minimal left (right) ideal not contained in Q.*

(iv) *$eSe - Q$ is a group.*

(v) *Every idempotent in $SeS - Q$ is a Q-primitive idempotent.*

(Let A and B be ideals (right, left or two-sided) of S. The expression "A is a minimal ideal not contained in B" means that A is a minimal member among the ideals of S which are not contained in B.)

Proof. (i) $\Rightarrow$ (ii): Suppose that e is a Q-primitive idempotent in S. Let M be an ideal of S contained in SeS. Assume that $M \not\subset Q$, then, since Q has the property $\mathscr{E}$, there is an idempotent $f \in M$ such that $f \notin Q$. As $f = f^3 \in f(SeS)f = (fSe)(eSf)$, f can be written in the form $f = ab$, where $a \in fSe$ and $b \in eSf$. From this it follows that

$$(ba)^2 = b(ab)\,a = bfa = ba\,.$$

Hence ba is an idempotent in eSe. Moreover, ba is not contained in Q. For, otherwise we have $f = f^2 = a(ba)\,b \in aQb \subset Q$, a contradiction. Therefore, ba is an idempotent in $eSe - Q$, and so ba must coincide with e. It follows that

$$SeS = SbaS = SbfaS \subset SfS \subset M\,.$$

Hence $M = SeS$, and so M is a minimal ideal not contained in Q.

(ii) $\Rightarrow$ (iii): Suppose that SeS is a minimal ideal not contained in Q. Let L be a left ideal of S contained in Se. Assume that $L \not\subset Q$, then there is an idempotent $f \in L$ such that $f \notin Q$. From $Se \supset L \supset Sf$, it follows that $SeS \supset SfS$. As $SfS \not\subset Q$, the minimality of SeS implies that $SeS = SfS$. Therefore there exist elements a and b such that $e = afb$. Here, we may assume that $a \in eSf$ and $b \in fSe$. Now, $Sf \subset Se$ implies $f = fe$; hence for any positive integer n,

$$a^n f b^n = (a^{n-1}f)\,(afb)\,b^{n-1} = a^{n-1}(fe)\,b^{n-1} = a^{n-1}fb^{n-1} = \cdots = afb = e\,.$$

Using Lemma 1 in [6] or Lemma 1.1.4 in [8], we can find an idempotent $g \in \Gamma(a)$ and an element $h \in \Gamma(b)$ such that $e = gfh$. We note $gf = g$, hence

$$e = ge = gfe = gf = g$$

and

$$e = g = gf\,.$$

This implies $e \in Sf$, and so $Se \subset Sf \subset L$. Hence $Se = L$, so that L is a minimal left ideal not contained in Q.

Analogously it can be shown that eS is a minimal right ideal not contained in Q.

(iii) $\Rightarrow$ (iv): Suppose that Se is a minimal left ideal not contained in Q.

We shall first show that for any $x \in eSe - Q$ there exists a left inverse x' in $eSe - Q$ with $x'x = e$. As $L(x) \subset Se$ and $L(x) \not\subset Q$, it follows that $L(x) = Se$. Therefore, $x = e$ or there is an element x'' in S such that $x''x = e$. In the former case x is the identity element of $eSe - Q$. In the later case let $x' = ex''e$. Then $x' \in eSe$ and $x'x = (ex''e)\,x = ex''(ex) = e(x''x) = e^2 = e$. It is clear that $x' \notin Q$, because $e \notin Q$.

We shall next show that if x and y are in $eSe - Q$, then $xy \in eSe - Q$. Let x' and y' be left inverses of x and y in $eSe - Q$, respectively. Clearly, $xy \in eSe$. If $xy \in Q$, then we have

$$e = (y'x')\,(xy) \in Q\,,$$

a contradiction. Hence $xy \in eSe - Q$, and we can conclude that $eSe - Q$ is a group.

(iv) $\Rightarrow$ (i): Trivial.

(v) $\Rightarrow$ (i): Trivial.

(ii) $\Rightarrow$ (v): Let f be any idempotent in $SeS - Q$, we have to show that f is a Q-primitive idempotent. Since SeS is a minimal ideal not contained in Q and SfS is an ideal of S contained in SeS but not contained in Q, it follows that $SfS = SeS$. This implies that SfS is also a minimal ideal not contained in Q. From the equivalency of (i) and (ii), we can conclude that f is a Q-primitive idempotent.

III. Q-DIVISORS

We begin this section with the following lemma.

Lemma 3.1. *If Q_0 is a semiprime ideal of S and $x, y \in S$ such that $xSy \subset Q_0$, then $ySx \subset Q_0$.*

Proof. Let $Q_0 = \bigcap\{P_\alpha : \alpha \in \Lambda\}$ be an expression of Q_0 as an intersection of prime ideals (see Proposition 1.5). If P_α is any prime ideal in $\{P_\alpha : \alpha \in \Lambda\}$, then $xSy \subset \subset Q \subset P_\alpha$ implies that $x \in P_\alpha$ or $y \in P_\alpha$. From this it follows that $ySx \subset P_\alpha$. Since P_α is taken arbitrary, we can conclude that $ySx \subset \bigcap\{P_\alpha : \alpha \in \Lambda\} = Q_0$.

We now define the concept of Q-divisor in S.

Definition 3.2. Let Q be a q-ideal and A a non-empty subset of S. The *Q-divisor* of A is defined to be the set

$$(Q : A) = \{x \in S : xSa \subset Q \text{ for all } a \in A\}\,.$$

In view of Lemma 3.1, we can easily see that

$$(Q : A) = \{x \in S : aSx \subset Q \text{ for all } a \in A\}\,.$$

If $A = \{a\}$, we abbreviate the notation $(Q : \{a\})$ by $(Q : a)$.

From the definition of Q-divisor one can quickly verify the following proposition.

Proposition 3.3. *Let Q be a q-ideal and A a non-empty subset of S. Then each of the following assertions is true.*

(i) *$(Q : A)$ is an ideal of S containing Q.*
(ii) *$A \subset Q$ if and only if $(Q : A) = S$.*
(iii) *If Q is closed, then $(Q : A)$ is closed.*
(iv) *If Q is open and if S and A are compact, then $(Q : A)$ is open.*

Lemma 3.4. *If S is compact and Q a q-ideal of S, then $(Q : A)$ is a q-ideal for any non-empty subset A.*

Proof. We shall first show that for any element $a \in A$, $(Q : a)$ is a q-ideal. If $a \in Q$, then $(Q : a) = S$, and so $(Q : a)$ is certainly a q-ideal. Suppose that $a \notin Q$, and let M be an ideal of S not contained in $(Q : a)$. Then there exists an element $x \in M$ such that $xSa \not\subset Q$. Since Q is an intersection of open prime ideals, we can find an open prime ideal P containing Q such that $xSa \not\subset P$. Hence $J(x) \not\subset P$, and $J(x) - P$ contains an idempotent e. As P is a prime ideal and $e, a \notin P$, it follows that $eSa \not\subset P$. Therefore $eSa \not\subset Q$, and so we obtain $e \notin (Q : a)$. Thus M contains an idempotent e which is not contained in $(Q : a)$. In view of Theorem 2.6, $(Q : a)$ is a q-ideal.

From $(Q : A) = \bigcap\{(Q : a) : a \in A\}$, $(Q : A)$ is also expressed as an intersection of open prime ideals. This completes the proof.

We shall next prove the following theorem which is an extension of Shum's Theorem 2.8 in [9].

Theorem 3.5. *Let S be a compact semigroup and Q a q-ideal of S. If e is a Q-primitive idempotent, then $(Q : e)$ is an open prime ideal of S, more precisely, $(Q : e) = J_0(S - e)$.*

Moreover, $(Q : e)$ is a minimal prime ideal containing Q, that is, if P' is a prime ideal (not necessarily open) such that $Q \subset P' \subset (Q : e)$, then $P' = (Q : e)$.

Proof. We shall first show that $(Q : e) = J_0(S - e)$. From $eSe \not\subset Q$, it is clear that $e \notin (Q : e)$, and therefore $(Q : e) \subset J_0(S - e)$. To prove $J_0(S - e) = (Q : e)$, let us assume that $J_0(S - e) \not\subset (Q : e)$ and seek a contradiction. Since $(Q : e)$ is a q-ideal (see Lemma 3.4), $J_0(S - e)$ contains an idempotent f such that $f \notin (Q : e)$. From $f \notin (Q : e)$ it follows that $eSf \not\subset Q$ and so $eSfS \not\subset Q$. Therefore there exists an idempotent $g \in eSfS$ such that $g \notin Q$. We have now

$$(ge)^2 = g(eg)\, e = gge = ge\,,$$

$$e(ge) = (eg)\, e = ge$$

and

$$(ge)\, e = ge\,.$$

Hence ge is an idempotent in eSe. Furthermore, $ge \notin Q$. For, otherwise we have

$$g = g^2 = (eg)^2 = e(ge)\, g \subset eQg \subset Q\,,$$

a contradiction. Thus ge is an idempotent in $eSe - Q$, therefore ge must coincide with e, i.e. $ge = e$. From this it follows that

$$e = ge \in eSfSe \subset J(f) \subset J_0(S - e)\,,$$

so that we arrived at a contradiction. Hence we have $(Q : e) = J_0(S - e)$.

Let us next assume that P' is a prime ideal such that $Q \subset P' \subset (Q : e)$. Let x be an arbitrary element in $(Q : e)$. Then $xSe \subset Q \subset P'$ and so $x \in P'$ or $e \in P'$ since P' is a prime ideal. But $e \notin (Q : e)$ and $P' \subset (Q : e)$ imply that $e \notin P'$. Hence we have $x \in P'$, therefore $(Q : e) \subset P'$. Thus $(Q : e)$ is a minimal prime ideal containing Q. This completes the proof of the theorem.

Suppose that P is an open prime ideal of S (containing a q-ideal Q). We say that P is a *minimal open prime ideal* (*containing* Q) if P is a minimal member among the open prime ideals of S (which contain Q).

The following theorem gives a partial converse of the preceding theorem.

Theorem 3.6. *Let S be a compact semigroup and Q a q-ideal of S. If P is a minimal open prime ideal containing Q, then $P = S$ or $P = (Q : e)$ for some Q-primitive idempotent e.*

Proof. Suppose that $P \neq S$, then according to Proposition 1.8 there exists an idempotent e such that $P = J_0(S - e)$. We shall show that e is a Q-primitive idempotent. Certainly $e \notin Q$, so let us assume that e is not a Q-primitive idempotent and seek a contradiction. By Theorem 2.9, there exists an ideal M such that $M \subset SeS$, $M \neq SeS$ and $M \not\subset Q$. Let f be an idempotent in M such that $f \notin Q$. Then the open prime ideal $J_0(S - f)$ contains Q and it is properly contained in $J_0(S - e) = P$, because $SfS \subset M \subset SeS$, $M \neq SeS$ (in this connection see Lemma 9 in [7]). This contradicts to the minimality of P, and e must be a Q-primitive idempotent. Hence we have $(Q : e) = J_0(S - e) = P$.

According to Theorem 2.2 and the fact that every open prime ideal contains the radical (see Proposition 1.7), we have the following immediate corollary to the preceding theorems:

Corollary 3.7. *Let S be a compact semigroup and N the radical of S. If e is a non-zero primitive idempotent, then $(N : e)$ is a minimal open prime ideal of S. Conversely, if P is a minimal open prime ideal of S, then $P = S$ or $P = (N : e)$ for some non-zero primitive idempotent e.*

The following theorem is a generalization of Corollary 1 to Theorem 3.1 in [9] and Lemma 3.1 in [1].

Theorem 3.8. *Let S be a compact semigroup and Q a proper q-ideal of S. Then Q is the intersection of all Q-divisors $(Q : e)$, where e runs through the set $E - Q$.*

Proof. Since S itself is an ideal not contained in Q, there is an idempotent e in S such that $e \notin Q$. Hence we have $E - Q \neq \emptyset$. Let Q' be the intersection of all the sets $(Q : e)$, where e runs through $E - Q$. Clearly $Q' \supset Q$, and so to prove $Q' = Q$ let us assume that $Q' \not\subset Q$ and seek a contradiction. Let f be an idempotent in Q' such that $f \notin Q$. Then, since $f \in E - Q$, we obtain $Q' \subset (Q : f)$ and so $f \in (Q : f)$. This implies $fSf \subset Q$, so that $f \in Q$. Thus we arrived at a contradiction, which completes the proof.

Corollary 3.9. *Let S be compact and $S \neq N$, then N is the intersection of all N-divisors $(N : e)$, where e runs through the set of non-zero idempotents.*

We now proceed to consider the existence of Q-primitive idempotents. Let us recall that open semiprime ideals in a compact semigroup are q-ideals (Theorem 2.7).

Lemma 3.10. *Let S be a compact semigroup and Q an open semiprime ideal of S. If M is an ideal of S not contained in Q, then M has a Q-primitive idempotent.*

Proof. Take an element x from $M - Q$ and consider the ideal $J(x)$. $J(x)$ is a compact ideal such that $J(x) \subset M$ and $J(x) \not\subset Q$. To prove the lemma it is enough to show that $J(x)$ has a Q-primitive idempotent. Therefore we may assume, without loss of generality, that M itself is a compact ideal.

Let $\mathscr{M}$ be the set of all compact ideals contained in M but not contained in Q. $\mathscr{M}$ is partially ordered by inclusion and is nonvoid, because $M \in \mathscr{M}$. We now assert that the intersection of an arbitrary chain $\mathscr{C}$ in $\mathscr{M}$ is an element of $\mathscr{M}$, because ideals in $\mathscr{C}$ are compact and Q is open. By Zorn's lemma, $\mathscr{M}$ has a minimal element, say M_1. Namely, M_1 is a compact ideal contained in M but not in Q such that if M' is a compact ideal contained in M but not in Q and if $M' \subset M_1$, then $M' = M_1$. We shall show that the ideal M_1 is a minimal ideal not contained in Q. Suppose that M'' is an ideal of S such that $M'' \subset M_1$ and $M'' \not\subset Q$. For any $y \in M'' - Q$, $J(y)$ is a compact ideal contained in M_1 (hence in M) but not contained in Q. Hence $J(y)$ is an element of $\mathscr{M}$. Since M_1 is a minimal element of $\mathscr{M}$, we obtain $J(y) = M_1$. Therefore we have $M_1 = J(y) \subset M''$ and so $M_1 = M''$. Thus M_1 is a minimal ideal not contained in Q. Take an idempotent e from $M_1 - Q$. Clearly SeS coincides with M_1, therefore SeS is a minimal ideal of S not contained in Q. From Theorem 2.9 we can conclude that e is a Q-primitive idempotent. This completes the proof of the lemma.

We can now easily prove the following theorem.

Theorem 3.11. *Let S be a compact semigroup and Q a proper open semiprime ideal of S. If P' is an open prime ideal containing Q, then there exists a minimal prime ideal P containing Q such that $P \subset P'$. Furthermore, P has the form $P = J_0(S - e) = (Q : e)$ for some Q-primitive idempotent e.*

Proof. Suppose, first, that $P' \neq S$. In view of Proposition 1.8, P' has the form $P' = J_0(S - f)$ for some idempotent f. The ideal $J(f)$ is not contained in Q, and so, by Lemma 3.10, there exists a Q-primitive idempotent e in $J(f)$. Using Theorem 3.5, we can conclude that $(Q : e) = J_0(S - e)$ and $(Q : e)$ is a minimal prime ideal containing Q. According to Lemma 9 in [7], from $J(e) \subset J(f)$ it follows that $J_0(S - e) \subset P'$. Therefore $P = (Q : e) = J_0(S - e)$ is a required prime ideal.

If $P' = S$, S contains a Q-primitive idempotent e by Lemma 3.10. In this case, it is evident that the ideal $P = J_0(S - e) = (Q : e)$ is also a required prime ideal. This completes the proof of the theorem.

We conclude this section with the following theorem and its corollary, which generalize the result of Hoo and Shum [1; Theorem 4.6].

Theorem 3.12. *Let S be a compact semigroup and Q a proper open semiprime ideal of S. Then Q is the intersection of all Q-divisors $(Q : e)$, where e runs through the set of Q-primitive idempotents.*

Proof. For any Q-primitive idempotent e it is evident that $Q \subset (Q : e)$.

Since Q is a q-ideal, Q has the form $Q = \bigcap P'_\alpha$, where P'_α runs over a set of open prime ideals containing Q. By Theorem 3.11, for any P'_α we can find a minimal prime ideal P_α containing Q such that $P_\alpha \subset P'_\alpha$ and has the form $P_\alpha = (Q : e_\alpha)$, where e_α is a Q-primitive idempotent. Therefore we have

$$Q = \bigcap P_\alpha = \bigcap (Q : e_\alpha) .$$

Hence we can conclude that Q is the intersection of all Q-divisors $(Q : e)$, where e runs through the set of Q-primitive idempotents.

A semigroup S is said to be an *N-semigroup* if N is an open set in S.

As an immediate consequence of the preceding theorem we have the following corollary.

Corollary 3.13. *Let S be a compact N-semigroup. If $N \neq S$, then N is the intersection of all N-divisors $(N : e)$, where e runs through the set of non-zero primitive idempotents.*

IV. EXAMPLES

4.1. An ideal B of S is said to be *compressed* (Shum [9]) or *completely semiprime* if $a^2 \in B$ implies that $a \in B$, a being an element of S. The following is an example of a finite semigroup possessing a prime ideal which is not a compressed ideal ([8], p. 51).

Let T be the semigroup consisting of five elements e, f, a, b and 0 with multiplication table

	e	f	a	b	0
e	e	0	0	b	0
f	0	f	a	0	0
a	a	0	0	f	0
b	0	b	e	0	0
0	0	0	0	0	0

Then $\{0\}$ is a prime ideal of T which is not a compressed ideal.

4.2. We shall give here an example of a finite semigroup possessing a semiprime ideal which is not a prime ideal.

Let T be the semigroup described in the preceding example. Let S be the set of all pairs (x, y) with $x, y \in T$. Under the discrete topology and the componentwise multiplication S becomes a semigroup. Namely, S is the direct product of the *two* semigroups T and T. By 0^* we denote the zero element of S, that is, $0^* = (0, 0)$.

We shall show that the ideal $\{0^*\}$ of S is a semiprime ideal which is not prime. Let $A = \{(x, 0) : x \in T\}$ and $B = \{(0, y) : y \in T\}$. It is not difficult to see that A and B are ideals of S distinct from $\{0^*\}$. And straighforward calculation shows that $AB = \{0^*\}$. Therefore $\{0^*\}$ is not a prime ideal of S. Next, suppose that $z^* = (x, y)$ is an element of S such that $z^*Sz^* = \{0^*\}$. From this it follows that $xTx = \{0\}$ and $yTy = \{0\}$. Since $\{0\}$ is a prime ideal of the semigroup T (see Example 4.1), we obtain $x = 0$ and $y = 0$, and so $z^* = 0^*$. Hence $\{0^*\}$ is a semiprime ideal of S.

4.3. Let Λ be an arbitrary infinite set of indices. To each element $\alpha \in \Lambda$ we associate a copy T_α of the semigroup T described in Example 4.1, and denote by S the direct product (Cartesian product) of all the semigroups T_α, $\alpha \in \Lambda$. Then S, endowed with the product topology, is a compact, non-discrete semigroup. We denote by π_α the projection from S onto T_α. It is well-known that π_α is an open continuous homomorphism. The zero element of S is denoted by 0^*, i.e. 0^* is an element of S such that $\pi_\alpha(0^*) = 0$ for all $\alpha \in \Lambda$.

In exactly the same fashion with Example 4.2, it follows that $\{0^*\}$ is a semiprime ideal of S which is not prime. Let $P_\alpha = \pi_\alpha^{-1}(0)$. Using the fact that $\{0\}$ is an open prime ideal of the semigroup T, it is not difficult to see that P_α is an open prime ideal of S. It is also easy to see that the intersection of all P_α's coincides with $\{0^*\}$. From this we can conclude that $\{0^*\}$ is a q-ideal of S. It is noticed that $\{0^*\}$ is not open, since the index set Λ is infinite.

Furthermore if we denote by x_α^* the element of S such that $\pi_\alpha(x_\alpha^*) = x$ and $\pi_\beta(x_\alpha^*) = 0$ for $\beta \neq \alpha$ where x is an element of T, then it can easily be shown that e_α^* and f_α^* are $\{0^*\}$-primitive idempotents for every $\alpha \in \Lambda$.

4.4. Let S be the closed unit interval of real numbers. S is a compact commutative semigroup under the topology induced from reals and the multiplication defined by

$$xy = \min(x, y), \quad x, y \in S.$$

For any number a $(0 < a \leqq 1)$, the set $P_a = \{x \in S : 0 \leqq x < a\}$ is an open prime ideal of S. Let b be any element in S. The intersection of open prime ideals P_a, $b < a \leqq 1$, coincides with the closed ideal $\bar{P}_b$, and therefore $\bar{P}_b$ is a q-ideal of S. In this case, however, there are no $\bar{P}_b$-primitive idempotents.

References

[1] *C. S. Hoo* and *K. P. Shum:* On compact N-semigroups, Czechoslovak Math. J. *24* (*99*) (1974), 552—562.

[2] *R. J. Koch:* Remarks on primitive idempotents in compact semigroups with zero, Proc. Amer. Math. Soc. *5* (1954), 828—833.

[3] *R. J. Koch* and *A. D. Wallace:* Maximal ideals in compact semigroups, Duke Math. J. *21* (1954), 681—685.

[4] *N. H. McCoy:* The theory of rings, Chelsea Publishing Company, New York (1973),

[5] *K. Numakura:* On bicompact semigroups with zero, Bull. Yamagata University (Natural. Science), *4* (1951), 405 —412.

[6] *K. Numakura:* On bicompact semigroups, Math. J. Okayama University, *1* (1952), 99—108.

[7] *K. Numakura:* Prime ideals and idempotents in compact semigroups, Duke Math. J. *24* (1957), 671—680.

[8] *A. B. Paalman - de Miranda:* Topological semigroups, Mathematical Centre Tracts, Amsterdam (1970).

[9] *K. P. Shum:* On compressed ideals in topological semigroups, Czechoslovak Math. J. *25* (*100*) (1975), 261—273.

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