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ON EULERIAN SUBGRAPHS OF COMPLEMENTARY GRAPHS

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Let G be a graph in the sense of [1] or [2]. We denote by $V(G)$, $E(G)$, $\bar{G}$ and $L(G)$ its vertex set, edge set, complement, and line graph, respectively. The cardinality of $V(G)$ is referred to as the order of G. If $v_1, \ldots, v_n$ $(n \geqq 1)$ are distinct vertices which do not belong to G, then we denote by $G_{(v_1,\ldots,v_n)}$ the graph with the properties

$$V(G_{(v_1,\ldots,v_n)}) = V(G) \cup \{v_1, \ldots, v_n\}$$

and

$$E(G_{(v_1,\ldots,v_n)}) = E(G)\,.$$

As usual, we say that a graph F is eulerian if it is nontrivial and connected, and contains a closed trail passing through every edge of F. It is well-known (see, for example, Theorem 3.1 in [1] or Theorem 7.1 in [2]) that a connected nontrivial graph is eulerian if and only if each of its vertices has an even degree.

Let G be a nontrivial graph. We shall say that a subgraph F of G is eulerian if F is an eulerian graph. Clearly, a nontrivial subgraph F of G is eulerian if and only if there exists a closed trail T in G such that F and T have the same vertices and edges. We shall define the number $\operatorname{eul}(G)$. If G contains no eulerian subgraph, then we put $\operatorname{eul}(G) = 2$. If there exists an eulerian subgraph of G, then we denote by $\operatorname{eul}(G)$ the maximum integer among the orders of eulerian subgraphs of G. Obviously, G contains an eulerian subgraph if and only if $\operatorname{eul}(G) \geqq 3$.

The observations made in the following remark will be very useful for us.

Remark. Let F be a graph isomorphic to the complete bipartite graph $K(2, p-2)$, where $p \geqq 3$. If p is even, then F is eulerian. Assume that p is odd. Then no spanning subgraph of F is eulerian. On the other hand, if $p \geqq 5$, then F contains a subgraph which is isomorphic to $K(2, p-3)$, and therefore eulerian. Let u and v be the vertices of degree $p-2$. Obviously, $F + uv$ is eulerian. If w_1 and w_2 are distinct vertices of F which are different from both u and v, then $F + w_1w_2 - uw_1 - vw_2$ is also eulerian.

Thus, we have obtained the following results: Let G be a graph of order $p \geqq 3$. If G contains a proper subgraph isomorphic to $K(2, p-2)$, then $\operatorname{eul}(G) = p$. If G is isomorphic to $K(2, p-2)$, then $\operatorname{eul}(G) = p$ if and only if p is even.

Before stating the main result of the present paper we shall define a certain class of graphs.

Let b_1, b_2, b_3 and b_4 be distinct vertices. We denote by Q the path with $V(Q) = \{b_1, b_2, b_3, b_4\}$ and $E(Q) = \{b_1b_2, b_2b_3, b_3b_4\}$. Obviously, the graphs Q and $\bar{Q}$ are isomorphic.

Let $i, j \in \{1, \ldots, 4\}$ such that $i < j$, and let G be a graph such that $V(G) \cap V(Q) = \emptyset$. We denote by $Q_{ij}(G)$ the graph with

$$V(Q_{ij}(G)) = V(G) \cup V(Q)$$

and

$$E(Q_{ij}(G)) = E(G) \cup E(Q) \cup \{b_iv;\ v \in V(G)\} \cup \{b_jw;\ w \in V(G)\}\,.$$

Let G be a graph such that $V(G) \cap V(Q) = \emptyset$. It is clear that $\overline{Q_{12}(G)}$ is isomorphic with $Q_{13}(\bar{G})$, that $\overline{Q_{13}(G)}$ is isomorphic with $Q_{12}(\bar{G})$, and that $\overline{Q_{23}(G)}$ is isomorphic with $Q_{23}(\bar{G})$.

Let G be a graph of order $\geqq 4$. Assume that there exists a graph G' and an isomorphism $f : G' \to G$ such that one of the following conditions holds:

(0) G' is identical with Q;

(1) there exists a complete graph G_1 of even order such that $V(G_1) \cap V(Q) = \emptyset$, and G' is identical with $Q_{12}(G_1)$;

(2) there exists a graph G_2 of even order such that $V(G_2) \cap V(Q) = \emptyset$, $E(G_2) = \emptyset$, and G' is identical with $Q_{13}(G_2)$;

(3) there exists a graph G_3 such that $V(G_3) \cap V(Q) = \emptyset$, and G' is identical with $Q_{23}(G_3)$.

Then we shall say that G is an excluding graph and that the set $f(V(Q))$ of vertices in G is a basic quadruple in G. We denote by Exc the class of all excluding graphs. It is easy to see that $G \in \mathrm{Exc}$ if and only if $\bar{G} \in \mathrm{Exc}$. Moreover, if $G \in \mathrm{Exc}$ and B is a basic quadruple in G, then B is also a basic quadruple in $\bar{G}$.

Now we are ready to prove the main result of this paper:

Theorem. *Let G be a graph of order $p \geqq 4$. If $G \in \mathrm{Exc}$, then* $\mathrm{eul}(G) = p - 2 = \mathrm{eul}(\bar{G})$. *If $G \notin \mathrm{Exc}$, then either* $\mathrm{eul}(G) \geqq p - 1$ *or* $\mathrm{eul}(\bar{G}) \geqq p - 1$.

Proof. First, let $p = 4$. Since the complete graph of order four has precisely six edges, we assume without loss of generality that $|E(G)| \geqq 3$. If G contains a cycle, then $G \notin \mathrm{Exc}$, and $\mathrm{eul}(G) \geqq 3$. Assume that G does not contain a cycle. Since $|E(G)| \geqq 3$, G is a tree. If G is a path, then $\bar{G}$ is isomorphic to G, and thus $G \in \mathrm{Exc}$ and $\mathrm{eul}(G) = 2 = \mathrm{eul}(\bar{G})$. If G is not a path, then it is a star, and thus $G \notin \mathrm{Exc}$ and $\mathrm{eul}(\bar{G}) = 3$. Hence, for $p = 4$ the result of the theorem is proved.

Now, let $p = n \geqq 5$. Assume that for $p = n - 1$ the result of the theorem is proved. If $G \in \mathrm{Exc}$, then it follows from the definition of an excluding graph that $\mathrm{eul}(G) = p - 2 = \mathrm{eul}(\bar{G})$.

Now, let $G \notin \mathrm{Exc}$. Consider an arbitrary vertex u_1 of G. Obviously, $\overline{G - u_1}$ is identical with $\bar{G} - u_1$. We distinguish a number of cases:

Case 1. Assume that $G - u_1 \in \mathrm{Exc}$. Let B be a basic quadruple of $G - u_1$. Then $\bar{G} - u_1 \in \mathrm{Exc}$, and B is also a basic quadruple of $\bar{G} - u_1$. Without loss of generality we assume that in G the vertex u_1 is adjacent to at least two vertices of B. If in G the vertex u_1 is adjacent to at least three vertices of B, then $\mathrm{eul}\,(G) \geqq p - 1$. If both in G and in $\bar{G}$ the vertex u_1 is adjacent to two vertices of B, then either $\mathrm{eul}\,(G) \geqq p - 1$ or $\mathrm{eul}\,(\bar{G}) \geqq p - 1$ (otherwise $G \in \mathrm{Exc}$, which is a contradiction).

Case 2. Assume that $G - u_1 \notin \mathrm{Exc}$. Thus $\bar{G} - u_1 \notin \mathrm{Exc}$. According to the induction assumption either $\mathrm{eul}\,(G - u_1) \geqq p - 2$ or $\mathrm{eul}\,(\bar{G} - u_1) \geqq p - 2$. Without loss of generality we assume that $\mathrm{eul}\,(G - u_1) \geqq p - 2$. If $\mathrm{eul}\,(G) \geqq p - 1$, then the theorem is proved. Let $\mathrm{eul}\,(G) \leqq p - 2$. Since $\mathrm{eul}\,(G - u_1) \leqq \mathrm{eul}\,(G)$, we have that $\mathrm{eul}\,(G - u_1) = p - 2$. Then there exists $u_2 \in V(G - u_1)$ such that $G - u_1 - u_2$ contains a spanning eulerian subgraph, say F. We shall prove that $\mathrm{eul}\,(\bar{G}) \geqq p - 1$.

Let $i \in \{1, 2\}$. Denote

$$R_i = \{v \in V(G - u_1 - u_2);\ u_i v \in E(G)\}\,,$$

$$R_{12} = \{v \in V(G - u_1 - u_2);\ u_1 v,\ u_2 v \in E(G)\}$$

and

$$S_{12} = \{v \in V(G - u_1 - u_2);\ u_1 v,\ u_2 v \in E(\bar{G})\}\,.$$

Moreover, we denote $m = |S_{12}|$. Assume that there exist distinct $v_1, v_2 \in R_i$ such that $v_1 v_2 \in E(G)$. If $v_1 v_2 \in E(F)$ then $F_{(u_i)} + u_i v_1 + u_i v_2 - v_1 v_2$ is an eulerian subgraph of G, and thus $\mathrm{eul}\,(G) \geqq p - 1$; a contradiction. If $v_1 v_2 \notin E(F)$, then $F_{(u_i)} + u_i v_1 + u_i v_2 + v_1 v_2$ is an eulerian subgraph of G; a contradiction. This implies that R_i is an independent set of vertices in G.

Case 2.1. Assume that $u_1 u_2 \in E(G)$. Therefore $R_{12} = \emptyset$ (otherwise there exists $v \in V(G - u_1 - u_2)$ such that $F_{(u_1, u_2)} + u_1 u_2 + u_1 v + u_2 v$ is an eulerian subgraph of G, and thus $\mathrm{eul}\,(G) = p$, which is a contradiction). We have that $R_1 \cup R_2$ is an independent set of vertices in G (otherwise there exist distinct vertices $v_1, v_2 \in V(G - u_1 - u_2)$ such that $u_1 v_1,\ u_2 v_2,\ v_1 v_2 \in E(G)$, and thus $\mathrm{eul}\,(G) = p$, which is a contradiction). Since F contains a cycle, we have that $m \geqq 2$. Clearly, $\bar{G} - (R_1 \cup R_2)$ contains a spanning subgraph isomorphic to $K(2, m)$.

Case 2.1.1. Assume that $R_1 \cup R_2 = \emptyset$. Then $m = p - 2$. This implies that $\mathrm{eul}\,(\bar{G}) \geqq p - 1$.

Case 2.1.2. Assume that $R_1 \cup R_2$ contains precisely one vertex, say w. Without loss of generality we assume that $u_1 w \in E(G)$. Since $R_{12} = \emptyset$, we have that $u_2 w \in E(\bar{G})$. If p is even, w is adjacent with precisely one vertex in $\bar{G}$ and $\bar{G} - w$ is a complete bipartite graph, then $\bar{G} \in \mathrm{Exc}$, which is a contradiction. If either p is odd, or w is adjacent with at least two vertices in $\bar{G}$, or $\bar{G} - w$ is not a complete bipartite graph, then it is easy to see that $\mathrm{eul}\,(\bar{G}) \geqq p - 1$.

Case 2.1.3. Assume that $|R_1 \cup R_2| \geqq 2$. Since $R_1 \cup R_2$ is an independent set of vertices in G, we have that the subgraph of $\bar{G}$ induced by $R_1 \cup R_2$ is a complete graph. If $R_1 \neq \emptyset \neq R_2$, then $\bar{G}$ contains a $u_1 - u_2$ path P with the property that $V(P) = R_1 \cup R_2 \cup \{u_1, u_2\}$. If either $R_1 = \emptyset$ or $R_2 = \emptyset$, then $\bar{G}$ contains a cycle C such that either $V(C) = R_2 \cup \{u_1\}$ or $V(C) = R_1 \cup \{u_2\}$, respectively. This implies that $\text{eul}(\bar{G}) \geqq p - 1$.

Case 2.2. Assume that $u_1u_2 \notin E(G)$. Then $u_1u_2 \in E(\bar{G})$.

Case 2.2.1. Assume that $R_{12} = \emptyset$. Then $R_1 \cap R_2 = \emptyset$.

Let $m = 0$. Then $R_1 \cup R_2 = V(G - u_1 - u_2)$. Since R_1 and R_2 are independent sets of vertices in G, we have that $G - u_1 - u_2$ contains no cycle of odd length. Since F is eulerian, there exist distinct vertices v_1, v_2, v_3 and v_4 of $G - u_1 - u_2$ such that $v_1v_2,\ v_2v_3,\ v_3v_4 \in E(F)$. Without loss of generality we assume that $v_1 \in R_1$. Hence $v_2, v_4 \in R_2$ and $v_3 \in R_1$. Thus

$$F_{(u_1,u_2)} + u_1v_1 + u_1v_3 + u_2v_2 + u_2v_4 - v_1v_2 - v_3v_4$$

is a spanning eulerian subgraph of G, which is a contradiction. Therefore $m \geqq 1$. Clearly, $\bar{G} - (R_1 \cup R_2) - u_1u_2$ contains a spanning subgraph isomorphic with $K(2, m)$.

Case 2.2.1.1. Assume that either $|R_1| \neq 1$ or $|R_2| \neq 1$. Without loss of generality we assume that $|R_1| \geqq |R_2|$. If $R_1 = \emptyset$, then $\text{eul}(\bar{G}) = p$. If $|R_1| = 1$, then $R_2 = \emptyset$, and thus $\text{eul}(\bar{G}) \geqq p - 1$.

Let $|R_1| \geqq 2$. Then there exists a cycle $C_{(1)}$ in $\bar{G}$ such that $V(C_{(1)}) = R_1 \cup \{u_2\}$. If $|R_2| \geqq 2$, then analogously there exists a cycle $C_{(2)}$ in $\bar{G}$ such that $V(C_{(2)}) = R_2 \cup \{u_1\}$. This implies that if $|R_2| \neq 1$, then $\text{eul}(\bar{G}) = p$, and if $|R_2| = 1$, then $\text{eul}(\bar{G}) \geqq p - 1$.

Case 2.2.1.2. Assume that $|R_1| = 1 = |R_2|$. Let w_1 and w_2 be vertices such that $R_1 = \{w_1\}$ and $R_2 = \{w_2\}$. Clearly $u_1w_2,\ u_2w_1 \in E(\bar{G})$. Since $G \notin \text{Exc}$, we assume without loss of generality that w_1 is adjacent to at least two vertices in $\bar{G}$. It is easy to see that $\text{eul}(\bar{G}) \geqq p - 1$.

Case 2.2.2. Assume that $R_{12} \neq \emptyset$. Then $|R_{12}| = 1$ (otherwise $\text{eul}(G) = p$). It is not difficult to see that $R_1 \cup R_2$ is an independent set of vertices in G. This implies that $m \geqq 2$. Since $\bar{G} - (R_1 \cup R_2) - u_1u_2$ contains a spanning subgraph isomorphic to $K(2, m)$, we have that $\bar{G} - (R_1 \cup R_2)$ contains a spanning eulerian subgraph.

Case 2.2.2.1. Assume that $|R_1 \cup R_2| \neq 2$. If $|R_1 \cup R_2| = 1$, then $\text{eul}(G) \geqq p - 1$. Let $|R_1 \cup R_2| \geqq 3$. Then there exist distinct $v_1, v_2 \in (R_1 \cup R_2) - R_{12}$. Obviously, the subgraph of $\bar{G}$ induced by $R_1 \cup R_2$ contains a spanning $v_1 - v_2$ path. Since in $\bar{G}$ the vertex v_1 is adjacent to u_1 or u_2 and the vertex v_2 is also adjacent to u_1 or u_2, we have that $\text{eul}(\bar{G}) = p$.

Case 2.2.2.2. Assume that $|R_1 \cup R_2| = 2$. Let w_1 and w_2 be the vertices of $R_1 \cup R_2$. Without loss of generality we assume that $w_1 \in R_{12}$ and that $u_1w_2 \in E(\bar{G})$. Obviously, $w_1w_2 \in E(\bar{G})$ but $u_1w_1,\ u_2w_1,\ u_2w_2 \notin E(\bar{G})$. If in $\bar{G}$ the vertex w_1 is adjacent to at least two vertices or the vertex w_2 is adjacent to at least three vertices, then $\text{eul}(\bar{G}) \geqq p - 1$.

Assume that in $\bar{G}$ the vertex w_1 is adjacent only to w_2, and the vertex w_2 is adjacent only to w_1 and u_1. Since $\bar{G} \notin$ Exc, we have that either p is odd or the graph $\bar{G} - u_1 - u_2 - w_1 - w_2$ is not complete. It is not difficult to see that $G - u_1$ contains a spanning eulerian subgraph and thus $\text{eul}(G) \geqq p - 1$, which is a contradiction.

Thus the proof of the theorem is complete.

By a covering subgraph of a graph G we shall mean such a subgraph F of G that every edge of G is incident with a vertex of F. Let G be a connected graph with $|E(G)| \geqq 3$, and let G be no star. HARARY and NASH-WILLIAMS [3] have proved that $L(G)$ is hamiltonian if and only if there exists a covering eulerian subgraph of G.

The theorem we have just proved offers a new proof for the following result originally presented in [4]:

Corollary. *Let G be a graph of order $p \geqq 5$. Then there exists a graph $G' \in \{G, \bar{G}\}$ such that G' is connected and $L(G)$ is hamiltonian.*

Proof. First, let $G \in$ Exc. Since $p \geqq 5$, it is easy to see that either G or $\bar{G}$ contains a covering eulerian subgraph. Since both G and $\bar{G}$ are connected, the result follows.

Next, let $G \notin$ Exc. According to Theorem, either $\text{eul}(G) \geqq p - 1$ or $\text{eul}(\bar{G}) \geqq p - 1$. Without loss of generality we assume that $\text{eul}(G) \geqq p - 1$. The case when $\text{eul}(G) = p$ is obvious. Let $\text{eul}(G) = p - 1$. Then there exists a covering eulerian subgraph of G. Therefore, $L(G)$ is hamiltonian. If G is connected, the result follows. Now, let us assume that G is disconnected. Then it contains precisely one vertex of degree 0, say a vertex u. This implies that $\bar{G}$ contains a spanning star. If $\bar{G}$ is a star, then $L(\bar{G})$ is hamiltonian. Assume that $\bar{G}$ is no star. Consider a maximum matching M in $\bar{G} - u$. Let H be the subgraph of $\bar{G}$ induced by M. Then the graph H' with the properties that $V(H') = V(H) \cup \{u\}$ and

$$E(H') = E(H) \cup \{uv;\ v \in V(H)\}$$

is a covering subgraph of $\bar{G}$. Since H' is eulerian, $L(\bar{G})$ is hamiltonian. Hence the corollary follows.

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