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SYSTEMS OF UNARY ALGEBRAS WITH COMMON ENDOMORPHISMS II

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Part I of this paper has been submitted to Czech. Math. Journ.; for references, cf. Part I. In this Part II the definitions and denotations from Part I will be used. It will be proved that card $\mathrm{Eq}(f) \leqq c$ is valid for each $f \in F$. A constructive description of all elements of the set $\mathrm{Eq}(f)$ will be given.

4. UNARY OPERATIONS EQUIVALENT WITH RESPECT TO ENDOMORPHISMS

Let I be a nonempty set and for each $\iota \in I$ let (A_ι, f_ι) be a connected monounary algebra. Assume that $A_\iota \cap A_\varkappa = \emptyset$ for each $\iota, \varkappa \in I$, $\iota \neq \varkappa$. We denote by $\bigcup_{\iota \in I}(A_\iota . f_\iota)$ the monounary algebra (B, g), where $B = \bigcup_{\iota \in I} A_\iota$ and $g(x) = f_\iota(x)$ for each $x \in A_\iota$, $\iota \in I$.

Now let (A, f) be a monounary algebra. A system of connected monounary algebras $\{(A_\iota, f_\iota)\}_{\iota \in I}$ such that

$$(A, f) = \bigcup_{\iota \in I}(A_\iota, f_\iota)$$

will be called a component partition of (A, f). Obviously, each monounary algebra (A, f) has a uniquely determined component partition.

For each $n \in N$, $n > 1$ we denote by $\mathcal{O}_2(n)$ the class of all algebras belonging to $\mathcal{O}_2$ and having a cycle with period n. The symbols $\mathcal{O}_{21}(n)$ and $\mathcal{O}_{20}(n)$ have an analogous meaning.

Let (A, f) and (A, g) be monounary algebras. Suppose that $\{(A_\iota, f_\iota)\}_{\iota \in I}$ and $\{(A_\iota, g_\iota)\}_{\iota \in I}$ are the component partitions of (A, f) and (A, g), respectively. We shall consider the following conditions:

(a) If there exists $\iota \in I$ with $(A_\iota, f_\iota) \in \mathcal{K} \cup \mathcal{N}_1$, then $g_\varkappa = f_\varkappa$ for each $\varkappa \in I$.

(b) If there exists $\iota \in I$ with $(A_\iota, f_\iota) \in \mathcal{N}_2$ and $g_\iota = f_\iota$, then $g_\varkappa = f_\varkappa$ for each $\varkappa \in I$.

(c) If there exist $\iota, \varkappa \in I$ and $z \in A_\iota$ such that $(A_\iota, f_\iota) \in \mathcal{O}_1 \cup \mathcal{O}_{21}(2)$, $f_\iota^2(z) \neq z$, $s_f(z) = \infty$, $(A_\varkappa, f_\varkappa) \in \mathcal{N}_2$, then $g_\lambda = f_\lambda$ for each $\lambda \in I$.

(d) If there exist $\iota, \varkappa \in I$, $2 < p_\iota \in N$ such that $(A_\iota, f_\iota) \in \mathcal{O}_{21}(p_\iota)$ and $(A_\varkappa, f_\varkappa) \in \mathcal{N}_2$, then $g_\lambda = f_\lambda$ for each $\lambda \in I$.

(e) If there exists $\iota \in I$ with $(A_\iota, f_\iota) \in \mathcal{N}_2$ and $g_\iota \neq f_\iota$, then for each $\varkappa \in I$, whenever $1 < p_\varkappa \in N$ and $(A_\varkappa, f_\varkappa) \in \mathcal{O}_{20}(p_\varkappa)$, the relation $g_\varkappa = f_\varkappa^{p_\varkappa - 1}$ holds.

(f) If there exist $\iota \in I$ and $1 < p_\iota \in N$ such that $(A_\iota, f_\iota) \in \mathcal{O}_{21}(p_\iota)$, then the following conditions are fulfilled:

(f1) whenever $\varkappa \in I$, $1 < p_\varkappa \in N$, p_ι is divisible by $p_\varkappa$ and $(A_\varkappa, f_\varkappa) \in \mathcal{O}_2(p_\varkappa)$, then $g_\varkappa = f_\varkappa$;

(f2) whenever $\lambda \in I$, $1 < p_\lambda \in N$, p_λ is divisible by p_ι and $(A_\lambda, f_\lambda) \in \mathcal{O}_{20}(p_\lambda)$, then there exists $n \in N$ such that $0 < n < p_\lambda$, n and p_λ are relatively prime, $n \equiv 1 \pmod{p_\iota}$ and $g_\lambda = f_\lambda^n$.

(g) If there are $\iota, \varkappa \in I$, $1 < p_\iota \in N$, $1 < p_\varkappa \in N$, $(A_\iota, f_\iota) \in \mathcal{O}_{20}(p_\iota)$, $(A_\varkappa, f_\varkappa) \in \mathcal{O}_{20}(p_\varkappa)$, where p_ι is divisible by $p_\varkappa$, then there exists $n \in N$ such that $0 < n < p_\iota$, n and p_ι are relatively prime and $g_\iota = f_\iota^n$, $g_\varkappa = f_\varkappa^n$.

Lemma 10. *Let (A, f) be a monounary algebra nad let $\{(A_\iota, f_\iota)\}_{\iota \in I}$ be the component partition of (A, f). Suppose that $g \in F(A)$ and that g and f are equivalent with respect to endomorphisms. Then we have:*

(i) *$\{(A_\iota, g_\iota)\}_{\iota \in I}$ is the component partition of (A, g) (g_ι is the operation g reduced to the set A_ι);*

(ii) *the conditions* (a)–(g) *are fulfilled;*

(iii) *$g_\iota \operatorname{eq} f_\iota$ for each $\iota \in I$.*

Proof. From Lemma 4 of Part I it follows that if (A_ι, f_ι) is a connected component of (A, f) and if g_ι is the operation g reduced to A_ι, then (A_ι, g_ι) is a connected component of (A, g). Thus $\{(A_\iota, g_\iota)\}_{\iota \in I}$ is the component partition of (A, g).

Let $\iota \in I$ and let H be a mapping of A_ι into A_ι. Put $\bar{H}(x) = H(x)$ for each $x \in A_\iota$, $\bar{H}(x) = x$ for each $x \in A - A_\iota$. The mapping $\bar{H}$ is a homomorphism with respect to f if and only if H is a homomorphism with respect to f_ι. Analogously, $\bar{H}$ is a homomorphism with respect to g if and only if H is a homomorphism with respect to g_ι. Since $g \operatorname{eq} f$, we obtain that H is a homomorphism with respect to g_ι if and only if H is a homomorphism with respect to f_ι, i.e., $g_\iota \operatorname{eq} f_\iota$. Let us remark that according to the results of § 3, the relation $g_\iota \neq f_\iota$ can hold only if $(A_\iota, f_\iota) \in \mathcal{O}_{20} \cup \mathcal{N}_2$.

a) Let the assumption of the condition (a) be fulfilled. Suppose that $\varkappa \in I$, $1 < p_\varkappa \in N$, $(A_\varkappa, f_\varkappa) \in \mathcal{O}_{20}(p_\varkappa)$. Let $x \in A_\iota$. If $z \in A_\iota$, then there are $m, n \in N$ with $z \in f_\iota^{-m}(f_\iota^n(x))$. Let y belong to the cycle of $(A_\varkappa, f_\varkappa)$. Put

$$H(z) = z \quad \text{for each} \quad z \in A - A_\iota\,,$$

$$H(z) = f_\varkappa^{kp_\varkappa + n - m}(y) \quad \text{for each} \quad z \in f_\iota^{-m}(f_\iota^n(x))\,, \quad \text{where } k$$

is an integer such that $\quad kp_\varkappa + n - m \geqq 0\,.$

The mapping H is a homomorphism with respect to f, thus H is a homomorphism with respect to g. Then we have

$$f_\varkappa(y) = f_\varkappa(H(x)) = H(f_\iota(x)) = H(g_\iota(x)) = g_\varkappa(H(x)) = g_\varkappa(y),$$

and according to Thm. 2 the relation $f_\varkappa = g_\varkappa$ is valid.

Let $\lambda \in I$ be such that $(A_\lambda, g_\lambda) \in \mathcal{N}_2$. From Lemma 5 it follows that there are distinct elements $\{x_i\}_{i\in Z}$ in A_λ such that $f_\lambda(x_i) = x_{i+1}$ for each $i \in Z$. Put

$$G(z) = z \quad \text{for each} \quad z \in A - A_\iota,$$
$$G(z) = x_{n-m} \quad \text{for each} \quad z \in f_\iota^{-m}(f_\iota^n(x)).$$

The mapping G is a homomorphism with respect to f, thus G is a homomorphism with respect to g and hence

$$f_\lambda(x_0) = x_1 = G(f_\iota(x)) = G(g_\iota(x)) = g_\lambda(G(x)) = g_\lambda(x_0).$$

Then according to Thm. 1 we have $g_\lambda = f_\lambda$. Thus we have proved that the condition (a) is valid.

b) The condition (b) can be proved analogously.

c) Let the assumption of the condition (c) be fulfilled. If $g_\varkappa \neq f_\varkappa$, then Lemma 5 and Thm. 1 imply that there are distinct elements $\{x_i\}_{i\in Z}$ in $A_\varkappa$ such that $f_\varkappa(x_i) = x_{i+1}$, $g_\varkappa(x_i) = x_{i-1}$ for each $i \in Z$, $f_\varkappa(y) \in \{x_i : i \in Z\}$ for each $y \in A_\varkappa$. Since $s_f(z) = \infty$, $f_\iota^2(z) \neq z$, there exist distinct elements $\{z_j\}_{j\in N_0}$ in A_ι with $z_0 = z$, $f_\iota(z_j) = z_{j-1}$ for each $j \in N$. Put

$$H(y) = y \quad \text{for each} \quad y \in A - A_\varkappa,$$
$$H(y) = f_\iota^{i-1}(z) \quad \text{for each} \quad y \in f_\varkappa^{-1}(x_i), \quad i \in N,$$
$$H(y) = z_{-i+1} \quad \text{for each} \quad y \in f_\varkappa^{-1}(x_i), \quad i \in Z, \quad i \leqq 0.$$

The mapping H is a homomorphism with respect to f, hence H is a homomorphism with respect to g. There exists a least positive integer n such that $f^n(z)$ belongs to the cycle of (A_ι, f_ι). According to Thm. 3 we have $g_\iota = f_\iota$. Then we obtain

$$f_\iota^{n-1}(z) = H(x_{n-1}) = H(g_\varkappa(x_n)) = g_\iota(H(x_n)) = g_\iota(f_\iota^n(z)) = f_\iota(f_\iota^n(z)) = f_\iota^{n+1}(z),$$

which is a contradiction. Hence $g_\varkappa = f_\varkappa$ and the condition (b) yields $g_\lambda = f_\lambda$ for each $\lambda \in I$.

d) Let the assumption of the condition (d) be fulfilled and let z be an element of A_ι such that $f_\iota^{p_\iota}(z) = z$. If $g_\varkappa \neq f_\varkappa$, then according to Lemma 5 and Thm. 1 there are distinct elements $\{x_i\}_{i\in Z}$ in $A_\varkappa$ with $f_\varkappa(x_i) = x_{i+1}$, $g_\varkappa(x_i) = x_{i-1}$ for each $i \in Z$, $f_\varkappa(y) \in \{x_i : i \in Z\}$ for each $y \in A_\varkappa$. Put

$$H(y) = y \quad \text{for each} \quad y \in A - A_\varkappa,$$
$$H(y) = f_\iota^{kp_\iota+i-1}(z) \quad \text{for each} \quad y \in f_\varkappa^{-1}(x_i), \quad i \in Z,$$

where k is an integer such that $kp_\iota + i - 1 \geqq 0$.

The mapping H is a homomorphism with respect to f and hence H is a homomorphism with respect to g. Thm. 3 implies $g_\iota = f_\iota$, thus we have

$$H(g_\varkappa(x_0)) = H(x_{-1}) = f_\iota^{p_\iota - 1}(z) \neq f_\iota(z) = g_\iota(z) = g_\iota(H(x_0)),$$

and this is a contradiction. Hence $g_\varkappa = f_\varkappa$ and then the condition (b) yields that $g_\lambda = f_\lambda$ for each $\lambda \in I$.

e) Let the assumption of the condition (e) be fulfilled. Then there are distinct elements $\{x_i\}_{i \in Z}$ in A_ι such that $f_\iota(x_i) = x_{i+1}$, $g_\iota(x_i) = x_{i-1}$ for each $i \in Z$, $f_\iota(y) \in \{x_i : i \in Z\}$ for each $y \in A_\iota$. Let x be an element belonging to the cycle of $(A_\varkappa, f_\varkappa)$. Put

$$H(y) = y \quad \text{for each} \quad y \in A - A_\iota,$$

$$H(y) = f_\varkappa^{kp_\varkappa + i - 1}(x) \quad \text{for each} \quad y \in f_\iota^{-1}(x_i), \quad i \in Z,$$

where k is an integer such that $kp_\varkappa + i - 1 \geqq 0$.

The mapping H is a homomorphism with respect to f, hence H is a homomorphism with respect to g and we obtain

$$g_\varkappa(x) = g_\varkappa(H(x_0)) = H(g_\iota(x_0)) = H(x_{-1}) = f_\varkappa^{p_\varkappa - 1}(x).$$

According to Thm. 2, $g_\varkappa = f_\varkappa^{p_\varkappa - 1}$ holds.

f) Let the assumption of the condition (f) be fulfilled. Suppose that $(A_\varkappa, f_\varkappa) \in \mathcal{O}_2(p_\varkappa)$, where p_ι is divsible by $p_\varkappa$, $1 < p_\varkappa \in N$. Let x and z be elements belonging to the cycles of (A_ι, f_ι) and $(A_\varkappa, f_\varkappa)$, respectively. Put

$$H(y) = y \quad \text{for each} \quad y \in A - A_\iota,$$

$$H(y) = f_\varkappa^{p_\varkappa - i}(z) \quad \text{for each} \quad y \in f_\iota^{-i}(x), \quad i \in N_0,$$

where k is an integer such that $kp_\varkappa - i \geqq 0$.

From the fact that p_ι is divisible by $p_\varkappa$ it follows that the mapping H is correctly defined. The mapping H is a homomorphism with respect to f and thus H is a homomorphism with respect to g. Since $f_\iota = g_\iota$ (cf. Thm. 3), we have

$$g_\varkappa(z) = g_\varkappa(H(x)) = H(g_\iota(x)) = H(f_\iota(x)) = f_\varkappa(H(x)) = f_\varkappa(z).$$

Then Thm. 2 yields $g_\varkappa = f_\varkappa$.

Suppose that $(A_\lambda, f_\lambda) \in \mathcal{O}_{20}(p_\lambda)$, where p_λ is divisible by p_ι, $1 < p_\lambda \in N$. Because of $g_\lambda \operatorname{eq} f_\lambda$, according to Thm. 2 there exists $n \in N$ such that $0 < n < p_\lambda$, n and p_λ are relatively prime and $g_\lambda = f_\lambda^n$. Let u be an element belonging to the cycle of (A_λ, f_λ); we set

$$G(y) = y \quad \text{for each} \quad y \in A - A_\lambda,$$

$$G(y) = f_\iota^{i-1}(x) \quad \text{for each} \quad y \in f_\lambda^{-1}(f_\lambda^i(u)), \quad i \in N, \quad 0 < i \leqq p_\lambda.$$

The mapping G is a homomorphism with respect to f, hence G is a homomorphism with respect to g and we get

$$f_\iota(x) = g_\iota(x) = g_\iota(H(u)) = H(g_\lambda(u)) = H(f^n_\lambda(u)) = f^n_\iota(H(u)) = f^n_\iota(x) .$$

Thus $n \equiv 1 \pmod{p_\iota}$.

g) Let the assumption of the condition (g) be fulfilled. From Thm. 2 it follows that there is $n \in N$ such that $0 < n < p_\iota$, n and p_ι are relatively prime and $g_\iota = f^n_\iota$. Let x and z be elements belonging to the cycles of (A_ι, f_ι) and $(A_\varkappa, f_\varkappa)$, respectively. Put

$$H(y) = y \quad \text{for each} \quad y \in A - A_\iota ,$$
$$H(y) = f^{i-1}_\varkappa(z) \quad \text{for each} \quad y \in f^{-1}_\iota(f^i_\iota(x)) , \quad i \in N , \quad 0 < i \leqq p_\iota .$$

The mapping H is a homomorphism with respect to f, thus H is a homomorphism with respect to g and we have

$$g_\varkappa(z) = g_\varkappa(H(x)) = H(g_\iota(x)) = H(f^n_\iota(x)) = f^n_\varkappa(H(x)) = f^n_\varkappa(z) .$$

Then, according to Thm. 2, $g_\varkappa = f^n_\varkappa$ is valid.

Lemma 11. *Let (A, f) be a monounary algebra and let $\{(A_\iota, f_\iota)\}_{\iota \in I}$ be the component partition of (A, f). Suppose that $g \in F(A)$ and that $H : A \to A$ is a homomorphism with respect to f. If the conditions* (i)–(iii) *from Lemma* 10 *are fulfilled, then H is a homomorphism with respect to g.*

Proof. We have to prove that the relation $H(g(x)) = g(H(x))$ holds for each $x \in A$. Let $\iota \in I$ and let $x \in A_\iota$. From [7] (Thm 7.1) it follows that there exists $\varkappa \in I$ with $H(A_\iota) \subseteq A_\varkappa$. If $f_\iota = g_\iota$ and $f_\varkappa = g_\varkappa$, then the assertion is obvious. Let us remark that if $\lambda \in I$, $g_\lambda \neq f_\lambda$, then $(A_\lambda, f_\lambda) \in \mathscr{N}_2 \cup \mathscr{O}_{20}$. Hence it suffices to assume that either $f_\iota \neq g_\iota$ or $f_\varkappa \neq g_\varkappa$ holds.

Suppose that $f_\iota = g_\iota$. If $(A_\iota, f_\iota) \in \mathscr{K} \cup \mathscr{N}_1 \cup \mathscr{N}_2$, then according to the conditions (a) and (b) the relation $g_\varkappa = f_\varkappa$ holds. If $(A_\iota, f_\iota) \in \mathscr{O}_1$, then $(A_\varkappa, f_\varkappa) \in \mathscr{O}_1$ and $g_\varkappa = f_\varkappa$. Let $1 < p_\iota \in N$, $(A_\iota, f_\iota) \in \mathscr{O}_2(p_\iota)$. Then $(A_\varkappa, f_\varkappa) \in \mathscr{O}_1 \cup \mathscr{O}_2(p_\varkappa)$, where p_ι is divisible by $p_\varkappa$, $1 < p_\varkappa \in N$. If $(A_\varkappa, f_\varkappa) \in \mathscr{O}_1$, then $g_\varkappa = f_\varkappa$. If $(A_\iota, f_\iota) \in \mathscr{O}_{21}$, then the condition (f) implies that $g_\iota = f_\iota$. If $(A_\iota, f_\iota) \in \mathscr{O}_{20}$, then the condition (g) and the fact that $g_\iota = f^1_\iota$ yield $g_\varkappa = f_\varkappa$.

Assume that $f_\iota \neq g_\iota$. Hence the assumption of none of the conditions (a)–(d) is fulfilled. Let $(A_\iota, f_\iota) \in \mathscr{N}_2$. From Thm. 1 it follows that $A_\iota = \bigcup_{i \in Z}(\{x_i\} \cup B_i)$, where $x_i \neq x_j$, $B_i \cap B_j = \emptyset$ for each $i, j \in Z$, $i \neq j$, $f_\iota(b_i) = x_{i+1}$, $g_\iota(b_i) = x_{i-1}$ for each $b_i \in \{x_i\} \cup B_i$. Then $(A_\varkappa, f_\varkappa) \in \mathscr{N}_2 \cup \mathscr{O}_1 \cup \mathscr{O}_{21}(2) \cup \mathscr{O}_{20}$. First let $(A_\varkappa, f_\varkappa) \in \mathscr{N}_2$. Thus according to the condition (b), $g_\varkappa \neq f_\varkappa$ and $A_\varkappa = \bigcup_{i \in Z}(\{y_i\} \cup D_i)$, where $y_i \neq y_j$, $D_i \cap D_j = \emptyset$ for each $i, j \in Z$, $i \neq j$, $f_\varkappa(d_i) = y_{i+1}$, $g_\varkappa(d_i) = y_{i-1}$ for each $d_i \in \{y_i\} \cup D_i$, $i \in Z$. Since H is a homomorphism with respect to f, there exists $k \in Z$ such that $H(x_i) = y_{i+k}$, $H(B_i) \subseteq \{y_{i+k}\} \cup D_{i+k}$ for each $i \in Z$. Then H is also a homomorphism with respect to g.

If $(A_\varkappa, f_\varkappa) \in \mathcal{O}_1$, $y \in A_\varkappa$ with $f_\varkappa(y) = y$, then according to the condition (c), we have $H(x_i) = y$, $H(B_i) \subseteq f_\varkappa^{-1}(y)$ for each $i \in Z$ and then H is a homomorphism with respect to g as well. If $(A_\varkappa, f_\varkappa) \in \mathcal{O}_{21}(2)$, then according to the condition (c) there exist distinct elements $y_1, y_2 \in A_\varkappa$ such that $f_\varkappa^2(y_1) = y_1$, $f_\varkappa^2(y_2) = y_2$, $H(x_{2i}) = y_1$, $H(x_{2i+1}) = y_2$, $H(B_{2i}) \subseteq f_\varkappa^{-1}(y_2)$, $H(B_{2i+1}) \subseteq f_\varkappa^{-1}(y_1)$ for each $i \in Z$. Then H is also a homomorphism with respect to g, since

$$H(g_\iota(b_{2i})) = H(x_{2i-1}) = y_2 = f_\varkappa(H(b_{2i})) = g_\varkappa(H(b_{2i})),$$

$$H(g_\iota(b_{2i+1})) = H(x_{2i}) = y_1 = f_\varkappa(H(b_{2i+1})) = g_\varkappa(H(b_{2i+1}))$$

$$\text{for each} \quad b_{2i} \in B_{2i} \cup \{x_{2i}\}, \quad b_{2i+1} \in B_{2i+1} \cup \{x_{2i+1}\}, \quad i \in Z.$$

Let $(A_\varkappa, f_\varkappa) \in \mathcal{O}_{20}(p_\varkappa)$, $1 < p_\varkappa \in N$, and let $z = H(x_0)$. Then $H(x_i) = f_\varkappa^{kp_\varkappa + i}(z)$, $H(B_i) \subseteq f_\varkappa^{-1}(f_\varkappa^{kp_\varkappa + i + 1}(z))$ for each $i \in Z$, where k is an integer such that $kp_\varkappa + i \geqq 0$. From Thm. 2 it follows that $g_\varkappa(y) = g_\varkappa(f_\varkappa^j(z))$ for each $y \in f_\varkappa^{-1}(f_\varkappa^{j+1}(z))$. Since from the condition (e) we get $g_\varkappa = f_\varkappa^{p_\varkappa - 1}$, we obtain

$$g_\varkappa(H(b_i)) = g_\varkappa(f_\varkappa^{kp_\varkappa + i}(z)) = f_\varkappa^{p_\varkappa - 1}(f_\varkappa^{kp_\varkappa + i}(z)) = f_\varkappa^{(k+1)p_\varkappa + i - 1}(z) =$$
$$= H(x_{i-1}) = H(g_\iota(b_i))$$
$$\text{for each} \quad b_i \in B_i \cup \{x_i\}, \quad i \in Z.$$

Now suppose that $(A_\iota, f_\iota) \in \mathcal{O}_{20}(p_\iota)$, $1 < p_\iota \in N$. Then either $(A_\varkappa, f_\varkappa) \in \mathcal{O}_1$ or there exists $1 < p_\varkappa \in N$ with $(A_\varkappa, f_\varkappa) \in \mathcal{O}_2(p_\varkappa)$, where p_ι is divisible by $p_\varkappa$. Let y be an element belonging to the cycle of (A_ι, f_ι). Put $z = H(y)$. Then z belongs to the cycle of $(A_\varkappa, f_\varkappa)$. If $(A_\varkappa, f_\varkappa) \in \mathcal{O}_1$, then $H(f_\iota^i(y)) = z$, $H(u) \in f_\varkappa^{-1}(z)$ for each $u \in f_\iota^{-1}(f_\iota^i(y))$, $i \in N_0$. For each $u \in A_\iota$, $g_\iota(u)$ belongs to the cycle of (A_ι, f_ι) and hence we have

$$H(g_\iota(u)) = z = f_\varkappa(H(u)) = g_\varkappa(H(u)).$$

If $(A_\varkappa, f_\varkappa) \in \mathcal{O}_{21}$, then $g_\varkappa = f_\varkappa$ and from the condition (f) it follows that there exists $n \in N$ such that $0 < n < p_\iota$, n and p_ι are relatively prime, $n \equiv 1 \pmod{p_\varkappa}$ and $g_\iota = f_\iota^n$. Further, for each $u \in A_\iota$, the element $f_\varkappa(H(u)) = H(f_\iota(u))$ belongs to the cycle of $(A_\varkappa, f_\varkappa)$. Hence

$$H(g_\iota(u)) = H(f_\iota^n(u)) = f_\varkappa^n(H(u)) = f_\varkappa(H(u)) = g_\varkappa(H(u)).$$

If $(A_\varkappa, f_\varkappa) \in \mathcal{O}_{20}(p_\varkappa)$, then the condition (g) implies that there exists $n \in N$ such that $0 < n < p_\iota$, n and p_ι are relatively prime, and $g_\iota = f_\iota^n$, $g_\varkappa = f_\varkappa^n$. Then for each $v \in A_\iota$ we have

$$H(g_\iota(v)) = H(f_\iota^n(v)) = f_\varkappa^n(H(v)) = g_\varkappa(H(v)).$$

Hence we have proved that $H(g(x)) = g(H(x))$ is valid for each $x \in A_\iota$, $\iota \in I$.

Let us denote by $(a')-(g')$ the conditions that we obtain from the conditions (a)$-$(g) by interchanging the operations f and g.

Lemma 12. *Let (A, f) be a monounary algebra, $g \in F(A)$ and let the conditions* (i), (ii) *and* (iii) *from Lemma* 10 *be fulfilled. Then the conditions* (a′)–(g′) *hold.*

Proof. a) If $\iota \in I$, $(A_\iota, g_\iota) \in \mathscr{K} \cup \mathscr{N}_1$, then it follows from $g_\iota \operatorname{eq} f_\iota$ and from Thm. 3 that $g_\iota = f_\iota$; from this and from the condition (a) we have $g_\varkappa = f_\varkappa$ for each $\varkappa \in I$.

b) The conditions (b) and (b′) are identical.

c) If there exist $\iota, \varkappa \in I$ and $z \in A_\iota$ such that $(A_\iota, g_\iota) \in \mathscr{O}_1 \cup \mathscr{O}_{21}(2)$, $g_\iota^2(z) \neq z$, $s_g(z) = \infty$, $(A_\varkappa, g_\varkappa) \in \mathscr{N}_2$, then Thm. 1 yields $(A_\varkappa, f_\varkappa) \in \mathscr{N}_2$. Since according to Thm. 3 we have $g_\iota = f_\iota$, we obtain from the condition (c) that $g_\lambda = f_\lambda$ for each $\lambda \in I$.

d) If there are $\iota, \varkappa \in I$ such that $(A_\iota, g_\iota) \in \mathscr{O}_{21}(p_\iota)$, $2 < p_\iota \in N$, $(A_\varkappa, g_\varkappa) \in \mathscr{N}_2$, then according to Thm. 3 the relation $g_\iota = f_\iota$ holds and according to Thm. 1 we have $(A_\varkappa, f_\varkappa) \in \mathscr{N}_2$. Then from the condition (d) we get $g_\lambda = f_\lambda$ for each $\lambda \in I$.

e) If there exists $\iota \in I$ such that $(A_\iota, g_\iota) \in \mathscr{N}_2$, $g_\iota \neq f_\iota$ and if $\varkappa \in I$, $1 < p_\varkappa \in N$, $(A_\varkappa, g_\varkappa) \in \mathscr{O}_{20}(p_\varkappa)$, then according to Thm. 1 and Thm. 2 we have $(A_\iota, f_\iota) \in \mathscr{N}_2$ and $(A_\varkappa, f_\varkappa) \in \mathscr{O}_{20}(p_\varkappa)$. Hence the condition (e) yields $g_\varkappa = f_\varkappa^{p_\varkappa - 1}$. Since

$$(p_\varkappa - 1) \, . \, (p_\varkappa - 1) \equiv 1 \pmod{p_\varkappa},$$

according to Remark 1 (after Thm. 2) we have $f_\varkappa = g_\varkappa^{p_\varkappa - 1}$.

f) If there exist $\iota \in I$, $1 < p_\iota \in N$ such that $(A_\iota, g_\iota) \in \mathscr{O}_{21}(p_\iota)$, then Thm. 3 implies $g_\iota = f_\iota$. If $\varkappa \in I$, $1 < p_\varkappa \in N$, $(A_\varkappa, g_\varkappa) \in \mathscr{O}_2(p_\varkappa)$ and if p_ι is divisible by $p_\varkappa$, then according to Thms. 2 and 3 we have $(A_\varkappa, f_\varkappa) \in \mathscr{O}_2(p_\varkappa)$; hence from the condition (f1) it follows that $g_\varkappa = f_\varkappa$. If $\lambda \in I$, $1 < p_\lambda \in N$, $(A_\lambda, g_\lambda) \in \mathscr{O}_{20}(p_\lambda)$ and p_λ is divisible by p_ι, then according to Thm. 2 we obtain $(A_\lambda, f_\lambda) \in \mathscr{O}_{20}(p_\lambda)$. Thus it follows from the condition (f2) that there exists $n \in N$ such that $0 < n < p_\lambda$, n and p_λ are relatively prime, $n \equiv 1 \pmod{p_\iota}$ and $g_\lambda = f_\lambda^n$. Hence according to Remark 1 there exists $m \in N$ such that $0 < m < p_\lambda$, m and p_λ are relatively prime, $m \, . \, n \equiv 1 \pmod{p_\lambda}$ and $f_\lambda = g_\lambda^m$. Then obviously $m \equiv 1 \pmod{p_\iota}$.

g) If $\iota, \varkappa \in I$, $1 < p_\iota \in N$, $1 < p_\varkappa \in N$, $(A_\iota, g_\iota) \in \mathscr{O}_{20}(p_\iota)$, $(A_\varkappa, g_\varkappa) \in \mathscr{O}_{20}(p_\varkappa)$ and if p_ι is divisible by $p_\varkappa$, then it follows from Thm. 2 that $(A_\iota, f_\iota) \in \mathscr{O}_{20}(p_\iota)$, $(A_\varkappa, f_\varkappa) \in \mathscr{O}_{20}(p_\varkappa)$. From the condition (g) we obtain that there exists $n \in N$ such that $0 < n < p_\iota$, n and p_ι are relatively prime and $g_\iota = f_\iota^n$, $g_\varkappa = f_\varkappa^n$. According to Remark 1 there exists $m \in N$ such that $0 < m < p_\iota$, m and p_ι are relatively prime, $mn \equiv 1 \pmod{p_\iota}$ and $f_\iota = g_\iota^m$. Then also $f_\varkappa = f_\varkappa^{nm} = g_\varkappa^m$.

Theorem 4. *Let (A, f) be a monounary algebra and let $\{A_\iota, f_\iota)\}_{\iota \in I}$ be the component partition of (A, f). Further let $g \in F(A)$. The operations f and g are equivalent with respect to endomorphisms if and only if the conditions* (i), (ii), (iii) *from Lemma* 10 *are fulfilled.*

Proof. If $f \operatorname{eq} g$, then according to Lemma 10, the conditions (i), (ii) and (iii) are valid. Conversely, suppose that the conditions (i)–(iii) are fulfilled. If a mapping $H : A \to A$ is a homomorphism with respect to f, then according to Lemma 11 the mapping H is a homomorphism with respect to g. From Lemma 12 it follows that the conditions (a′)–(g′) are fulfilled. Hence, if a mapping $H : A \to A$ is a homomorphism with respect to g, then H is a homomorphism with respect to f (this follows from Lemma 11 by interchanging f and g). Thus the operations f and g are equivalent with respect to endomorphisms.

5. THE CARDINALITY OF Eq(f)

Let A be a nonempty set and let $f \in F(A)$. Let c be the cardinality of the continuum. We shall prove that the relation

$$\text{card Eq}(f) \leqq c$$

holds (independently of the cardinality of A).

Let us consider the following cases:

(1) First assume that (A, f) is a connected monounary algebra. Then it follows from Thms. 1–3 that the set Eq(f) is finite.

Further, assume that the algebra (A, f) fails to be connected and that $\{(A_\iota, f_\iota)\}_{\iota \in I}$ is the component partition of (A, f).

(2) If there exists $\iota \in I$ such that $(A_\iota, f_\iota) \in \mathscr{K} \cup \mathscr{N}_1$, then Thm. 4 (cf. the condition (a) from Lemma 10) yields that card Eq$(f) = 1$.

Now suppose that $(A_\iota, f_\iota) \notin \mathscr{K} \cup \mathscr{N}_1$ for each $\iota \in I$.

(3) Let there be $\iota \in I$ with $(A_\iota, f_\iota) \in \mathscr{N}_2$. Suppose that $h \in$ Eq(f) and let h_ι be the operation h reduced to A_ι (cf. Thm. 4). Hence according to Thm. 1 we have card Eq$(f_\iota) = 2$. If $h_\iota = f_\iota$, then it follows from Thm. 4 (cf. the condition (b) from Lemma 10) that $h_\varkappa = f_\varkappa$ for each $\varkappa \in I$; if $h_\iota \neq f_\iota$, then Thm. 4 (cf. the condition (e)) implies that, for each $\varkappa \in I$, the operation $h_\varkappa$ is uniquely determined. Hence card Eq$(f) \leqq 2$.

Further, assume that $(A_\iota, f_\iota) \notin \mathscr{N}_2$ for each $\iota \in I$.

(4) According to the assumption we have $(A_\iota, f_\iota) \in \mathscr{O}_1 \cup \mathscr{O}_2$ for each $\iota \in I$. Denote

$$I(1) = \{\iota \in I : (A_\iota, f_\iota) \in \mathscr{O}_1\},$$
$$I(n) = \{\iota \in I : (A_\iota, f_\iota) \in \mathscr{O}_2(n)\} \quad \text{for each} \quad 1 < n \in N,$$
$$(B^{(n)}, g^{(n)}) = \bigcup_{\iota \in I(n)} (A_\iota, f_\iota) \quad \text{for each} \quad n \in N.$$

Let $h \in$ Eq(f) and let $h^{(1)}$ be the operation h reduced to the set $B^{(1)}$. Thm. 3 implies that $h^{(1)} = g^{(1)}$.

Let $1 < n \in N$ and let $h^{(n)}$ be the operation h reduced to the set $B^{(n)}$. Let $\iota, \varkappa \in I(n)$. From Thm. 2 it follows that $h_\iota = f_\iota^i$ for some $i \in \{1, 2, \ldots, n-1\}$. Moreover, from Thm. 4 (cf. the conditions (f), (g)) we obtain that $h_\varkappa = f_\varkappa^i$. Hence the operation

$h^{(n)}$ is uniquely determined by h_ι; therefore there exists only a finite number of possibilities for $h^{(n)}$. Since

$$(A, h) = \bigcup_{n \in N}(B^{(n)}, h^{(n)})$$

and since the set $\{(B^{(n)}, h^{(n)}) : n \in N\}$ is countable, we infer that we have at most c possibilities for the operation h.

We have proved

Theorem 5.1. *Let A be a nonempty set, $f \in F(A)$. Then* card $\mathrm{Eq}(f) \leqq c$.

Theorem 5.2. *There exists a countable set A and a unary operation f on A such that* card $\mathrm{Eq}(f) = c$.

Proof. Let $\{p_\iota : \iota \in N\}$ be the set of all positive primes greater than 2. Let $\{A_\iota\}_{\iota \in N}$ be a system of mutually disjoint sets such that card $A_\iota = p_\iota$ for each $\iota \in N$. For each $\iota \in N$ we define a unary operation f_ι on A_ι in such a way that A_ι is the cycle of (A_ι, f_ι). Put $(A, f) = \bigcup_{\iota \in N}(A_\iota, f_\iota)$. Then card $A = \aleph_0$. For each $M \subseteq N$ we denote by g_M the unary operation on A by putting

$$g_M(x) = f_\iota^2(x) \quad \text{for each} \quad x \in A_\iota\,, \quad \iota \in M\,,$$

$$g_M(x) = f_\iota(x) \quad \text{for each} \quad x \in A_\iota\,, \quad \iota \in N - M\,.$$

Then Thm. 4 implies that g_M eq f. For $M_1, M_2 \subseteq N$, $M_1 \neq M_2$ we have $g_{M_1} \neq g_{M_2}$. Since the system of all subsets of the set N has the cardinality c, we obtain card $\mathrm{Eq}(f) \geqq c$. Hence according to Thm. 5.1 the relation card $\mathrm{Eq}(f) = c$ is valid.

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