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COMPLETE DISTRIBUTIVITY AND α-CONVERGENCE

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1. Introduction. A totally ordered group (notation: o-group) is well-known to be a Hausdorff topological group and topological lattice (and thus a *topological o-group*) in its interval topology. The interval topology is *compatible* with the order: i.e., if $\bigwedge_{j\in J} |x_j| = e$ for each cofinal subset J of the directed index set I, then the net $(x_i)_{i\in I}$ converges to e where e denotes the identity. Furthermore, if G is any topological o-group then sets of the form $\{x \in G \mid a < x < b\}$ for $a, b \in G$ are open so that the topology in G lies between the interval and the discrete. As ELLIS [4] has remarked, various authors have attempted to generalize these results to lattice-ordered groups (notation: l-groups) and these attempts have been largely unsuccessful. For example, JAKUBÍK has shown that the interval topology of a representable l-group is Hausdorff group topological if and only if the group is an o-group [8]. On the other hand, order convergence (which in the totally ordered case derives from the interval topology) does not in general derive from a topology and in fact is topological only in rather special cases [5, 6, 7]. We shall here generalize results of PAPANGELOU [11, 12] and Ellis [4] to show that in an arbitrary completely distributive l-group G (and only in a completely distributive l-group) the topology from which α-convergence derives makes G into a Hausdorff topological group and topological lattice (so that G is called a *topological l-group* or tl-group and the topology a *tl-topology*) which reduces to the interval topology in the totally ordered case (Theorem 2, 3). This topology for G has fewer open sets than any other tl-topology for G (Theorem 4). The tl-topology from which α-convergence in G derives is (1) compatible with the order (Theorem 5) and (2) is such that G can be continuously embedded by a one-to-one lattice homomorphism π in the Cartesian product of topological chains of the form G/N where each N is a topologically closed prime convex l-subgroup of G and G/N is the collection of right cosets. In fact, the closed prime convex l-subgroups N may be chosen so that each G/N has precisely the interval topology and $\pi : G \to G\pi$ is a homeomorphism (Theorem 2). Conversely, if an arbitrary l-group G has a tl-topology with properties (1) and (2), then G is completely distributive and the given topology is precisely the topology from which α-convergence derives (Corollary 8). For basic terminology see [1, 7, 9].

2. Preliminaries. A net $(x_i)_{i\in I}$ in a lattice L is said to *α-converge* to $x \in L$ (notation: $\alpha - \lim_{i\in I} x_i = x$) if x is the only element of L which satisfies

$$x = \bigvee_{i \geqq i_0} (x_i \wedge x) = \bigwedge_{i \geqq i_0} (x_i \vee x)$$

for every $i_0 \in I$. We say that α-convergence *derives* from the topology T on L (and thus that α-convergence is *topological*) if a net $(x_i)_{i\in I}$ in L converges to x if and only if $\alpha - \lim_{i\in I} x_i = x$. The lattice L is said to be *completely distributive* if

$$\bigwedge_{k\in K} \big(\bigvee_{j\in J} x_{kj}\big) = \bigvee_{f\in F} \big(\bigwedge_{k\in K} x_{kf(k)}\big)$$

holds whenever $\{x_{kj} \mid k \in K, j \in J\}$ is a doubly-indexed subset of L for which all the indicated joins and meets exist and $F = J^K$. It is well-known and easy to see that if L is totally ordered then L is completely distributive and α-convergence on L derives from the interval topology.

A convex l-subgroup M of an l-group G is called *L-closed* if whenever $\{g_i \mid i \in I\} \subseteq M$ and $\bigvee_{i\in I} g_i$ exists then $\bigvee_{i\in I} g_i \in M$. In that case, the natural map $\pi : G \to G/M$ preserves all suprema and infema [2] and is said to be *regular*. The *distributive radical* $D(G)$ is the intersection of the L-closures of the minimal prime convex l-subgroups of G. It was shown in [3] that G is completely distributive if and only if $D(G) = \{e\}$ where e denotes the identity of G. It was shown in [10] that if G is a *tl*-group and M is L-closed then M is (topologically) closed. Thus, if $T_1(G)$ denotes the intersection of the closures of the minimal prime convex l-subgroups of G then $T_1(G) \subseteq D(G)$.

For completeness we shall present a somewhat different proof of one direction of a fundamental result of Ellis. We shall use the following result of [12].

Theorem 1 (Papangelou). *Let G be an l-group. If $\alpha - \lim_{i\in I} x_i = e$ then for each cofinal subset J of I, $\bigwedge_{j\in J} |x_j| = e$. If G is completely distributive the converse also holds.*

Theorem 2 (Ellis [4]). *If α-convergence in an l-group G is topological then G is completely distributive. Conversely, let G be a completely distributive l-group and let $\{N_\beta \mid \beta \in B\}$ be any collection of L-closed prime convex l-subgroups of G with $\bigcap_{\beta\in B} N_\beta = \{e\}$. For $\beta \in B$ let G/N_β denote the chain of right cosets of N_β and give G/N_β the interval topology. Let the full product $\prod_{\beta\in B} (G/N_\beta)$ be ordered componentwise and be given the Cartesian topology T. Then α-convergence derives from the topology that G inherits from T via the natural one-to-one lattice homomorphism $\pi : G \to \prod_{\beta\in B} (G/N_\beta)$. Thus, G is a topological lattice and if representable even a topological group.*

Proof. If $(g_i)_{i\in I}$ is a net in G with $\alpha - \lim_{i\in I} g_i = e$ then for each cofinal subset J of I and for each L-closed N_β, $\beta \in B$, $\bigwedge_{j\in J} N_\beta|g_j| = N_\beta$. To show by way of contradiction that the net $(g_i)_{i\in I}$ convergences to e in the topology G inherits from T suppose there exists $\beta \in B$ and an interval $U = \{N_\beta z \mid N_\beta y < N_\beta z < N_\beta y'\}$ about N_β in G/N_β such that it is not true that $(N_\beta g_i)_{i\in I}$ is eventually in U. Then there exists a cofinal subset J of I with say $N_\beta g_j \leqq N_\beta y$, $j \in J$. Since for $j \in J$, $N_\beta y < N_\beta \leqq N_\beta y g_j^{-1}$ we have

$$N_\beta y g_j < N_\beta g_j \leqq N_\beta y < N_\beta y g_j^{-1}$$

so that $N_\beta y = N_\beta y(\bigwedge_{j\in J} |g_j|) = \bigwedge_{j\in J} N_\beta y|g_j| = \bigwedge_{j\in J} N_\beta y g_j^{-1} \geqq N_\beta > N_\beta y$ for the desired contradiction. Since $\alpha - \lim_{i\in I} g_i = g$ is equivalent to $\alpha - \lim_{i\in I} g_i g^{-1} = e$, it follows immediately that $\alpha - \lim_{i\in I} g_i = g$ implies that the net $(g_i)_{i\in I}$ converges to g in the topology G inherits from T. If on the other hand $(g_i)_{i\in I}$ is eventually in each T-neighborhood of g so for $\beta \in B$, $\alpha - \lim_{i\in I} N_\beta g_i = N_\beta g$, the fact that each $\pi_\beta : G \to G/N_\beta$ is regular guarantees that $\alpha - \lim_{i\in I} g_i = g$.

3. A compatible group topology. We now present a proof that every completely distributive l-group is a tl-group in the topology from which α-convergence derives.

Theorem 3. *An l-group G is completely distributive if and only if α-convergence derives from a topology with which G is a tl-group.*

Proof. By Theorem 2, if α-convergence is topological then G is completely distributive. Now suppose G is completely distributive. By Theorem 2, G is a topological lattice in the topology which derives from α-convergence. Since it follows from the definitions that $\alpha - \lim_{i\in I} x_i = x$, $\alpha - \lim_{i\in I} x_i^{-1} = x^{-1}$, $\alpha - \lim_{i\in I} x_i c = xc$ and $\alpha - \lim_{i\in I} cx_i = cx$ are equivalent for $(x_i)_{i\in I}$ an arbitrary net in G and $x, c \in G$, it only remains to show that if $(x_i)_{i\in I}$ and $(y_j)_{j\in J}$ are nets in G with $\alpha - \lim_{i\in I} x_i = \alpha - \lim_{j\in J} y_j = e$ then $\alpha - \lim_{(i,j)\in I\times J} x_i y_j = e$ where $I \times J$ is ordered component-wise. The results from [12] used below carry over to the non-Abelian case.

Since $\alpha - \lim_{i\in I} x_i = e$ and $\alpha - \lim_{j\in J} y_j = e$ we have $\alpha - \lim_{i\in I} |x_i| = e$ and $\alpha - \lim_{j\in J} |y_j| = e$ [12, Corollary 3.5]. By [12, Proposition 3.3], $\alpha - \lim_{(i,j,k)\in I\times J\times I} |x_i| \cdot |y_j|\,|x_k| = e$ so by [12, Proposition 3.1], $\alpha - \lim_{(i,j)\in I\times J} |x_i|\,|y_j|\,|x_i| = e$. We show that this implies $\alpha - \lim_{(i,j)\in I\times J} |x_i y_j| = e$ so by [12, Proposition 3.6], $\alpha - \lim_{(i,j)\in I\times J} x_i y_j = e$ as desired.

According to [12, Proposition 3.2], if $(z_k)_{k\in K}$ is a net in G with $z_k \geqq e$ for every k then $\alpha - \lim_{k\in K} z_k = e$ if and only if for each $z > e$ there exist $k_0 \in K$ and $u_0 \in G$ such that

$$z > u_0 \geqq z_k \wedge z$$

for all $k \geqq k_0$. Now $\alpha - \lim_{(i,j)\in I\times J} |x_i|\,|y_j|\,|x_i| = e$ so if $z > e$ there exist $i_0 \in I$, $j_0 \in J$, $u_0 \in G$ such that

$$z > u_0 \geqq |x_i|\,|y_j|\,|x_i| \wedge z$$

for all $(i, j) \geqq (i_0, j_0)$. But $|x_i y_j| \leqq |x_i|\,|y_j|\,|x_i|$ so

$$z > u_0 \geqq |x_i y_j| \wedge z$$

so we do in fact have $\alpha - \lim_{(i,j)\in I\times J} |x_i y_j| = e$.

Theorems 4 and 5 make it still more clear that the topology of α-convergence is a suitable generalization to completely distributive l-groups of the interval topology in o-groups.

Theorem 4. *Let G be a completely distributive tl-group. Then the topology for G lies between the discrete and the topology from which α-convergence derives.*

Proof. Let $\{N_\beta \mid \beta \in B\}$ be a collection of L-closed and so topologically closed prime convex l-subgroups of G with $\bigcap_{\beta\in B} N_\beta = \{e\}$. If each G/N_β is given the projection topology and $\prod_{\beta\in B} (G/N_\beta)$ the Cartesian topology, the natural map $\pi : G \to \prod_{\beta\in B} (G/N_\beta)$ is continuous. Since each G/N_β has at least the open sets of the interval topology the result follows from Theorem 2.

Theorem 5. *Let G be a completely distributive l-group. Then any topology (not necessarily a tl-topology) for G whose convergence is implied by α-convergence is compatible with the order on G. In particular, the topology from which α-convergence derives is compatible.*

Proof. Suppose $\bigwedge_{j\in J} |g_j| = e$ for each cofinal subset J of the directed index set I. By Theorem 1, $\alpha - \lim_{i\in I} g_i = e$ so by hypothesis $(g_i)_{i\in I}$ converges to e.

The next result provides a converse for an arbitrary l-group leaving open the question of the necessity of complete distributivity in Theorem 5.

Theorem 6. *Let G be a compatible tl-group. If $(g_i)_{i\in I}$ is a net in G with $\alpha - \lim_{i\in I} g_i = g$ then the net converges to g.*

Proof. Since $\alpha - \lim_{i\in I} g_i g^{-1} = e$, $\bigwedge_{j\in J} |g_j g^{-1}| = e$ for every cofinal subset J of I. By compatibility $(g_i g^{-1})_{i\in I}$ converges to e so $(g_i)_{i\in I}$ converges to g.

Theorems 4 and 6 together show that every completely distributive compatible tl-group has precisely the topology from which α-convergence derives. The following lemma allows us to strengthen that result.

Lemma 7. *In a compatible tl-group each closed l-subgroup is L-closed.*

Proof. Let N be a closed l-subgroup and let $g = \bigvee_{i \in I} g_i$ with $S = \{g_i \mid i \in I\}$ a set of elements in N. Let the set of all elements which can be written in the form hg^{-1} where each h is the supremum of finitely many elements of S be indexed by itself and thus be considered to be a net in N. By compatibility this net converges to e so the net of elements h converges to g forcing $g \in N$.

Corollary 8. *Let G be a compatible tl-group. Then $T_1(G) = D(G)$. Thus if $T_1(G) = \{e\}$, then G is completely distributive and α-convergence derives from the topology on G.*

4. Additional Remarks. Let G be an l-group and define a set X to be closed in G if X contains with every α-convergent net the α-limit. If we call the resulting topology the *α-topology*, then in general, every α-convergent net converges in the α-topology and in particular, if G is completely distributive then α-convergence derives from the α-topology. It is natural to enquire about the α-topology in the case where G is not completely distributive. Unfortunately, an example of FLOYD [6] shows that we cannot in general expect the α-topology to be a tl-topology, for his l-group has no σ-compatible tl-topology and the α-topology is easily seen to satisfy this weaker form of compatibility.

One may also try to topologize non-completely distributive l-groups by observing that every l-group may be embedded in a completely distributive l-group and thus inherits a tl-topology. But in addition to the limitations of Corollary 8 the following example shows that the resulting topology depends upon the embedding. It also shows that a completely distributive l-subgroup of a completely distributive l-group need not be a regular sublattice, thus answering a question raised in [4].

Let G be the l-group of all those integer valued functions f on the set $\{1, 2, \ldots\}$ which have the property that for some integer c, $f(n) = c$ for all but finitely many n. These functions are to be added and ordered component-wise. Then G is completely distributive and so inherits from itself the α-topology. But if $N_i = \{f \mid f(i) = 0\}$ $i = 1, 2, \ldots$ and $N_0 = \{f \mid f(n) = 0$ except for finitely many $n\}$ then G may be embedded in the completely distributive l-group $\prod_{i=0}^{\infty} (G/N_i) \equiv H$. But the α-topology on H does not cut down to the α-topology on G and if the functions $f_i \in G$ are defined by $f_i(j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta then $\bigvee_{i=1}^{\infty} f_i$ in G is not the same as $\bigvee_{i=1}^{\infty} f_i$ in H.

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