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A NOTE ON UPPER EMBEDDABLE GRAPHS

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In the present note only graphs in the sense of the book [1] are considered (multiple edges or loops are not allowed). Let G be a graph. Its vertex set, its edge set, and the number of its components will be denoted by $V(G)$, $E(G)$, and $c(G)$. respectively. If U is a nonempty subset of $V(G)$, then $\langle U \rangle_G$ denotes the subgraph of G induced by U, and $E(U, G)$ denotes the set of edges $e \in E(G)$ with the property that e is incident with exactly one vertex of U. Define $\beta(G) = |E(G)| - |V(G)| + c(G)$.

0. Let G be a connected graph. As was shown in [4], for no integer $n > [\beta(G)/2]$, there exists a 2-cell (or cellular [7]) embedding of G onto the orientable surface of genus n. G is said to be *upper embeddable* if there exists a 2-cell embedding of G onto the orientable surface of genus $[\beta(G)/2]$. (For various details concerning the concept of upper embeddability and related subjects the reader is referred to [6]).

If H is a graph, then we denote by $b(H)$ the number of components F of H with the property that $\beta(F)$ is odd.

We now state two characterizations of upper embeddable graphs:

Theorem 0. *Let G be a connected graph. Then the following statements are equivalent*:

(I) *G is upper embeddable*;

(II) *there exists a spanning tree T of G such that for at most one component F of $G - E(T)$, $|E(F)|$ is odd*;

(III) for every $A \subseteq E(G)$,

$$b(G - A) + c(G - A) - 2 \leqq |A| . \qquad (*)$$

The equivalence (I) $\Leftrightarrow$ (II) was found independently by Jungerman [2] and Xuong [8]. The equivalence (II) $\Leftrightarrow$ (III) follows immediately from the results proved in [3]. In the present note two results will be deduced from the equivalence (I) $\Leftrightarrow$ (III).

1. Let G be a graph, and let n be a positive integer. We shall say that G is *oddly n-edge-connected* if it is connected, and for every nonempty proper subset U of $V(G)$ with the properties that $\langle U \rangle_G$ is connected, $\beta(\langle U \rangle_G)$ is odd, and no component of $G - E(U, G)$ is a tree, it holds that $|E(U, G)| \geqq n$.

Theorem 1. *Every oddly 4-edge-connected graph is upper embeddable.*

Proof. On the contrary, we assume that there exists an oddly 4-edge-connected graph G which is not upper embeddable. It follows from the implication (III) ⇒ (I) that there exists $A \subseteq E(G)$ such that $(*)$ does not hold, and that A is minimal in the sense that for every $A' \subseteq E(G)$, if $|A'| < |A|$, then $b(G - A') + c(G - A') - 2 \leqq \leqq |A'|$. This implies that $c(G - A) = b(G - A)$, and that every component of $G - A$ is an induced subgraph of G. Moreover, we get that $c(G - A) \geqq 2$.

Consider an arbitrary component F of $G - A$. It is clear that F is a component of $G - E(V(F), G)$. Since $c(G - A) = b(G - A)$, we have that $\beta(F)$ is odd and that no component of $G - E(U, G)$ is a tree. Since G is oddly 4-edge-connected, $|E(V(F), G)| \geqq 4$.

This implies that $2|A| \geqq 4b(G - A)$, and thus $(*)$ holds, which is a contradiction. Hence, the theorem follows.

We say that a graph G is cyclically n-edge-connected ($n \geqq 1$) if it is connected, and for every nonempty proper subset U of $V(G)$ with the property that neither of the graphs $\langle U \rangle_G$ and $\langle V(G) - U \rangle_G$ is a forest, it holds that $|E(U, G)| \geqq n$. It is clear that every cyclically n-edge-connected graph ($n \geqq 1$) is oddly n-edge-connected.

Corollary (Payan and Xuong [5]). *If a graph is cyclically 4-edge-connected, then it is upper embeddable.*

Note that Payan and Xuong [5] proved the result in the corollary without utilizing the implication (III) ⇒ (I), but their proof is rather difficult. Theorem 1 is stronger than the corollary: the graphs in Figs. 1 and 2 can serve as examples of oddly 4-edge-connected graphs which are not cyclicaliy 4-edge-connected.

2. We shall say that a connected graph G is *absolutely upper embeddable* if every graph which is spanned by G is upper embeddable. According to the definition, every absolutely upper embeddable graph is upper embeddable. The trees of diameter $\geqq 5$ and the graph in Fig. 1 can serve as examples of upper embeddable graphs which are not absolutely upper embeddable.

Let H be a graph. We denote by $i(H)$ the number of components F of H with the property that either $\beta(F)$ is odd or F is a non-complete graph. Obviously, $i(H) \geqq \geqq b(H)$.

The following theorem gives a characterization of absolutely upper embeddable graphs:

Theorem 2. *A connected graph G is absolutely upper embeddable if and only if*

$$(**) \qquad i(G - A) + c(G - A) - 2 \leqq |A|,$$

for every $A \subseteq E(G)$.

Proof. (1) We first assume that there exists $A \subseteq E(G)$ such that $(**)$ does not hold. We shall assume that A is minimal in the sense that for every $A_0 \subseteq E(G)$, if $|A_0| < < |A|$, then $i(G - A_0) + c(G - A_0) - 2 \leqq |A_0|$. This implies that every component of $G - A$ is an induced subgraph of G. We wish to prove that G is not absolutely upper embeddable.

Consider a graph H obtained from G in such a way that one new edge is inserted into each component F of $G - A$ with the property that F is non-complete and $\beta(F)$ is even. Since every component of $G - A$ is an induced subgraph of G, no new edge of H belongs to A. Since $b(H - A) = i(G - A)$ and $c(H - A) = c(G - A)$, we have that $b(H - A) + c(H - A) - 2 > |A|$. According to the implication (I) $\Rightarrow$ (III), H is not upper embeddable. The desired result follows.

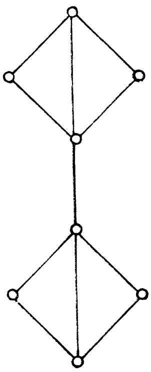

Fig. 1.

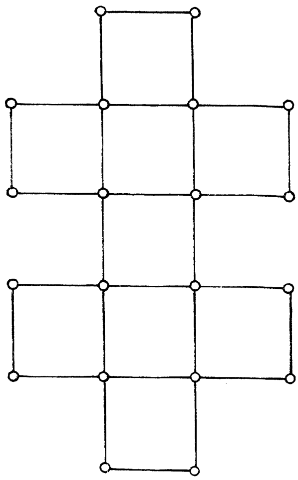

Fig. 2.

(2) We now assume that G is not absloutely uppeı embeddable. We wish to prove that there exists $A \subseteq E(G)$ such that $(**)$ does not hold. There exists a graph H' which is spanned by G and which is not upper embeddable. According to the implication (III) $\Rightarrow$ (I), there exists $A' \subseteq E(H')$ such that $b(H' - A') + c(H' - A') - 2 >$ $> |A'|$. Put $A = A' \cap E(G)$.

Consider an arbitrary component F' of H'. Obviously, $b(F') + c(F') \leqq 2$. Denote $F = \langle V(F') \rangle_G$. If $c(F) = 1$, then $i(F) \geqq i(F') \geqq b(F')$, and thus $i(F) + c(F) \geqq$ $\geqq b(F') + c(F')$. If $c(F) \geqq 2$, then $i(F) + c(F) \geqq c(F) \geqq b(F') + c(F')$.

This observation implies that $(**)$ does not hold, which completes the proof.

It can be easily deduced from Theorem 2 that the graph in Fig. 2 is absolutely upper embeddable.

References

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Added in proofs. The equivalence (I) $\Leftrightarrow$ (II) also follows immediately from the results in *N. P. Homenko, N. A. Ostroverkhy,* and *V. A. Kusmenko:* The maximum genus of graps (in Ukrainian, English summary). φ-peretvorennya grafiv (N. P. Homenko, ed.), IM AN URSR, Kiev 1973, pp. 180—210.

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