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Jarmila Hedlíková

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TERNARY SPACES, MEDIA, AND CHEBYSHEV SETS

JARMILA HEDLÍKOVÁ, Bratislava

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0. INTRODUCTION

A ternary space is a ternary structure common to metric spaces (metric betweenness), to partially ordered sets (order betweenness), to modular lattices (lattice betweenness), to vector spaces over a partially ordered field or over a ternary field (algebraic betweenness), to media (these are ternary algebras given by five of the identities satisfied in modular lattices by the operation $(abc) = (a \wedge (b \vee c)) \vee \vee (b \wedge c) = (a \vee (b \wedge c)) \wedge (b \vee c)$), and to other algebraic and geometric structures.

A Chebyshev set in a ternary space is a subset C of it containing for every point x a point between x and all points of C. In Section 1, after discussing some examples of ternary spaces, we prove basic properties of Chebyshev sets in ternary spaces. J. R. Isbell has developed the theory of Chebyshev sets in I-media (these are media satisfying a stronger convexity condition than media).

Media are investigated in Section 2 via their interval and betweenness structure and similar results to Isbell's ones for I-media are obtained. The Jordan-Hölder theorem for chains is a consequence of the fact that a medium is "locally" a modular lattice. The ternary space of a medium does not determine it; but the structure of the medium can be described by its intervals.

Section 3 is devoted to a study of Chebyshev sets in special ternary spaces. The Chebyshev sets are characterized in discrete media (using this modular lattices can be distinguished in the class of discrete lattices via Chebyshev sets), in modular lattices, and in partially ordered sets; in general media, the Chebyshev ideals are characterized. With any discrete medium one can associate a medium on the same points and with the same ternary space, the Chebyshev sets of which are ideals. Further results: the Chebyshev sets in a discrete ternary space form a complete lattice; only a one-dimensional vector space can have a nontrivial Chebyshev set.

1. TERNARY SPACES

Let T be a set together with a ternary relation abc. If $a, b, c \in T$ and abc, b is said to be *between* a and c. A *segment* (ab) is defined as the set of all $x \in T$ between a and b, i.e. $(ab) = \{x \in T\colon axb\}$. A subset A of T is called a *chain* if $A = \{a_i\}_{i \in I}$ where $(I, \leqq)$ is an ordered set and $a_i a_j a_k$ for all $i, j, k \in I$, $i \leqq j \leqq k$. An element a_i $(i \in I)$ is an *endelement* of the chain A if i is a least or a greatest element in $(I, \leqq)$. The notation $x_0 x_1 \ldots x_n$ $(x_0, x_1, \ldots, x_n \in T$ and $n > 1)$ means that $x_i x_j x_k$ for all $i, j, k \in \{0, 1, \ldots, n\}$, $i \leqq j \leqq k$ (we also use such a notation with an infinite number of members, e.g. $x_0 x_1 x_2 \ldots x$, $x_0 x_1 x_2 \ldots y_2 y_1 y_0$, and so on). If $a, b, c, d \in T$ are distinct and abc, bcd, cda, dab, then a, b, c, d are said to form a *rectangle*. The *length* of a chain $a_0 a_1 \ldots a_n$ $(a_0, a_1, \ldots, a_n$ are distinct) is n.

A *ternary space* is a set T together with a ternary relation abc satisfying

(T_1) $abc \Rightarrow cba$,

(T_2) $abc, acb \Leftrightarrow b = c$,

(T_3) $abc, acd \Rightarrow bcd$,

(T_4) $abc, acd \Rightarrow abd$.

T is called *trivial* if there are no distinct elements $a, b, c \in T$ such that abc.

First, some basic results about chains.

1.1. Lemma. *In a ternary space*

(1) $x_0 x_1 \ldots x_n,\ x_0 x_n x \Rightarrow x_0 x_1 \ldots x_n x$,

(2) $x_0 x_1 \ldots x_n,\ x_{i-1} x x_i$ *for some* $i \in \{1, 2, \ldots, n\} \Rightarrow x_0 x_1 \ldots x_{i-1} x x_i \ldots x_n$.

Proof. Let $x_0 x_1 \ldots x_n$, $x_0 x_n x$, and $i, j \in \{0, 1, \ldots, n\}$, $i \leqq j$. Then (1) follows from the following two implications

$$x_0 x_i x_n,\ x_0 x_n x \Rightarrow x_i x_n x\,, \quad x_i x_j x_n,\ x_i x_n x \Rightarrow x_i x_j x\,.$$

(2) for $n = 2$ is obvious. If $n > 2$, then (one can assume $i \neq 1$) using induction we get $x_1 x_2 \ldots x_{i-1} x x_i \ldots x_n$ which with $x_0 x_1 x_n$ gives the required result by (1). ∎

Characterizations of chains in more general spaces than ternary spaces can be derived e.g. from [18] and [12] (cf. [1]). Results from these papers can be summarized in the following assertion.

1.2. Theorem. *Let T be a space given by conditions* (T_1), (T_3), *and*

(T_0) $aba \Rightarrow a = b$.

Consider the following conditions (A is a nonempty subset of T):

(3) *at least one of the relations abc, acb, bac holds,*

(4) *A does not contain a rectangle,*

(5) $abc,\ bcd,\ b \neq c \Rightarrow abcd$.

Then A is a chain if and only if it satisfies (3) *and* (4) *or* (3) *and* (5). *If A has not exactly four elements, then it is a chain if and only if it satisfies* (3).

The concept of a ternary space is sufficiently general to allow many various examples. We list some of them.

Every metric space (M, d) with metric betweenness $abc \Leftrightarrow d(a, b) + d(b, c) =$

$= d(a, c)$ is a ternary space (cf. [11] and [13], Part II). Clearly, this example also includes a connected graph with its natural metric.

E. Pitcher and M. F. Smiley ([13], Part II) proved that lattice betweenness $abc \Leftrightarrow \Leftrightarrow (a \wedge b) \vee (b \wedge c) = b = (a \vee b) \wedge (b \vee c)$ satisfies $(T_1)-(T_3)$ in every lattice $(L, \wedge, \vee)$, and that a lattice is modular if and only if its betweenness satisfies (T_4). Thus a modular lattice with lattice betweenness is a ternary space. L.M. Kelly [7] characterized those ternary spaces which are modular lattices such that space betweenness and lattice betweenness coincide. A comparison of algebraic (see below), metric, and lattice betweenness can be found in [17].

Further examples of ternary spaces are provided by ternary algebras called media which are considered in Section 2. These algebras include modular lattices.

One can obtain a ternary space also from a partially ordered set $(P, \leqq)$ using order betweenness $abc \Leftrightarrow a = b$ or $b = c$ or $a < b < c$ or $c < b < a$.

W. Prenowitz and J. Jantosciak [16] introduced the concept of a join space which covers the theories of many geometries. We shall consider two examples of vector spaces which are special join spaces and which include vector spaces over an ordered field.

A *join space* is a set J with a *join operation* $\circ : J^2 \to P(J)$ such that

(J_1) $a \circ b \neq \emptyset$,

(J_2) $a \circ b = b \circ a$,

(J_3) $(a \circ b) \circ c = a \circ (b \circ c)$,

(J_4) $a \mid b \cap c \mid d \neq \emptyset \Rightarrow a \circ d \cap b \circ c \neq \emptyset$,

(J_5) $a \mid b \neq \emptyset$,

where $a \mid b = \{x \in J\colon a \in b \circ x\}$ and $A \circ B = \bigcup(a \circ b\colon a \in A,\ b \in B)$ for $A, B \subseteq J$. Its betweenness is given by $abc \Leftrightarrow b \in a \circ c$.

1.3. Lemma. *A join space $(J, \circ)$ is a ternary space if and only if $a \circ a = a$ and $a \in a \circ b$ for all $a, b \in J$.*

Proof. (T_1): This is obvious from (J_2). (T_2): By (J_2), abb. If abc and acb, then by (J_2), $b \in c \circ a$ and $c \in b \circ a$; this means $a \in b \mid c \cap c \mid b$, hence by (J_4), $b \circ b \cap c \circ \circ c \neq \emptyset$, that is $b = c$. (T_3), (T_4): If abc and acd, then $b \in a \circ (a \circ d)$; hence by (J_3), $b \in a \circ d$. From $a \in b \mid c \cap c \mid d$ by (J_4) we get $b \circ d \cap c \circ c \neq \emptyset$, whence $c \in b \circ d$. ∎

There are generalizations of the concept of an ordered field in two different directions – a partially ordered field and a ternary field. The only fields which are both partially ordered and ternary fields are just the ordered fields.

A *partially ordered field* is a field $(F, +, \cdot)$ containing a nonempty subset P (the set of positive elements) closed under addition and division (see [4]). The corresponding partial order is given by $a \leqq b \Leftrightarrow a = b$ or $b - a \in P$. F is *ordered* if for every $x \in F$ either $x \in P$, $-x \in P$ or $x = 0$. An equivalent definition: $(F, \leqq)$ is a partially ordered set such that $a < b$ implies $a + c < b + c$ for all $c \in F$ and $a/c < b/c$ for all $c > 0$.

A field $(F, +, \cdot)$ is called a *ternary field* if it is a nontrivial ternary space and for all $a, b, c, x \in F$

(F_1) $abc \Rightarrow a + x \quad b + x \quad c + x$,

(F_2) $abc \Rightarrow a . x \quad b . x \quad c . x$.

These properties can be immediately extended to chains, i.e. for all $n > 1$ and $a_0, a_1, \ldots, a_n, x \in F$, $a_0 a_1 \ldots a_n$ implies $a_0 + x\, a_1 + x \ldots a_n + x$ and $a_0 . x\, a_1 . x \ldots \ldots a_n . x$. Orderability is equivalent to the condition (3); this is clear e.g. from the following lemma.

1.4. Lemma. *The condition* (5) *is satisfied in any ternary field* $(F, +, \cdot)$.

Proof. Subtracting c from the terms in the relation abc (or bcd) and dividing by $b - c$ one has $x10$ (or $10y$) where $x = (a - c)/(b - c)$ (or $y = (d - c)/(b - c)$). Multiplying the terms in $01x$ by y/x we obtain $0\ y/x\ y$ which with $10y$ gives $10\ y/x$. This multiplied by x implies $x0y$ which with $x10$ gives $x10y$. Finally, from the last relation (multiply by $b - c$ and then add c) one has $abcd$. ∎

In view of 1.4 one can see that the concept of a ternary field is essentially the same as the concept of a "partially ordered field" in the sense of [14] (W. Prenowitz [14] used the name partially ordered field because he presumed (erroneously) that his concept coincides with that of D. W. Dubois [4]). The only difference between these two concepts is that W. Prenowitz [14] used strict betweenness (abc implies that a, b, c are distinct). By 1.4, it is also clear that in his definition of a "partially ordered field" (which is given by strict betweenness, (T_1), (T_3), (T_4), (F_1), (F_2) for $x \neq 0$, nontriviality and the property abc, bcd imply abd, acd) the last condition can be omitted.

We will use only the following three properties of a ternary field proved in [14] (those wanting to know more about ternary fields are referred to the cited paper):

(6) $0a1 \Rightarrow 0 \quad 1 - a \quad 1$,

(7) $0a1,\ 0b1 \Rightarrow 0 \quad a . b \quad 1$,

(8) $0a1,\ 0b1 \Rightarrow 0 \quad a \quad a + b$.

If $(F, +, \cdot)$ is simultaneously a ternary field and a partially ordered field such that the corresponding ternary spaces coincide, then F is an ordered field. For, subtracting 1 from the terms in the relation 012 and then multiplying by any $x \in F$, we have $-x0x$, hence either $x > 0$, $x < 0$ or $x = 0$.

1.5. Theorem. *A vector space V over a ternary field F is a join space satisfying $a \circ a = a$ and $a \in a \circ b$ under the join operation $a \circ b = \{c \in V\colon c = x . a + (1 - x) . b$ for some $x \in F$ such that $0x1\}$.*

Proof. By (6), one has another expression for $a \circ b$, $a \circ b = \{c \in V\colon c = x . a + y . b$ for some $x, y \in (01)$ such that $x + y = 1\}$. Now, the only nontrivial conditions are (J_3) and (J_4).

To prove (J_3) assume $d \in (a \circ b) \circ c$; this means that there is $e \in a \circ b$ such that $d \in e \circ c$. One can write $e = x . a + y . b$ and $d = z . e + t . c$ for some $x, y, z, t \in$

$\in (01)$ such that $x + y = z + t = 1$. Hence $d = x . z . a + y . z . b + t . c$. For $s = 1 - x . z = t + y . z$ one has $0ts$ by (7) and (8). If $s = 0$, then $t = 0$; this means $d = e$, hence $d \in a \circ (b \circ c)$. If $s \neq 0$, then $d = (1 - s) . a + s . f$ where $f = u . b + v . c$, $u = y . z/s$, and $v = t/s$. Clearly, $u + v = 1$, and by (7), $0\ x . z\ 1$. $0ts$ after dividing by s gives $0v1$. Thus $d \in a \circ (b \circ c)$.

To prove (J_4) assume there is $e \in a \mid b \cap c \mid d$; this means $a \in b \circ e$ and $c \in d \circ e$, hence $a = x . b + y . e$ and $c = z . d + t . e$ for some $x, y, z, t \in (01)$ such that $x + y = z + t = 1$. The cases $y = 0$ and $t = 0$ are trivial. Otherwise, one can eliminate e: $e = (a - x . b)/y = (c - z . d)/t$, hence $t . a + y . z . d = y . c + x . t . b$. Since $t + y . z = y + x . t$, $f = (t . a + y . z . d)/(t + y . z) = (y . c + x . t . b)/(y + x . t)$. Now, it is not difficult to show that $f \in a \circ d \cap b \circ c$. ■

Thus in view of 1.3 a vector space V over a ternary field F with algebraic betweenness $abc \Leftrightarrow b = x . a + (1 - x) . c$ for some $x \in F$ such that $0x1$ is a ternary space. The same is valid for a vector space V over a partially ordered field F (namely, 1.5 can be proved in an essentially similar way (cf. [16], Section 3 and [15], Appendix) for this vector space). The following lemma is true also for a partially ordered field F.

1.6. Lemma. *In a vector space V over a ternary field F the condition* (5) *is satisfied.*

Proof. If abc and bcd, then $b = x . a + (1 - x) . c$ and $c = y . b + (1 - y) . d$ where $x, y \in (01)$. Substituting b in the second expression we get $s . c = x . y . a + (1 - y) . d$ where $s = 1 - y + x . y$. Since $0\ 1 - y\ 1$ and $0\ x . y\ 1$ (by (6) and (7)), $0\ x . y\ s$ by (8); hence if $s = 0$, then $x = 0$ which means that $b = c$. Otherwise, $c = z . a + t . d$ where $z = x . y/s$ and $t = (1 - y)/s$. Clearly, $z + t = 1$. Dividing by s in $0\ x . y\ s$ one has $oz1$. Thus acd, hence abd. ■

A join space J which is also a ternary space need not satisfy the condition (5) (e.g. a four-element Boolean lattice with lattice betweenness).

The next two examples are suitable modifications of examples II and III in [16], Section 3 (cf. Section 12) given here to show join spaces satisfying $a \circ a = a$ and $a \in a \circ b$.

Let V be a vector space over a partially ordered field F. The ray $\boldsymbol{a}$ determined by an element $a \in V$ is defined as $\{x . a: x \in F, x > 0\}$. Define on $R = \{\boldsymbol{a}: a \in V\}$ a join operation $\cdot$: $\boldsymbol{a} \cdot \boldsymbol{b}$ is the set of all rays determined by the elements of $a \circ b$. Then $(R, \cdot)$ is a join space satisfying $\boldsymbol{a} \cdot \boldsymbol{a} = \boldsymbol{a}$ and $\boldsymbol{a} \in \boldsymbol{a} \cdot \boldsymbol{b}$. It is called the ray space of V.

Let V be a left module over a division ring D. The linear manifold a^* determined by an element $a \in V$ is the set $\{x . a: x \in D, x \neq 0\}$. Define on the set S of all linear manifolds a join operation $\cdot$: $a^* \cdot a^* = a^*$ and $a^* \cdot b^*$ for $a^* \neq b^*$ is the set of all linear manifolds determined by the elements of $a^* + b^* = \{c + d: c \in a^*, d \in b^*\}$ together with a^* and b^*. Then $(S, \cdot)$ is a join space satisfying $a^* \cdot a^* = a^*$ and $a^* \in a^* \cdot b^*$. It is called the linear manifold space of V.

J. R. Isbell [6] has introduced the concept of a Chebyshev set (in a quite strong sense) in ternary algebras which we call here I-media (the name media is used here for the somewhat more general concept of ternary algebras). This concept, in a natural way, can be introduced in a ternary space such that in the case of an I-medium it coincides with Isbell's one (see Section 3). We list some basic properties of Chebyshev sets with very simple proofs (cf. [6], 1.6, 1.7, 1.8).

A subset C of a ternary space T is called *convex* if $(ab) \subseteq C$ for all $a, b \in C$.

A subset C of a ternary space T is called a *Chebyshev* set if for each $x \in T$ there is $x_C \in C$ such that xx_Cc for all $c \in C$ (note the obvious fact that such x_C is unique). The map $x \mapsto x_C$ is then called *projection* upon C. A Chebyshev set is *trivial* if it is a singleton or the whole space.

Every Chebyshev set is convex, for if $a, b \in C$ and $x \in (ab)$, then ax_Cxx_Cb, hence $x = x_C \in C$.

1.7. Lemma. *If C is a Chebyshev set and $x, y \in T$, then $(xy) \cap C = (x_Cy_C)$ provided $(xy) \cap C$ is nonempty. In particular, $(xc) \cap C = (x_Cc)$ for all $c \in C$.*

Proof. For all $a \in (xy) \cap C$ we have xay, xx_Ca, and yy_Ca, hence xx_Cay_Cy. The assertion is now obvious. ∎

1.8. Lemma. *A relative Chebyshev set D in a Chebyshev set C is Chebyshev, and $x_D = x_{C_D}$ for all $x \in T$.*

Proof. If $x \in T$ and $d \in D$, then from x_C, $d \in C$ it follows that xx_Cd and $x_Cx_{C_D}d$, which imply $xx_{C_D}d$. ∎

1.9. Theorem. *A nonempty intersection of two Chebyshev sets is a Chebyshev set, the projections commute, and the composite is the projection upon the intersection.*

Proof. Let C, D be Chebyshev sets with $C \cap D \neq \emptyset$. With respect to 1.8 it suffices to show that $x_D \in C \cap D$ for all $x \in C$. Take $a \in C \cap D$. Then xx_Da and since $x, a \in C$, $x_D \in C$. ∎

For investigations in Sections 2 and 3 some further notions are necessary.

In a ternary space T, a chain A (with endelements a, b) is a *maximal chain* (with endelements a, b) if there is no chain $B \neq A$, $B \supseteq A$ (with endelements a, b). A chain A is called a *saturated chain* if $a, b \in A$, $x \in T$, axb, and $A \cup \{x\}$ is a chain imply $x \in A$. These two concepts are connected in the following sense: a chain A with endelements a, b is a maximal chain with endelements a, b if and only if it is saturated. A simple induction yields the following assertion. A chain $a_0a_1\ldots a_n$ is a maximal chain with endelements a, b if and only if $(a_{i-1}a_i) = \{a_{i-1}, a_i\}$ for all $i \in \{1, \ldots, n\}$. A segment (ab) is an *edge* if $a \neq b$ and $(ab) = \{a, b\}$. A ternary space is called *discrete* if all chains with endelements are finite.

Two segments (ab) and (cd) are called *transposed* (*e-transposed*) segments if $a, c \in (bd)$ $b, d \in (ac)$ (and $(ad) = \{a, d\}, (bc) = \{b, c\}$) or $a, d \in (bc)$ $b, c \in (ad)$ (and $(ac) = \{a, c\}, (bd) = \{b, d\}$). They are called *projective* (*e-projective*) if there are segments $(x_0y_0), \ldots, (x_ny_n)$, $x_0 = a$, $y_0 = b$, $x_n = c$, $y_n = d$ such that the segments $(x_{i-1}y_{i-1})$, (x_iy_i) are transposed (e-transposed) for $i = 1, \ldots, n$.

Now, some graph-theoretic remarks. By a *graph* G we mean a pair (V, E) where V is a nonempty set (the vertex set of G) and $E \subseteq \{\{a, b\} : a, b \in V \text{ are distinct}\}$ (the edge set of G). A *path* (of length n) from $a \in V$ to $b \in V$ is a sequence of edges $\{a = a_0, a_1\}, \{a_1, a_2\}, \ldots, \{a_{n-1}, a_n = b\}$ $(n \geqq 1)$. It is called a *simple* path fi $a_0, a_1, \ldots, a_n$ are distinct. The *distance* $d(a, b)$ between two distinct vertices a, b is defined as the length of a shortest path from a to b (if there is a path from a to b), $d(a, a) = 0$. A graph G is *connected* if two distinct vertices a, b are connected by a path. In this case, (V, d) is a metric space. Observe that a simple path $\{a_0, a_1\}, \{a_1, a_2\}, \ldots\ldots, \{a_{n-1}, a_n\}$ $(n > 1)$ in a graph G is a shortest path from a_0 to a_n if and only if $a_0 a_1 \ldots a_n$ (in the sense of metric betweenness).

The graph of a ternary space T is the graph (T, E) where $E = \{(ab): a, b \in T$ and (ab) is an edge$\}$. Clearly, the graph of a discrete ternary space is connected. By the graph of a lattice L we mean the graph (L, E) where $\{a, b\} \in E \Leftrightarrow a$ covers b or b covers a.

A lattice is called *discrete* if all chains with endelements are finite; let us observe that by [9], Lemma 3.1, we have an alternative definition: a lattice is discrete if and only if all chains with comparable endelements are finite (here, a chain in a lattice is considered in the sense of lattice betweenness). Let us also observe that for the definition of discreteness of a lattice we can as well use chains in the sense of order betweenness.

2. MEDIA

Here, an interesting case of ternary spaces which includes modular lattices is presented. A study of media introduced here is mainly inspired by Isbell's work [6] using a somewhat different approach.

A *medium* is a set M with a ternary operation (abc) satisfying

(M_1) $(aab) = a$,

(M_2) $(abc) = (acb)$,

(M_3) $((abc)\,ab) = (abc)$,

(M_4) $((dbc)(abc)\,a) = (abc)$,

(M_5) $((ab(dbc))\,bc) = (ab(dbc))$.

An *I-medium* is a set M with a ternary operation (abc) satifying $(M_1)-(M_4)$ and

(I) $((a(bde)(cde))\,de) = (a(bde)(cde))$.

Clearly, each I-medium is a medium, in fact, (M_1) and (I) imply (M_5). The work [6] is devoted to a study of I-media and the reader is referred to this paper to find a motivation and a geometric explanation of the study. The identities $(M_1)-(M_4)$ and (I) are equivalent to embeddability in a lattice (see [6], (viii), 1.4), this means that an I-medium is isomorphic to a subset M of a lattice such that $(a \wedge (b \vee c)) \vee \vee (b \wedge c) = (a \vee (b \wedge c)) \wedge (b \vee c) \in M$ for all $a, b, c \in M$ and conversely, every such subset is an I-medium.

In a medium, an *interval* $[ab]$ is the set of all x such that $(xab) = x$. It is the same as the set of all elements of the form (yab). For if x is in $[ab]$, then it is (xab). Conversely, by (M_1) and (M_2), $(yab) = (ya(bab))$ which is in $[ab]$ by (M_5). Axiom (M_5) can be read: $b \in [ac]$ implies $[ab] \subseteq [ac]$.

A *segment* (ab) is the set of all x such that $(axb) = x$. Its elements will be said to be *between* a and b; in symbols, axb means $x \in (ab)$. Clearly, $(ab) \subseteq [ab]$ by (M_2) and (M_3), and for $x \in (ab)$ we get by (M_1), (M_2) and (M_4), $(bxa) = ((bxb)(axb)\,a) = (axb) = x$. Hence (ab) is the set of all elements of the form (ayb). (M_4) can be read: $(dbc)\,(abc)\,a$.

Note that in the case of an I-medium, $x \in [ab]$ is equivalent to $a \wedge b \leqq x \leqq \leqq a \vee b$, and $x \in (ab)$ is equivalent to $(a \wedge x) \vee (x \wedge b) = x = (a \vee x) \wedge \wedge (x \vee b)$ (here, the lattice from [6], 1.4, in which the I-medium is embedded, is considered).

2.1. Theorem. *Every medium is a ternary space.*

Proof. Its betweenness is given by $abc \Leftrightarrow (abc) = b$. (T_1) and (T_2) are obvious. (T_3): If abc and acd, then $(bcd) = ((bca)(dca)\,d) = (dca) = c$. (T_4): If abc and acd, then $x = (dab) \in [ab] \subseteq [ac]$, hence $(dcx) = (d(dca)(xca)) = (dca) = c$ and $(cbx) = (c(cab)(dab)) = (cab) = b$. The relations dcx and dxb imply, by (T_3), cxb which with cbx gives $x = b$. ∎

2.2. Lemma. *In a medium,*

(9) $x = (abc)$ *if and only if* $[ax] \cap [bc] = \{x\}$.

Proof. By (M_4), each interval $[bc]$ is a Chebyshev set. Clearly, $x = (abc) \in \in [ax] \cap [bc]$. For $y \in [ax] \cap [bc]$ one has axy, hence $y = x$. Conversely, if $[ax] \cap \cap [bc] = \{x\}$ and $y = (abc)$, then ayx, hence $y = x$. ∎

In particular, abc is equivalent to $[ab] \cap [bc] = \{b\}$, i.e. b *splits* a and c by Isbell's definition.

2.3. Corollary. *In a medium,*

(10) $d \in [a(abc)]$ *implies* $(dbc) = (abc)$,

(11) $b \in [ac]$ *implies* $(xab) = ((xac)\,ab)$.

Proof. If $d \in [a(abc)]$, then $[d(abc)] \cap [bc] \subseteq [a(abc)] \cap [bc] = \{(abc)\}$. If $b \in [ac]$, then $(xab) \in [ac]$, hence $(xac) \in [x(xab)]$. Thus $[(xac)(xab)] \cap [ab] \subseteq \subseteq [x(xab)] \cap [ab] = \{(xab)\}$. ∎

In other words, the following identities are true:

(12) $((da(abc))\,bc) = (abc)$,

(13) $(xa(bac)) = ((xac)\,a(bac))$.

Using 1.9 one can easily prove in media the identity

(14) $((ab(cde))\,d(cbe)) = ((ad(cbe))\,b(cde))$,

which S. A. Kiss proved in modular lattices (see [8], Theorem 2). Indeed, $e \in [be] \cap \cap [de]$ implies $((cbe)\, de) = ((cde)\, be) \in [b(cde)] \cap [d(cbe)]$, hence (14).

Also, in a medium, $((abc)\, de) = ((ade)\, bc)$ provided $[bc] \cap [de]$ is nonempty. Thus the identity $((abc)\, dc) = ((adc)\, bc)$ is always satisfied.

In any I-medium one has the identity

(15) $(a(bde)\,(cde)) = ((ade)\,(bde)\,(cde))$.

The left side of (15) is $((a(bde)\,(cde))\, de)$ by (I), which is the right side since $[(bde)\,(cde)] \cap [de]$ is nonempty.

2.4. Theorem. *Every segment* (ab) *in a medium* M *is a modular lattice under the operations* $x \wedge y = (xay)$ *and* $x \vee y = (xby)$.

Proof. Clearly, (ab) is partially ordered by $x \leqq y \Leftrightarrow axy$ with universal bounds a, b. For $x, y \in (a, b)$ we have $(xay) = ((bax)\, ay) = ((bay)\, ax) = (yax) \in (ab)$. (xay) is a common lower bound of x and y and if $z \in (ab)$ and $z \leqq x, y$, then $((xay)\, az) = (xaz) = z$, hence $z \leqq x \wedge y$. So (ab) is a semilatitce and, interchanging a and b, a lattice. As for the modularity, assume $x, y, z \in (ab)$, $y \leqq z$, $x \wedge y = x \wedge \wedge z$, and $x \vee y = x \vee z$. Obviously, $a(x \wedge y)\, yz(x \vee z)\, b$. Since $y \in [az]$, $y = = ((x \wedge y)\, yz) = ((x \wedge z)\, yz) = ((xza)\, zy) = (xzy)$ and, symmetrically, $z = (xyz)$, hence $y = z$. ■

A somewhat stronger form of Theorem 2.4 was proved by J. R. Isbell [6], 1.16, for I-media. The author does not know whether (ab) is a submedium of a medium M. If M is an I-medium, then $x \wedge y = (xay) = (yax) = (axy)$. Indeed $x, y \in [b(x \wedge y)]$ implies $(axy) \in [xy] \subseteq [b(x \wedge y)]$ which with $a(axy)\,(x \wedge y)\, b$ gives $(axy) = x \wedge y$ (cf. [6], 1.17).

Fix an element a of a medium M and define on M a relation $x \leqq_a y \Leftrightarrow axy$. Then as in 2.4, $(M, \leqq_a)$ is a partially ordered set with the least element a, in which every set $\downarrow b = \{x \in M : x \leqq_a b\}$ $(b \in M)$ is a modular lattice.

From [6], 1.16 we immediately have that transposed segments in an I-medium are isomorphic (as media). In media the author only knows (as follows from 2.4) that the map $x \mapsto (xcd)$ is a bijection of transposed segments (ab) and (cd). An important corollary of 2.4 is

2.5. Theorem. (Jordan-Hölder theorem for chains in media.) *If A, B are maximal chains with endelements a, b in a medium and A is finite, then B is finite and there is a bijection between the sets of all edges of A and B such that the corresponding edges are e-projective and for the members (xy) of the e-projectivities $x, y \in (ab)$ holds.*

Proof. The assertion immediately follows (by Theorem 2.4) from the Jordan-Hölder theorem for chains with comparable endelements in modular lattices (chains are considered in the sense of lattice betweenness) formulated in [9], 4.4.1. ■

Theorem 2.5 was proved for modular lattices (chains are considered in the sense of lattice betweenness) by M. Kolibiar ([9], 4.5) (cf. [6], 1.19).

Let us observe the following implication (it occurs in the proof of 2.4) which is valid in media: axb, ayb imply $(xay) = (yax)$.

In order to compare our results with Isbell's ones we present the following two lemmas which give alternative definitions of an edge and a chain (in that manner J. R. Isbell defined his concepts).

2.6. Lemma. *The segment (ab) in a medium is an edge if and only if the interval $[ab]$ is not a singleton but all its proper subintervals are.*

Proof. Let $a \neq b$, $(ab) = \{a, b\}$, $[xy] \subseteq [ab]$, and $x \neq y$. One can assume that $a \in [bx]$, hence $[ab] = [bx]$. If aby, then $b = y$, this means $[ab] = [xy]$. Otherwise, if $a \in [by]$, then $[ab] = [by]$. In this case, if $a \in [b(bxy)]$, then $x \in [ab] \subseteq [b(bxy)]$; hence bxy which gives $x = y$, a contradiction. The other possibility $ab(bxy)$ gives $b \in [xy]$ since $(bxy) \in [ab]$. Thus $[bx] \subseteq [xy]$, and hence $[ab] \subseteq [xy]$. Conversely, let $a \neq b$ and let all proper subintervals of $[ab]$ be singletons. If $x \in (ab)$ and $a \neq x$, then $[ax] = [ab]$; hence $b = (bax) = x$. Thus (ab) is an edge. ■

If (ab) is an edge, then $[ab] = [xy]$ for all distinct $x, y \in [ab]$. Thus (xy) is an edge. Obviously, $(xyz) = x$, $(yxz) = y$, $(zxy) = z$ for all distinct points $x, y, z \in [ab]$.

2.7. Lemma. *A subset $\{a_0, a_1, \ldots, a_n\}$ $(n > 1)$ of a medium is the chain $a_0a_1 \ldots a_n$ if and only if $a_0a_ia_n$ for all i, $a_j \in [a_0a_i]$ for $j \leqq i$, and $a_j \in [a_ia_n]$ for $j \geqq i$.*

Proof. Clearly, the chain $a_0a_1 \ldots a_n$ satisfies the condition. Conversely, let the above condition be satisfied. Then from $a_1 \in [a_0a_2]$ one has $a_2 = (a_na_0a_2) = (a_n(a_na_0a_2)(a_1a_0a_2)) = (a_na_2a_1)$. The relations $a_0a_1a_n$ and $a_1a_2a_n$ imply $a_0a_1a_2a_n$. A simple induction leads to $a_0a_1 \ldots a_n$. ■

2.8. Theorem (cf. [6], 1.15 and Corollary to 1.20). *A simple path $(a_0a_1), (a_1a_2), \ldots, (a_{n-1}a_n)$ $(n > 1)$ in the graph of a medium is a shortest path from a_0 to a_n if and only if $a_0a_1 \ldots a_n$.*

Proof. The "only if" part for $n = 2$ follows from the remark after 2.6. If $n > 2$ and the assertion is true for all $1 < k < n$, then $a_1a_2 \ldots a_n$. Consider the element $a = (a_na_0a_1)$. If $a = a_1$, then $a_0a_1 \ldots a_n$. If $a = a_0$, then $d(a_0, a_n) \leqq n - 2$, and if $a \neq a_0, a_1$, then $(aa_0), (aa_1)$ are edges, and hence $d(a_0, a_n) \leqq n - 1$, both giving a contradiction. To prove the "if" part assume $(b_0b_1), (b_1b_2), \ldots, (b_{m-1}b_m)$ to be a shortest path from a_0 to a_n. Then $b_0b_1 \ldots b_m$, hence $m = n$. ■

2.9. Corollary. *A medium is discrete if and only if its graph is connected.*

2.10. Corollary. (cf. [6], 1.20). *In a discrete medium,*

(16) $abc \Leftrightarrow d(a, b) + d(b, c) = d(a, c)$.

2.11. Theorem. *In a medium the identities*

(17) $(ab(cda)) = (a(bda)(cda))$,

(18) $(ab(cda)) = (ac(bda))$,

(19) $((abc)\,b(c(abc)\,b)) = (abc)$,

(20) $(ab(cab)) = (abc)$,

(21) $((abc)dc) = (ac(dcb))$,

(22) $((abc)\,bd) = (ab(cbd))$

are equivalent.

Proof. The implications (17) ⇒ (18) ⇒ (19) and (21) ⇒ (22) ⇒ (20) are obvious. (19) ⇒ (20): Denote $x = (abc)$ and $y = (cab)$. Since $x \in [ab]$, $(cbx) = (ybx) \in [by]$. Using (19) one has $x \in [b(cbx)] \subseteq [by] \subseteq [bc]$, hence $(aby) = x$. (20) ⇒ (21): If $x = (abc)$ and $y = (dbc)$, then $(xcd) = (xc(dxc)) = (xc(yxc)) = (xcy) = (acy)$. (20) ⇒ (17): Denote $x = (cad)$ and $y = (bad)$. Since $x \in [ad]$, $(bax) = (yax)$, and hence $(axb) = (ax(bax)) = (ax(yax)) = (axy)$. ∎

A medium is *taut* if one of the identites (17)–(22) is satisfied. Another equivalent condition: $(abc) \in [b(cab)]$.

2.12. Theorem. *In an I-medium tautness is equivalent to each of the identities*

(23) $((ade)\,b(cde)) = ((ade)\,(bde)\,(cde))$,

(24) $(a(bde)\,(cde)) = ((aed)\,b(ced))$.

Proof. Clearly, each of (23) and (24) implies (20). Conversely, by (15), $(ayz) = (xyz)$ and $(bxz) = (yxz)$, where $x = (ade)$, $y = (bde)$, and $z = (cde)$. Then $(ayz) = (xz(yxz)) = (xz(bxz)) = (xzb)$. ∎

Some remarks about the above mentioned identities. A ternary operation $(a \wedge (b \vee c)) \vee (b \wedge c) = (a \vee (b \wedge c)) \wedge (b \vee c)$ in modular lattices was observed already by S. A. Kiss [8] and J. Hashimoto [5]. In [8], (14) and the associative law (22) are proved for it. In [5], (24) is used to characterize bounded modular lattices by means of a ternary operation (see Theorem 2). In [3], this characterization is done for modular lattices only with a zero (see Satz 6). Finally, M. Kolibiar and T. Marcisová ([10], Theorem 1) showed that for the result the identity (21) suffices. Thus a taut medium with a suitable element 0 (this means for any a, b there is x with $(0ax) = a$ and $(0bx) = b$) is a modular lattice with zero 0. (19) is the identity which Isbell used to define tautness in I-media (see [6], 2.1). The simplest identity characterizing tautness is (20).

An immediate corollary of 2.12 is the following assertion: For a taut medium the property of being an I-medium is equivalent to each of (15), (23), and (24).

Using the Lemma in [10], one obtains a characterization of taut media by identities $(abb) = b$ and (21). Just observe that the two identities imply (22) (and so (M_5)). Denote $x = (dbc)$, $u = ((abc)\,bd)$, and $v = (ab(cbd))$. Then $v = (ab(cbx)) = ((axb)cb) = (ucb)$ and so $u = ((acb)\,ub) = (ab(ubc)) = v$ which proves (22). Hence $((abx)\,bc) = (ab(xbc)) = (abx)$ which is (M_5).

Taut I-media can be characterized by two identities: $(abb) = b$ and (24) (use the beginning of the proof of [5], Theorem 2 and then the above characterization). However, the author has no example of a taut medium which is not an I-medium.

A free taut medium on three generators a, b, c can be easily determined using (20). It has six elements and can be embedded in a free modular lattice on three generators (cf. [6], Corollary to 2.2). Denote $A = (abc)$ and similarly B, C (all symmetric relations will be omitted). Then $(aBc) = A$, hence $(aBC) = (aB(caB)) = (aBc) = A$. Clearly, aBb and aAB, hence $aABb$.

2.11 has the following consequence. A medium M is a Boolean algebra if and only if for each $a \in M$ there is $a' \in M$ such that $M = (aa')$. Clearly, the complementation in a Boolean algebra satisfies the above condition. Conversely, if $a, b, c \in M$, then bab', bcb' imply $(abc) = (cba)$. Hence (20) (and so (21)) is satisfied and thus (using [10], Corollary 2) $M = (aa')$ is a Boolean algebra for any $a \in M$.

We close this section with three interval characterizations of media.

2.13. Theorem. *The structure of a medium is determined by its intervals which satisfy the following conditions*

(H_1) $b \in [ab]$,

(H_2) $x \in [ab]$ *implies* $[xa] \subseteq [ab]$,

(H_3) *for each* a, b, c *there is* $x \in [bc]$ *such that for all* $y \in [bc]$, $[ax] \cap [xy] = \{x\}$,

(H_4) $[ab] \cap [bc] = \{b\}$ *implies* $b \in [ac]$.

Proof. Clearly, $(H_1)-(H_4)$ are satisfied in a medium. Conversely, assume the four conditions. Then $[ab] = [ba]$ by (H_1) and (H_2). By (H_4), the element x in (H_3) is uniquely determined; denote it by (abc). (M_1) and (M_2) are obviously satisfied; alike $[aa] = \{a\}$, hence $[ab]$ is the set of all x for which $x = (xab)$, it is the same as the set of all elements of the form (yab). Thus $(abc) \in [ab]$ and $(ab(dbc)) \in [bc]$; hence (M_3) and (M_5) are true. It remains to prove (M_4). Denote $x = (abc)$ and $y = (dbc)$. For all $t \in [xa]$, $[xt] \subseteq [xa]$ and hence $[yx] \cap [xt] \subseteq [yx] \cap [xa]$. Since $y \in [bc]$, the last expression is $\{x\}$ and thus $(yxa) = x$. ∎

A subset J of a medium is an *ideal* if $[ab] \subseteq J$ for all $a, b \in J$. J. R. Isbell [6], 1.3 showed that the structure of an I-medium is determined by its ideals. His result is rephrased in the following theorem.

2.14. Theorem. *The structure of an I-medium is determined by its intervals which satisfy the following conditions*

(I_1) $a, b \in [ab]$,

(I_2) $[xy] \subseteq [ab]$ *for all* $x, y \in [ab]$,

(I_3) *for each* a, b, c *there is a unique* $x \in [bc]$ *such that for all* $y \in [bc]$, $x \in [ay]$.

Axiom (I_2) is an interval form of Isbell's identity (I). Similarly, (H_2) is an interval form of the identity (M_5).

2.15. Theorem. *The structure of a taut medium is determined by its intervals which satisfy the following conditions*

(K_1) $a \in [aa]$,
(K_2) *for each* a, b, c *there is* x *such that* $[ax] \cap [bc] = \{x\}$,
(K_3) $[ax] \cap [bc] = \{x\}$ *implies* $[ab] \cap [bc] = [bx]$.

Proof. (K_1) and (K_2) are satisfied in any medium (cf. 2.2). If $[ax] \cap [bc] = \{x\}$, then by 2.2, $x = (abc)$. Since $x \in [ab] \cap [bc]$, $[bx] \subseteq [ab] \cap [bc]$. Take $y \in [ab] \cap [bc]$; then using (22) $(yb(abc)) = ((yba)\ bc) = (ybc) = y$, hence ($K_3$).
Conversely, let (K_1)–(K_3) be satisfied. The aim is to prove the identities $(abb) = b$ and (21). First, some helpful facts. By (K_2), for each a there is x such that $[ax] \cap [aa] = \{x\}$, hence by (K_3), $[aa] = [ax]$, and thus, using (K_1), $[aa] = \{a\}$. For each a, b there is x such that $[bx] \cap [aa] = \{x\}$, hence $x = a$ and $a \in [ba]$. By (K_3), from $[aa] \cap [ba] = \{a\}$ one has $[ab] \cap [ba] = [ba]$, thus $[ba] \subseteq [ab]$ and, interchanging a and b, we get $[ab] = [ba]$. If $c \in [ab]$, this means $[cc] \cap [ab] = \{c\}$, then by (K_3), $[ca] \cap [ab] = [ac]$, hence $[ac] \subseteq [ab]$. The relation $[ab] \cap [bc] = \{b\}$ implies $b \in [ac]$, for there is x with $[bx] \cap [ac] = \{x\}$ and, since $x \in [ab] \cap [bc]$, $x = b$. For given a, b, c, the x determined by (K_2) is unique. For, let y be another such element, then by (K_3), $[bx] = [by] \subseteq [bc]$, hence $[ay] \cap [xb] = \{y\}$, then by (K_3), $[xy] = [ax] \cap [xb] = \{x\}$, thus $y = x$; denote $x = (abc)$. Clearly, $(abb) = b$. Let (ab) denote the set $\{x\colon [ax] \cap [bx] = \{x\}\}$. We need the implication: $c \in (ab) \Rightarrow (ac) \subseteq (ab)$. To prove it assume $c \in (ab)$ and $d \in (ac)$. By (K_2) there is x with $[bx] \cap [ad] = \{x\}$. Since $x \in [ad] \subseteq [ac]$, $[xc] \cap [bc] \subseteq [ac] \cap [bc] = \{c\}$, hence $c \in [bx]$. Similarly, $[xd] \cap [cd] \subseteq [ad] \cap [cd] = \{d\}$, hence $d \in [cx] \subseteq [bx]$, and thus $x = d$, this means $d \in (ab)$. To prove (21), let us denote $x = (abc)$, $y = (dcb)$, $p = (xdc)$, and $q = (acy)$. First observe that $p \in (xc) \subseteq (ac)$. By ($K_3$), $[cq] = [ac] \cap [cy] = [ac] \cap [bc] \cap [cd] = [cx] \cap [cd] = [cp]$, hence $[aq] \cap [cp] = \{q\}$ which with $[ap] \cap [cp] = \{p\}$ gives $p = q$, this proves (21). Clearly, $[ab] = \{x\colon x = (xab)\}$. ■

3. CHEBYSHEV SETS

Basic properties of Chebyshev sets in abstract ternary spaces are already proved in Section 1 (see 1.7, 1.8, 1.9). We proceed with a study of Chebyshev sets in special ternary spaces.

3.1. Proposition. *Every projection upon a Chebyshev set* C *in a ternary space with a metric* d *satisfying the condition*
(25) abc *implies* $d(a, b) + d(b, c) = d(a, c)$,
is distance-decreasing, i.e. $d(x_C, y_C) \leqq d(x, y)$ *for all* x, y.

Proof. One may assume that $d(x, x_C) \leqq d(y, y_C)$. Since yy_Cx_C, by (25), $d(y, y_C) + d(y_C, x_C) = d(y, x_C) \leqq d(y, x) + d(x, x_C) \leqq d(y, x) + d(y, y_C)$, hence $d(x_C, y_C) \leqq d(x, y)$. ■

By 3.1, for every Chebyshev set C in a ternary space with a metric satisfying (25), x_C is the (unique) nearest point in C to a point x. However, the property of being a Chebyshev set is much stronger than the one of having the unique nearest point to every point.

Every discrete ternary space satisfying (25) satisfies the Jordan-Dedekind chain condition (all maximal chains with the same endelements have the same length).

3.2. Theorem. *A nonempty intersection D of any set A of Chebyshev sets in a discrete ternary space is a Chebyshev set.*

Proof. Let x be any point not in D, hence there is $C_1 \in A$ not containing x. Denote $x_1 = x_{C_1}$. Since xx_1d for all $d \in D$, we are ready if $x_1 \in D$. Otherwise, there is $C_2 \in A$ not containing x_1. Denote $x_2 = x_{1C_2}$. Since x_1x_2d for all $d \in D$, xx_1x_2d, hence we are ready if $x_2 \in D$. Continuing in such a way, we must be ready after a finite number of steps, since otherwise we get an infinite chain $xx_1x_2 \dots d$ for any $d \in D$, which is impossible. ∎

By 3.2, in a discrete ternary space T, the smallest Chebyshev set containing a given nonempty set S always exists, it is the Chebyshev set generated by S. Let $|ab|$ denote the Chebyshev set generated by $\{a, b\} \subseteq T$. Clearly, $a, b \in |ab| = |ba|$ and $|xy| \subseteq$ $\subseteq |ab|$ for all $x, y \in |ab|$. One can define on T a ternary operation by

(26) $[abc] = a_{|bc|}$.

Obviously, $|ab|$ is the set of all $x \in T$ such that $x = [xab]$. Thus $(M_1)-(M_3)$ and (I) are satisfied.

3.3. Proposition. *The operation (26) is an I-medium operation if and only if $|ab| \cap |bc| = \{b\}$ whenever $[abc] = b$.*

Proof. It remains to prove (M_4). Denote $x = [abc]$ and $y = [dbc]$. Since $|xy| \subseteq$ $\subseteq |bc|$, $[axy] = x$, hence $|ax| \cap |xy| = \{x\}$, and thus $[yxa] = x$. ∎

The following fact shows that our definition of a Chebyshev set (in the case of an I-medium) coincides with that of Isbell.

3.4. Proposition. *A subset C of a medium is a Chebyshev set if and only if for each x there is a unique $x_C \in C$ such that $x_C \in [xc]$ for all $c \in C$.*

Proof. A Chebyshev set C satisfies the condition. Conversely, if the condition is satisfied, then $y = (cxx_C) \in C$ for all $c \in C$ and any point x. For all $d \in C$, $x_C \in [xd]$, hence $y \in [xd]$. Since x_C is a unique such element, $y = x_C$, which means xx_Cc. ∎

We can produce a new medium M_C from a given medium M and a Chebyshev set C (this is Isbell's construction – see the proof of [6], 1.7). New intervals: $|ab|$ is $[ab] \cap C$ if $a, b \in C$, $[ab]$ otherwise. Hence $|ab|$ is a Chebyshev set contained in $[ab]$. A new operation is defined by (26). Then $(M_1)-(M_3)$ and (M_5) are obvious.

Observe the equivalence

$$(27)\quad abc \Leftrightarrow |ab| \cap |bc| = \{b\},$$

(abc implies $\{b\} = [ab] \cap [bc] \supseteq |ab| \cap |bc|$; the other implication is obvious). To prove (M_4) denote $\boldsymbol{x} = [abc]$ and $y = [dbc]$. Then axy implies $|a\boldsymbol{x}| \cap |\boldsymbol{x}y| = \{x\}$, hence $[y\boldsymbol{x}a] = \boldsymbol{x}$. We get a new medium M_C with the same ternary space (by (27)), hence with the same Chebyshev sets (cf. [6], 1.10). Clearly, C is an ideal in M_C and if M is an I-medium, M_C is an I-medium as well.

This process immediately applies to any set A of Chebyshev sets for which all $|ab|$ are Chebyshev sets (e.g. this is satisfied for any A if M is discrete – cf. 3.2); $|ab|$ is the intersection of $[ab]$ and all $C \in A$ containing a, b if there is such C, $[ab]$ otherwise. Thus the new medium M_A has the same ternary space (hence the same Chebyshev sets), each $C \in A$ is an ideal in M_A, and if M is an I-medium, M_A is an I-medium as well. Moreover, if M is a discrete medium, one can continue the process to completion (A is the set of all Chebyshev sets) (cf. [6], Remark after 1.22); in this case, the new medium is always an I-medium, all its Chebyshev sets are ideals.

3.5. Proposition. *A discrete ternary space T is the ternary space of a medium M if and only if*

$$(28)\quad abc \text{ implies } [abc] = b.$$

Proof. If T is the ternary space of a medium M, then M is discrete. The new medium has the same ternary space, hence (28). Conversely, let T satisfy (28). If $[abc] = b$ and $x \in |ab| \cap |bc|$, then $[xab] = x$ and abx, hence $[xba] = b$. Thus $|ab| \cap |bc| = \{b\}$ and the condition of 3.3 is satisfied, hence the ternary operation is an I-medium operation. By (28), its ternary space is T. ∎

3.6. Proposition (cf. [6], 1.22). *A nonempty convex set C in a discrete medium is Chebyshev if and only if $[ab] \subseteq C$ for any edge $(ab) \subseteq C$.*

Proof. Let a be the nearest point in C to a point x and let $c \in C$. If $d(a, c) = 1$, then $b = (xac) \in (xa) \cap [ac] \subseteq C$, hence $b = a$. If $d(a, c) > 1$, then (using induction) dax for all $d \in (ac) \subseteq C$, $d \neq c$. Hence, if $e = (cax) \neq c$, then $e = (eax) = a$. Otherwise, $d \in [ac] \subseteq [ax]$, hence $d = a$, a contradiction. ∎

By 3.6, every nonempty ideal in a discrete medium is a Chebyshev set.

Obviously, in a modular lattice L(considered as a medium) a convex set is the same as an ideal. This is equivalent to the property of being empty or a convex sublattice of L in the usual sense (this means order convex). Thus the following result also characterizes the Chebyshev sets in L. The result (for median algebras) occurs in 5.2 of [2]. Note that a median algebra is a medium satisfying the identity (cf. [2], 1.2)

$$(29)\quad (abc) = (bac).$$

Chebyshev sets in median algebras are fully studied in [2] and the reader is referred to the cited paper for an acquaintance with median algebras.

3.7. Theorem. *For a nonempty ideal C of a medium the following conditions are equivalent*:

(i) *C is a Chebyshev set,*

(ii) *$(xy) \cap C$ is either empty, or a segment for all x, y,*

(iii) *there is $c \in C$ such that $(xc) \cap C$ is a segment for all x.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is obvious by 1.7. (iii) $\Rightarrow$ (i): If x is any point, $d \in C$, and $(xc) \cap C = (uv)$, then $y = (xuv) \in [uv] \subseteq C$, hence $z = (xyd) \in C$ and from xuc, xyu, xzy one has $xzyuc$. Thus $z \in (uv)$ and hence $y = (x(xuv)(zuv)) = (xyz)$ which with xzy gives $y = z$. ∎

The following result shows that modular lattices can be characterized in the class of discrete lattices via Chebyshev sets. A characterization (of another kind) of distributive lattices in the class of discrete lattices is done in [2], 4.5.

3.8. Theorem. *If L is a discrete lattice and G is the graph of L, then the following conditions are equivalent*:

(i) *L is modular,*

(ii) *every convex sublattice of L is a Chebyshev set in G,*

(iii) *every interval $[a, b] = \{x \in L: a \wedge b \leqq x \leqq a \vee b\}$, is a Chebyshev set in G.*

Proof. If L is modular, then by 2.10, G has the same ternary space, hence the same Chebyshev sets. Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) follows from 2.9 and 3.6. (iii) $\Rightarrow$ (i): Suppose that $a, b \in L$ cover $c = a \wedge b$ and let $e \in L$ be an upper bound of a, b. Since $c \notin [a, e]$, $c_{[a,e]} = a$, hence cae. One has $d = a_{[b,e]} \neq b$, since otherwise abe and cae would give cab which is impossible. Thus $d(a, d) = d(b, d) = 1$ by adb and $d \neq a, b$, hence d covers a, b, this means $d = a \vee b$. Thus $a \vee b$ covers a, b. From this and the dual argument we infer that L is modular. ∎

In a partially ordered set $(P, \leqq)$, let $[a, b] = \{x \in P: a \leqq x \leqq b\}$, $\downarrow c = \{x \in P: x \leqq c\}$, and $\uparrow c = \{x \in P: x \geqq c\}$ for any $a, b, c \in P$, $a \leqq b$. One can easily characterize the Chebyshev sets in $(P, \leqq)$.

3.9. Theorem. *A subset C of a partially ordered set $(P, \leqq)$ is a Chebyshev set it and only if either $C = P$ or C is a singleton or there are $a, b \in P$, $a \leqq b$, such thaf $P = \downarrow a \cup [a, b] \cup \uparrow b$ and $C \in \{\downarrow a, \uparrow b, [a, b]\}$.*

Proof. If C has any of the above forms, then obviously C is a Chebyshev set. Conversely, let C be a nontrivial Chebyshev set, $x \in P - C$, and $c \in C$, $c \neq x_C = y$. Since xyc and x, y, c are distinct, $x < y < c$ or $x > y > c$. Thus y is the least element 0 or the greatest element 1 of C. Put $a = b = 0$ or $a = b = 1$ or $a = 0$ and $b = 1$ in cases C has not 1 or C has not 0 or C has 0 and 1, respectively. Then $C = \uparrow b$ or $C = \downarrow a$ or $C = [a, b]$, respectively. Clearly, $P = \downarrow a \cup [a, b] \cup \uparrow b$ in all three cases. ∎

3.10. Corollary. *If a partially ordered field F possesses a nontrivial Chebyshev set, then F is ordered.*

Proof. By 3.9, there are $a, b \in F$, $a \leqq b$, such that $F = \downarrow a \cup [a, b] \cup \uparrow b$. Thus for any $x \in F$, $a - x \leqq a$ or $a - x \geqq a$, hence $x \geqq 0$ or $x \leqq 0$. ∎

3.11. Theorem. *If a vector space V over a ternary field F possesses a nontrivial Chebyshev set C, then* $\dim V = 1$.

Proof. For $a, b \in C$, $a \neq b$, and $d \in V - C$ from $dd_C a$ and $dd_C b$ one has $d_C = x . d + y . a = z . d + t . b$ for some $x, y, z, t \in F \cap (01)$ such that $x + y = z + t = 1$. Hence $(z - x) . d = y . a - t . b$ and since $x \neq z$, $d = u . a + v . b$ where $u = y/(z - x)$ and $v = t/(x - z)$. Clearly, $u + v = 1$. If a, b are linearly independent, then using the above argument for $d = a + b$ and for $d = s . a$ $(s \in F)$, we obtain $a + b = u . a + v . b$, a contradiction, and $s . a = u . a + v . b$, hence $v = 0$, and so $d_C = d$ which is impossible. Thus $a + b, s . a \in C$. Since $u . a, v . b \in C$ are linearly independent, their sum $d \in C$, a contradiction.

Thus every $a, b \in C$, $a \neq b$, are linearly dependent and since every vector $d \in V - C$ is a linear combination of a, b, we conclude that $\dim V = 1$. ∎

3.11 is valid also for a vector space V over a partially ordered field F, the proof is substantially the same.

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Author's address: 814 73 Bratislava, Obrancov mieru 49, ČSSR (Matematický ústav SAV).