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NOTE ON SPECTRAL THEORY OF NONLINEAR OPERATORS: EXTENSIONS OF SOME SURJECTIVITY THEOREMS OF FUČÍK AND NEČAS

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INTRODUCTION

In the years 1968–1972 many interesting results were achieved in the field of spectral analysis of nonlinear operators, above all by J. Nečas, S. Fučík and W. Petryshin (see [4], [5], [6], [7]).

Some of these results have been collected by J. Nečas, S. Fučík, J. Souček and V. Souček in the second chapter of [1]. This chapter deals with the solution of nonlinear operator equations

$$(1) \qquad \lambda\, T(x) - S(x) = f \quad x \in X\,, \quad f \in Y$$

in dependence on the real parameter λ, where T and S are nonlinear operators defined on a real Banach space X with values in a real Banach space Y.

The most important thing is to establish for which numbers λ the equation (1) has a solution in X, for each $f \in Y$.

This problem is solved in [1], provided $S : X \to Y$ is an odd completely continuous operator and $T : X \to Y$ works "as the identity operator" (see Definition 1.1 of [1], Chapter II).

The reason why T and S must possess these properties is that the main theorems obtained in [1] are based on the classical degree theory of Leray-Schauder which, as is well-known, concerns completely continuous operators.

Thus, if we had another degree theory for a different class of operators, we could try to get, at least in part, the results of Fučík and Nečas for (1), without supposing S to be completely continuous.

This is the purpose of the present paper.

In fact, in a recent paper of mine (see [3]), extending Canfora's degree theory (see [2]), I have defined the topological degree for bounded weakly closed operators in Hilbert and Banach spaces.

To be more precise, if X is a separable Hilbert space, S_r the ball with centre $0 \in X$ and radius $r > 0$, I consider operators of the following type: $I - T : S_r \to X$, where I

is the identity and T is a bounded weakly closed mapping satisfying the boundary condition:

(2) there exist $K \in]0, 1[$ and $a \in X$ with $\|a\| = 1$ such that for all $x \in \partial S_r$ at least one of the following conditions is satisfied:

I $(x, T(x)) \leqq K\|x\|^2$;

II $\begin{cases} (a, x - T(x)) \geqq \quad 1 - K \text{ if } (a, x) > 0, \\ (a, x - T(x)) \leqq -(1 - K) \text{ if } (a, x) < 0. \end{cases}$

If X is a reflexive separable Banach space with a strictly convex dual X', T is again supposed bounded and weakly closed, but, instead of (2), it satisfies the boundary condition:

(3) there exists $\beta \in]0, 1[$ such that $\|T(x)\| \leqq \beta\|x\|$ for all $x \in \partial S_r$.

For these operators I define the topological degree $d(I - T; S_r; 0)$ and prove that it has the usual properties.

It is precisely on the basis of this degree theory that in the present paper the equation (1) is studied.

The results obtained are, in some cases, similar to those of [1] (for instance, see Theorems 2.1, 2.2 and 2.5) whereas in other cases, they are either weaker or not comparable.

Of course this is due to the properties of T which are different in the two theories.

We now outline the contents of this note.

In Section 1, after showing how the conditions (3) and I of (2) can be improved by putting K and B equal to 1, we introduce and recall some definitions and theorems.

In Section 2 the theorems for operators $\lambda T - S$ are proved and the results obtained are compared with those of [1].

In the closing Section 3 we use the theory of Section 2 for establishing an existence theorem for a quasilinear elliptic (2m-order) equation with bounded measurable coefficients.

1. PRELIMINARIES

Let X be a separable Hilbert space, S_r the closed ball with center $0 \in X$ and radius $r > 0$, and $T: S_r \to X$ a bounded, weakly closed mapping such that

$$(1.1) \qquad (x, T(x)) \leqq \|x\|^2 \quad \text{for all} \quad x \in \partial S_r .$$

We want to prove that we can define the topological degree $d(I - T; S_r; 0)$ and that this degree is different from 0.

Let us consider the sequence of mappings $\{T_n\}$ defined by $T_n = (n/(n + 1))\, T$. It is easy to see that, for all $n \in N$, T_n is a bounded weakly closed mapping satisfying the boundary condition

$$(1.2)\qquad (x, T_n(x)) = \left(x, \frac{n}{n+1}\, T(x)\right) \leqq \frac{n}{n+1}\, \|x\|^2 \quad \text{for all} \quad x \in \partial S_r\,.$$

Then, according to the theory of [3], we can define the degree $d(I - T_n; S_r; 0) \neq 0$. Moreover, we have

$$(1.3)\qquad d(I - T_n; S_r; 0) = d(I - T_m; S_r; 0) \quad \text{for all} \quad n, m \in N\,.$$

In fact, supposing $n > m$, we consider the family of mappings $\{T_t\}_{t\in[0,1]}$:

$$T_t : S_r \to X\,, \quad T_t(x) = t\, T_n(x) + (1 - t)\, T_m(x)\,.$$

Obviously, T_t is a weakly closed operator and there exists a ball S_R $(R > 0)$ such that $T_t(S_r) \subseteq S_R$ for all $t \in [0, 1]$.

On the other hand, since we have

$$(x, T_t(x)) = (x, t\, T_n(x) + (1 - t)\, T_m(x)) \leqq t\, \frac{n}{n+1}\, \|x\|^2 + (1 - t)\, \frac{m}{m+1}\, \|x\|^2 \leqq$$

$$\leqq \left(t\, \frac{n}{n+1} + (1 - t)\, \frac{n}{n+1}\right) \|x\|^2 = \frac{n}{n+1}\, \|x\|^2 \quad \text{for all} \quad x \in \partial S_r\,,$$

we can conclude that there exists $K \in \,]0, 1[$ (it is sufficient to take $K = n/(n + 1)$) such that

$$(1.4)\qquad (x, T_t(x)) \leqq K\|x\|^2 \quad \text{for all} \quad t \in [0, 1] \quad \text{and} \quad x \in \partial S_r\,.$$

The inequality (1.4), together with the other properties of the mappings T_t, allows us to say, by the homotopy degree property (see [3]), that $d(I - T_t; S_r; 0)$ is a constant on $[0, 1]$.

Thus it makes sense to define the topological degree of $I - T$ with respect to S_r and 0 as the number (different from 0)

$$(1.5)\qquad d(I - T; S_r; 0) = \lim d(I - T_n; S_r; 0) = d(I - T_n; S_r; 0)\,.$$

Now let X be a reflexive separable Banach space with a strictly convex dual X'.

Proceeding as we did in Hilbert spaces, we can define the degree $d(I - T; S_r; 0)$, where $T: S_r \to X$ is a bounded weakly closed mapping satisfying the condition

$$(1.6)\qquad \|T(x)\| \leqq \|x\| \quad \text{for all} \quad x \in \partial S_r\,.$$

In fact it is sufficient to consider again the sequence $\{T_n\}$ $(T_n = (n/(n + 1))\, T)$ of bounded weakly closed mapping such that

$$\|T_n(x)\| = \left\|\frac{n}{n+1}\, T(x)\right\| \leqq \frac{n}{n+1}\, \|x\| \quad \text{for all} \quad x \in \partial S_r\,,$$

and to observe that, for each n, the degree $d(I - T_n; S_r; 0)$ exists and is equal to 1 (see [3]).

Then we define $d(I - T; S_r; 0) = d(I - T_n; S_r; 0) = 1$.

Now it remains to be proved that the degree $d(I - T; S_r; 0)$ just defined (either in Hilbert spaces or in Banach spaces) satisfies still the property:

$$(1.7) \qquad d(I - T; S_r; 0) \neq 0 \Rightarrow \text{there exists } y \in S_r \text{ such that } T(y) = y\,.$$

Let us prove (1.7). First of all we notice that the sequence $\{T_n\}$ $(T_n = (n/(n+1))\,T)$ converges to T uniformly in the strong topology of X.

In fact, we have

$$(1.8) \qquad \|T(x) - T_n(x)\| = \left\|T(x) - \frac{n}{n+1}\,T(x)\right\| = \left(1 - \frac{n}{n+1}\right)\|T(x)\| \to 0$$

uniformly with respect to $x \in S_r$.

The limit (1.8) implies that $\{T_n\}$ converges to T uniformly also in the weak-topology. Hence, since (1.7) holds for T_n, there exists a sequence $\{y_n\} \subseteq S_r$ such that

$$(1.9) \qquad y_n = T_n(y_n) \quad \text{for all} \quad n \in N\,.$$

But X is reflexive and $\{y_n\}$ is bounded, so we can suppose $y_n \rightharpoonup y \in S_r$. On the other hand, as T is weakly continuous[1]) and $\{T_n\}$ converges to T in the weak topology, we have

$$(1.10) \qquad T_n(y_n) \rightharpoonup T(y)\ ^{2)}\,.$$

From (1.9) and (1.10) it follows that $y_n \rightharpoonup T(y)$ and thus $y = T(y)$. So (1.7) is proved.

Now let us recall some definitions and theorems.

Theorem 1.1. (See [3] Corollary 6.1.) *Let X be a reflexive, separable Banach space with a strictly convex dual X'. If $T: X \to X$ is a bounded weakly closed operator such that*

$$(1.11) \qquad \limsup_{\|x\|_X \to \infty} \frac{\|T(x)\|_X}{\|x\|_X} = \alpha \quad \alpha \in [0, 1[$$

then $I - T$ maps X onto X.

Definition 1.1. Let X and Y be two real Banach spaces. The mapping $T: X \to Y$ is said to be a (K, L, a)-*operator* if

i) T is bijective,

ii) there exist real numbets $K > 0$, $a > 0$, $L > 0$ such that

$$L\|x\|_X^a \leqq \|T(x)\|_Y \leqq K\|x\|_X^a \quad \text{for each} \quad x \in X\,.$$

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[1]) T bounded and weakly closed $\Rightarrow$ T weakly continuous.

[2]) We have $|\langle T_n(y_n) - T(y), \varphi\rangle| \leqq |\langle T_n(y_n) - T(y_n), \varphi\rangle| + |\langle T(y_n) - T(y), \varphi\rangle|$ for all $\varphi \in Y'$.

Lemma 1.1. *Let X and Y be two real Banach spaces, $T : X \to Y$ a (K, L, a)-operator and $S : X \to Y$ a mapping.*

Then, for each real number $\lambda \neq 0$, we have:

p') $\lim_{\|x\|_X \to \infty} \|\lambda T(x) - S(x)\|_Y = +\infty \Rightarrow \lim_{\|y\|_Y \to \infty} \|y - ST^{-1}(y/\lambda)\|_Y = +\infty$;

p'') *if the mapping $y \in Y \to y - ST^{-1}(y/\lambda) \in Y$ is onto, then the mapping $\lambda T - S : X \to Y$ is onto as well.*

Proof. The proof proceeds like that of Lemma 1.1 of [1], Chapter II.

In the forthcomming definitions and theorems we suppose that X and Y are real Banach spaces and α is a real positive number.

Definition 1.2. An operator $F : X \to Y$ is said to be *α-homogeneous* if $F(tx) = t^\alpha F(x)$ for each $t \geqq 0$ and all $x \in X$.

Definition 1.3. An operator $F : X \to Y$ is said to be *α-strongly* quasihomogeneous with respect to $F_0 : X \to Y$, if:

$$t_n \searrow 0 , \quad x_n \rightharpoonup x_0 \Rightarrow t_n^\alpha F\left(\frac{x_n}{t_n}\right) \to F_0(x_0) \in Y .$$

Now we introduce the following definitions:

Definition 1.4. An operator $F : X \to Y$ is said to be *α-weakly* quasihomogeneous with respect to $F_0 : X \to Y$, if:

$$t_n \searrow 0 , \quad x_n \rightharpoonup x_0 \Rightarrow t_n^\alpha F\left(\frac{x_n}{t_n}\right) \rightharpoonup F_0(x_0) \in Y .$$

Definition 1.5. Let T and S be two operators from X to Y and λ a real number different from 0; λ is said to be an *eigenvalue* for the couple (T, S) if there exists $x_0 \in X$, $x_0 \neq 0$ such that

$$\lambda T(x_0) - S(x_0) = 0 .$$

Definition 1.6. A mapping $F : X \to Y$ is said to be *regularly surjective* if $F(X) = Y$ and for any $R > 0$ there exists $r > 0$ such that $\|x\|_X \leqq r$ for all $x \in X$ with $\|F(x)\|_Y \leqq R$.

2. MAIN THEOREMS

Lemma 2.1. *Let X and Y be two reflexive Banach spaces. If $T : X \to Y$ is a weakly closed (K, L, a)-operator and $S : X \to Y$ is a bounded weakly closed mapping, then $ST^{-1} : Y \to Y$ is bounded and weakly closed as well.*

Proof. Let us prove that T^{-1} is weakly closed. In fact,

$$\left.\begin{array}{l} y_n \rightharpoonup y \\ T^{-1}(y_n) \rightharpoonup z \end{array}\right\} \Rightarrow \left.\begin{array}{l} y_n = T(x_n) \rightharpoonup y \\ T^{-1}(T(x_n)) = x_n \rightharpoonup z \end{array}\right\} \Rightarrow T(z) = y \Rightarrow z = T^{-1}(y).$$

The operator $ST^{-1} : Y \to Y$ is weakly closed as well. In fact, since T^{-1} is weakly continuous, because it is weakly compact[3]) and weakly closed, we have:

$$\left.\begin{array}{l} y_n \rightharpoonup y \\ (ST^{-1})(y_n) \rightharpoonup z \end{array}\right\} \Rightarrow \left.\begin{array}{l} T^{-1}(y_n) \rightharpoonup T^{-1}(y) \\ ST^{-1}(y_n) \rightharpoonup z \end{array}\right\} \Rightarrow (ST^{-1})(y) = z.$$

It is trivial to see that ST^{-1} is bounded.

Lemma 2.2. *Let X and Y be two real Banach spaces and $T: X \to Y$ a (K, L, a)-operator. If $S : X \to Y$ is a mapping such that*

$$\limsup_{\|x\|_X \to \infty} \frac{\|S(x)\|_Y}{\|x\|_X^a} = A \in [0, +\infty[\tag{2.1}$$

then for $|\lambda| > A/L$[4]) we have

$$\lim_{\|x\|_X \to \infty} \|\lambda\, T(x) - S(x)\|_Y = \infty. \tag{2.2}$$

Proof. Since $|\lambda| > A/L$ we can find a number $\gamma > 1$ such that $|\lambda| > A\gamma/L$. Let us suppose that (2.2) is not true. Then there exist $M > 0$ and a sequence $\{x_n\}$ with the property

$$\|x_n\|_X \to \infty \quad \text{and} \quad \|\lambda\, T(x_n) - S(x_n)\|_Y \leqq M \quad \text{for all} \quad n \in N. \tag{2.3}$$

This last relation implies that

$$\frac{\lambda\, T(x_n)}{\|x_n\|_X^a} - \frac{S(x_n)}{\|x_n\|_X^a} \to 0 \quad \text{for} \quad n \to \infty.$$

Hence, by (2.1), we have

$$\limsup_{n \to \infty} \frac{|\lambda|\, \|T(x_n)\|_Y}{\|x_n\|_X^a} = A. \tag{2.4}$$

On the other hand, since T is a (K, L, a)-operator we also have

$$\frac{|\lambda|\, \|T(x_n)\|_Y}{\|x_n\|_X^a} \geqq |\lambda|\, L > \gamma \frac{A}{L} L = \gamma A \quad \text{for all} \quad n \in N.$$

This is a contradiction with (2.4), which proves the assertion.

Theorem 2.1. *Let X and Y be two reflexive Banach spaces. Moreover, let Y be separable and have a strictly convex dual Y'. Let $T: X \to Y$ be a weakly closed*

[3]) As X is reflexive the bounded operator $T^{-1} : Y \to X$ is weakly compact.

[4]) In particular, if $A = 0$, (2.2) holds for all $\lambda \neq 0$.

(K, L, a)*-operator and* $S : X \to Y$ *a bounded weakly closed mapping such that*

(2.5) $$\lim_{\|x\| \to \infty} \frac{\|S(x)\|_Y}{\|x\|_X^a} = 0 .$$

Then, for each $\lambda \neq 0$, $\lambda T - S$ *is regularly surjective.*

Proof. Let us prove that $\lambda T - S$ maps X onto Y. By Lemma 1.1 it is sufficient to show that the mapping $I - ST^{-1}(\cdot/\lambda) : Y \to Y$ is surjective, and to prove this it is sufficient, in virtue of Theorem 1.1, to get

(2.6) $$\lim_{\|y\| \to \infty} \frac{\|ST^{-1}(y/\lambda)\|_Y}{\|y\|_Y} = 0 .$$

If (2.6) were not true, there would exist $\varepsilon > 0$ and a sequence $\{y_n\} \subset Y$ such that

(2.7) $$\lim_n \|y_n\|_Y = \infty \quad \text{and} \quad \left\|ST^{-1}\left(\frac{y_n}{\lambda}\right)\right\|_Y \geqq \varepsilon \|y_n\|_Y .$$

Then, since T is a (K, L, a)-operator, there would exist another sequence $\{x_n\} \subset X$ such that

(2.8) $$\lambda\, T(x_n) = y_n \quad \left(\text{hence} \quad x_n = T^{-1}\left(\frac{y_n}{\lambda}\right)\right),$$

$$|\lambda|\, L\|x_n\|_X^a \leqq \|\lambda\, T(x_n)\|_Y \leqq |\lambda|\, K\|x_n\|_X^a .$$

From (2.7) and (2.8) it would follow that

(2.9) $$\|x_n\|_X \to \infty \quad \text{and} \quad \|S(x_n)\|_Y \geqq \varepsilon\|\lambda\, T(x_n)\|_Y \geqq \varepsilon|\lambda|\, L\|x_n\|_X^a .$$

Obviously (2.9) contradicts (2.5); so (2.6) holds. At this point, to show that $\lambda T - S$ is regularly surjective, it is sufficient to prove that

(2.10) $$\text{for all} \quad R > 0\,, \quad \text{there exists} \quad r > 0 \quad \text{such that}$$
$$\|x\|_X \leqq r \quad \text{for each} \quad x \in X \quad \text{with} \quad \|\lambda\, T(x) - S(x)\|_Y \leqq R .$$

Suppose that there exist $R > 0$ and a sequence $\{x_n\} \subseteq X$ such that

(2.11) $$\|x_n\|_X \to \infty \quad \text{and} \quad \|\lambda\, T(x_n) - S(x_n)\|_Y \leqq R .$$

From (2.5), in virtue of lemma 2.2, it follows that

(2.12) $$\lim_{\|x\|_X \to \infty} \|\lambda\, T(x) - S(x)\|_Y = \infty \quad \text{for all} \quad \lambda \neq 0 .$$

The contradiction between (2.12) and (2.11) proves (2.10) and hence the assertion.

Theorem 2.2. *Let X and Y be two reflexive Banach spaces, with Y separable and Y' strictly convex. Moreover, let T and S be, respectively, a weakly closed* (K, L, a)*-*

operator and a bounded weakly closed mapping such that

$$(2.13) \qquad \limsup_{\|x\|_X \to \infty} \frac{\|S(x)\|_Y}{\|x\|_X^a} = A \in]0, +\infty[\, .$$

Then, for $|\lambda| > A/L$, *the operator* $\lambda T - S$ *is regularly surjective.*

Proof. Let us prove that $\lambda T - S$ maps X onto Y. Of course it is sufficient to consider such λ that

$$(2.14) \qquad \text{there exists} \quad \gamma > 1 \quad \text{with} \quad |\lambda| > \gamma(A/L) > A/L \, .$$

From (2.13), in virtue of Lemma 2.2, it follows that

$$(2.15) \qquad \lim_{\|x\|_X \to \infty} \|\lambda\, T(x) - S(x)\|_Y = +\infty \, ,$$

and hence, by Lemma 1.1, we get

$$(2.16) \qquad \lim_{\|y\|_Y \to \infty} \left\| y - ST^{-1}\left(\frac{y}{\lambda}\right)\right\|_Y = +\infty \, .$$

Let z_0 be a point of Y and $m > (\gamma + 1)/(\gamma - 1)$ an integer. The limit (2.16) guarantees that

(2.17) there exists $R > 0$ such that

$$\|y\|_Y \geqq R \Rightarrow \left\| y - ST^{-1}\left(\frac{y}{\lambda}\right)\right\|_Y > m\|z_0\| \geqq 0 \, .$$

On the other hand, for (2.13), once the number

$$\varepsilon = \left(\frac{m-1}{m+1} - \frac{1}{\gamma}\right)\gamma A > 0 \text{ [5]}$$

is fixed, there exists $R' > 0$ such that

$$(2.18) \qquad \|x\|_X \geqq R' \Rightarrow \|S(x)\|_Y < (A + \varepsilon)\, \|x\|_X^a \, .$$

Obviously we can suppose that in (2.17), R is greater than $K|\lambda|\, R'^a$. Then we have (Def. 1.1, ii)

$$(2.19) \qquad \|y\|_Y \geqq R \Rightarrow \left\| T^{-1}\left(\frac{y}{\lambda}\right)\right\|_X \geqq R' \, .$$

Now we consider the operator $[ST^{-1}(y/\lambda) + z_0] : S_R \subseteq Y \to Y$; it is bounded and weakly closed.

[5]) Since $m > (\gamma + 1)/(\gamma - 1)$, surely ε is positive. This is the reason why we must suppose $A > 0$; therefore for $A = 0$ it is necessary to provide a different proof, by Theorem 2.1.

By (2.17), (2.18) and (2.19) we also have

$$\left\|ST^{-1}\left(\frac{y}{\lambda}\right) + z_0\right\|_Y \leqq \left\|ST^{-1}\left(\frac{y}{\lambda}\right)\right\|_Y + \|z_0\|_Y < \left\|ST^{-1}\left(\frac{y}{\lambda}\right)\right\|_Y +$$

$$+ \frac{1}{m}\left\|y - ST^{-1}\left(\frac{y}{\lambda}\right)\right\|_Y \leqq \left(1 + \frac{1}{m}\right)\left\|ST^{-1}\left(\frac{y}{\lambda}\right)\right\|_Y + \frac{1}{m}\|y\|_Y \leqq$$

$$\leqq \left(1 + \frac{1}{m}\right)(A + \varepsilon)\left\|T^{-1}\left(\frac{y}{\lambda}\right)\right\|_X^a + \frac{1}{m}\|y\|_Y \leqq \left(1 + \frac{1}{m}\right)\frac{A + \varepsilon}{L|\lambda|}\|y\|_Y + \frac{1}{m}\|y\|_Y$$

$$\text{for all} \quad x \in \partial S_R\,.$$

Finally, we obtain

(2.20)

$$\left\|ST^{-1}\left(\frac{y}{\lambda}\right) + z_0\right\|_Y \leqq \left[\left(1 + \frac{1}{m}\right)\frac{A + \left(\frac{m-1}{m+1} - \frac{1}{\gamma}\right)\gamma A}{L|\lambda|} + \frac{1}{m}\right]\|y\|_Y \leqq \|y\|_Y \text{ [6]}$$

for all $y \in \partial S_R$, since $|\lambda| > \gamma A/L$.

The inequality (2.20) allows us to consider (see Section 1) the degree $d(I - ST^{-1}(y/\lambda) - z_0; S_R; 0)$ which is equal to 1. Thus, by the degree property (1.7), we can say that there exists $\bar{y} \in S_R$ such that $\bar{y} = ST^{-1}(\bar{y}/\lambda) + z_0$. Since z_0 is an arbitrary point of Y, this implies that $I - ST^{-1}(y/\lambda)$ is surjective and so $\lambda T - S$ is surjective as well.

Moreover, $\lambda T - S$ is regularly surjective. In fact, if this were not true, then proceeding like in the previous theorem we should find a sequence $\{x_n\} \subseteq X$ such that

$$\|x_n\|_X \to \infty \quad \text{and} \quad \|\lambda\, T(x_n) - S(x_n)\|_Y \leqq R\,,$$

a contradiction with (2.15); this completes the proof.

Theorem 2.3. *Let X be a reflective Banach space and Y a separable reflexive Banach space with a strictly convex dual Y'. Moreover, let $T: X \to Y$ be a weakly closed (K, L, a)-operator and $S: X \to Y$ a bounded weakly closed mapping.*

If S is an a-weakly quasihomogeneous mapping with respect to $S_0: X \to Y$, then there exists finite

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[6] $$\left(\frac{m+1}{m}\;\frac{A + \frac{m-1}{m+1}\gamma A - A}{L|\lambda|} + \frac{1}{m}\right) \leqq 1 \Leftrightarrow \frac{(m+1)\frac{m-1}{m+1}\gamma A}{mL|\lambda|} \leqq \frac{m-1}{m} \Leftrightarrow \frac{\gamma A}{L|\lambda|} \leqq 1 \Leftrightarrow$$

$$\Leftrightarrow |\lambda| \geqq \frac{\gamma A}{L}\,.$$

$$\limsup_{\|x\|_X \to \infty} \frac{\|S(x)\|_Y}{\|x\|_X^a} = A\,,$$

and for $|\lambda| > A/L$, $\lambda T - S$ *is regularly surjective.*

Proof. The second part of the assertion is a consequence of the previous theorems. Let us prove that there exists a finite

$$\limsup_{\|x\|_X \to \infty} \frac{\|S(x)\|_Y}{\|x\|_X^a}\,.$$

If this were not true, there would exist a sequence $\{x_n\} \subseteq X$ such that

$$\|x_n\|_X \to \infty \quad \text{and} \quad \frac{\|S(x_n)\|_Y}{\|x_n\|^a} > n \tag{2.21}$$

for all $n \in N$. Obviously we can suppose that the numerical sequence $\{\|x_n\|_X\}$ is increasing.

On the other hand, the bounded sequence $v_n = x_n/\|x_n\|_X$ would have a subsequence (denoted again by $\{v_n\}$) convergent, in the weak topology, to a point $v_0 \in X$.

Then, putting $t_n = 1/\|x_n\| \searrow 0$, by the a-weak quasihomogenity of S with respect to S_0 we should have

$$t_n^a S\left(\frac{v_n}{t_n}\right) = \frac{1}{\|x_n\|_X^a}\, S\left(\frac{x_n}{\|x_n\|_X}\, \|x_n\|_X\right) = \frac{S(x_n)}{\|x_n\|_X^a} \rightharpoonup S_0(v_0)\,;$$

i.e. the sequence $S(x_n)/\|x_n\|_X^a$ would be bounded. This is a contradiction with (2.21), so the assertion is proved.

Theorem 2.4. *Let X and Y be two reflexive Banach spaces with Y separable and Y' strictly convex. Let $T: X \to Y$ be an a-homogeneous weakly closed (K, L, a)-operator and $S: X \to Y$ an a-homogeneous bounded weakly closed mapping. Then*

i) *there exists a finite* $\limsup_{\|x\|_X \to \infty} \|S(x)\|_Y/\|x\|_X^a = A$ *and so* $\lambda T - S$ *is regularly surjective for* $|\lambda| > A/L$;

ii) $\lambda \neq 0$, $\lambda T - S$ *is regularly surjective* $\Rightarrow$ λ *is not an eigenvalue for the couple* (T, S).

Proof. Let us prove i). Suppose that the lim sup is not finite. Then there exists a sequence $\{x_n\} \subseteq X$ such that

$$\|x_n\|_X \to \infty \quad \text{and} \quad \frac{\|S(x_n)\|_Y}{\|x_n\|_X^a} > n\,. \tag{2.22}$$

If we put $v_n = x_n/\|x_n\|_X$ and consider a suitable subsequence, we have $v_n \rightharpoonup v_0 \in X$ and hence $S(v_n) \rightharpoonup S(v_0)$, because S is weakly continuous. Thus $\{S(v_n)\}$ is bounded; but the homogeneity of S and (2.22) yield

$$S(v_n) = S\left(\frac{x_n}{\|x_n\|_X}\right) = \frac{s(x_n)}{\|x_n\|_X^a} > n\,,$$

a contradiction with the boundedness of $\{S(v_n)\}$. This proves i). Let us prove ii). Let $\lambda T - S$ be regularly surjective; if λ were an eigenvalue for the couple (T, S), there would exist a point $x_0 \neq 0$ belonging to X, such that $\lambda T(x_0) - S(x_0) = 0$. Moreover, denoting by $\{t_n\}$ a sequence of real positive numbers convergent to 0, and putting $v_n = x_0/t_n$, we should have

$$\lim_{n\to\infty} \|v_n\|_X = +\infty\,. \tag{2.23}$$

Then, from the homogeneity of S and T, it would follow that

$$\lambda\, T(v_n) - S(v_n) = \frac{\lambda\, T(x_0) - S(x_0)}{t_n} = 0 \quad \text{for all} \quad n \in N\,,$$

and this would imply that

$$\|\lambda\, T(v_n) - S(v_n)\|_Y < \varepsilon \quad \text{for each} \quad \varepsilon > 0 \quad \text{and all} \quad n \in N\,,$$

whereas, by (2.23), for any $r > 0$ there exists $v_r \in N$ such that $\|v_n\|_X > r$ for all $n > v_r$. This contradicts the assumption that $\lambda T - S$ is regularly surjective; so λ is not an eigenvalue for (T, S) and the assertion is proved.

Remark 2.1. Under the assumptions of Theorem 2.4, from i) and ii) it follows that, for $|\lambda| > A/L$, λ is not an aigenvalue for the couple (T, S).

Theorem 2.5. *Let X and Y be two reflexive Banach spaces, with Y separable and Y' strictly convex. Let $T: X \to Y$ be a weakly closed (K, L, a)-operator and $S: X \to Y$ a bounded weakly closed mapping which is b-weakly quasihomogeneous with respect to $S_0: X \to Y$.*

Then, if a is greater than b and λ is different from 0, $\lambda T - S$ is regularly surjective.

Proof. By Theorem 2.1 it is sufficient to show that

$$\lim_{\|x\|_X\to\infty} \frac{\|S(x)\|_Y}{\|x\|_X^a} = 0\,. \tag{2.24}$$

Let us suppose that this is not true. Then it is possible to find $\varepsilon > 0$ and a sequence $\{x_n\} \subseteq X$ such that

$$\|x_n\|_X \to \infty \quad \text{and} \quad \frac{\|S(x_n)\|_Y}{\|x_n\|_X^a} \geqq \varepsilon \quad \text{for all} \quad n \in N\,. \tag{2.25}$$

If we put $v_n = x_n/\|x_n\|_X$ and suppose like in the previous theorems that the sequence $\{\|x_n\|_X\}$ is increasing and there exists a point $v_0 \in X$ such that $v_n \rightharpoonup v_0$, we have

$$(2.26) \qquad \frac{S(\|x_n\| \, v_n)}{\|x_n\|_X^b} = \frac{S(x_n)}{\|x_n\|_X^b} \rightharpoonup S_0(v_0)\,.$$

On the other hand, since a is greater than b, we obtain

$$\lim_{n \to \infty} \frac{\|x_n\|^b}{\|x_n\|_X^a} = 0\,.$$

From this last relation and from the boundedness of $\{S(x_n)/\|x_n\|_X^b\}$ (deduced from (2.26)) it follows that

$$\lim_{n \to \infty} \frac{\|x_n\|^b}{\|x_n\|_X^a} \frac{\|S(x_n)\|_Y}{\|x_n\|_X^b} = \lim_{n \to \infty} \frac{\|S(x_n)\|_Y}{\|x_n\|_X^a} = 0\,,$$

which is a contradiction with (2.25). This proves (2.24) and hence the assertion.

Remark 2.2. In the theory concerning the Fredholm alternative for nonlinear operators (see [1]) S. Fučík and J. Nečas obtained, on the basis of the Leray-Schauder degree theory, the following theorem (see [1] Theorem 1.1 page 56):

Theorem 2.6. *Let X and Y be two real Banach spaces. Let $T: X \to Y$ be an odd (K, L, a)-homeomorphism*[7]) *and $S: X \to Y$ an odd completely continuous operator. Then for each $\lambda \neq 0$ such that*

$$\lim_{\|x\| \to \infty} \|\lambda \, T(x) - S(x)\|_Y = \infty\,,$$

$\lambda T - S$ maps X onto Y.

This theorem is basic (in Fučík-Nečas's theory) for establishing other important theorems, like theorem analogous to Theorem 2.2 (but with S completely continuous, see [1] Theorem 1.2, page 57) and the theorem about the Fredholm alternative (see [1] Theorem 3.2, page 61).

Let us show by an example that we cannot extend Theorem 2.6 to bounded weakly closed operators.

Let H be a separable Hilbert space and $\{e_n\}_{n \in N}$ an orthonormal basis in H. Thus, for each $x \in X$, we have $x = \sum_i \alpha_i e_i$, where $\alpha_i = (x, e_i)$. Let us consider the transformation $L: X \to X$ defined by

$$L(x) = \sum_i \alpha_i e_{i+1} \quad \text{for all} \quad x \in X\,.$$

It is easy to see that L is a linear bounded weakly continuous operator such that $\|x\| = \|L(x)\|$ for all $x \in X$. Moreover, L is one-to-one but it does not map X onto X; in fact, if $x = \sum_i \alpha_i e_i$ is different from $y = \sum_i \beta_i e_i$, at least one of α_i is different from the corresponding β_i and, consequently, $L(x) \neq L(y)$. On the other hand, L does

[7]) A (K, L, a)-homeomorphism is a (K, L, a)-operator which is a homeomorphism (see [1] page 54).

not map X onto X because all $y \in Y$ with (y, e_1) different from 0 do not belong to $L(X)$; in fact $L(X)$ includes just those points $z \in X$ which have $(z, e_1) = 0$.

Now let us denote by I the identity operator. Of course I is a weakly closed $(1, 1, 1)$-homeomorphism and $I - L = S$ is still a linear bounded weakly continuous operator.

Moreover, if we put $\lambda = 1$, we have

(2.27)

$$\lim_{\|x\| \to \infty} \|\lambda I(x) - S(x)\| = \lim_{\|x\| \to \infty} \|x - (x - Lx)\| = \lim_{\|x\| \to \infty} \|Lx\| = \lim_{\|x\| \to \infty} \|x\| = \infty .$$

Thus all the assumptions of Theorem 2.6 are satisfied (with S bounded and weakly closed) but $\lambda I - S = L$ is not surjective.

Remark 2.3. In the spectral theory of [1] concerning completely continuous operators, S. Fučík and J. Nečas succeeded in proving the contrary of the assertion ii) of Theorem 2.4. They obtained the following theorem:

Theorem 2.7. *Let X and Y be two reflexive Banach spaces, $T: X \to Y$ an odd (K, L, a)-homeomorphism which is an a-homogeneous operator, and $S: X \to Y$ an odd completely continuous a-homogeneous operator.*

Then

(2.28) $\lambda(\neq 0)$ *is not an eigenvalue for the couple* $(T, S) \Rightarrow \lambda T - S$ *is regularly surjective.*

The proof of Theorem 2.7 (see [1] Theorem 3.2, page 61) is based, essentially, on the homogeneity property of T and S and on Theorem 2.6. From this we immediately understand that if we wanted to extend Theorem 2.7 to bounded weakly closed operators, we could not proceed like in [1] when proving (2.28), because Theorem 2.6 does not hold for bounded weakly closed operators. Then we could try to obtain (2.28) using another method; let us show, by an example, that this is not possible.

Let H be a separable Hilbert space and L, I, S, the operators defined in the previous remark. The number $\lambda = 1$ is not an eigenvalue for the couple (I, S). In fact, the equation $\lambda I(x) - S(x) = 0$ has only the solution $x = 0$, because $I - S = I - (I - L) = L$ is a linear injective operator.

Moreover, I and S are both 1-homogeneous operators, but $\lambda I - S = L$ does not map X onto X, as we have seen above. This means that Proposition (2.28) is not true in general, if we suppose S' only bounded and weakly closed.

3. AN APPLICATION

Let $x = (x_1, \ldots, x_n)$ be a point of R^n and Ω an open bounded subset of R^n with a boundary of class C^{2m}.

Consider the real Sobolev space $H^{2m,p}(\Omega) \cap H_0^{m,p}(\Omega) = \mathscr{H}$ (with $p > n > 1$) endowed with the norm

$$\|u\|_{\mathscr{H}} = \|u\| = \sum_{i=1}^{n} \|D_i^{2m} u\|_p + \|u\|_p \quad u \in \mathscr{H},$$

where $\|\cdot\|_p$ is the usual norm in $L^p(\Omega)$.

Let $\alpha_1(x, z), \ldots, \alpha_n(x, z), \gamma(x, z)$ be $n + 1$ functions satisfying the following conditions[8]):

(3.1) $\alpha_1, \ldots, \alpha_n, \gamma$ are measurable in x for all z and continuous in z uniformly with respect to x; for instance: for each $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$|z' - z''| = \sum_{|\alpha| \leqq 2m-1} |z'_\alpha - z''_\alpha| < \delta_\varepsilon \Rightarrow |\alpha_i(x, z') - \alpha_i(x, z'')| < \varepsilon$$
$$\text{for all} \quad x \in \Omega;$$

(3.2) $\alpha_1, \ldots, \alpha_n, \gamma$, are bounded and we have $0 < m \leqq \alpha_i(x, z) \leqq M < +\infty$, $i = 1, \ldots, n$, $0 < m \leqq \gamma(x, z) \leqq M < +\infty$ for all (x, z).

Let us put $\zeta(u) = \{D^\alpha u\}_{|\alpha| \leqq 2m-1}$ $\zeta(u)(x) = \{D^\alpha u(x)\}_{|\alpha| \leqq 2m-1}$ and consider the quasilinear elliptic operator

$$L(u) = (-1)^m \sum_{i=1}^{n} \alpha_i(x, \zeta(u))\, D_i^{2m} u + \gamma(x, \zeta(u))\, u .$$

In order to study the equation

$$Lu = f, \quad f \in L^p(\Omega), \quad u \in \mathscr{H} \tag{3.3}$$

we introduce the operator $T: \mathscr{H} \to L^p(\Omega)$ defined by

$$T(u) = (-1)^m \sum_{i=1}^{n} D_i^{2m} u + u \quad u \in \mathscr{H} .$$

It is well known that T is a continuous linear operator (thus bounded and weakly continuous) bijective between $\mathscr{H}$ and $L^p(\Omega)$. Therefore it is a weakly closed $(K_p, c_p, 1)$-operator, where K_p and c_p are "the best constants"[9]) such that

$$c_p \|u\|_{\mathscr{H}} \leqq \|Tu\|_p \leqq K_p \|u\|_{\mathscr{H}} \quad \text{for all} \quad u \in \mathscr{H} .$$

We explicitly remark that $c_p < 1$.

The next lemma will be very useful later on.

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[8]) We set $z = \{z_\alpha\}_{|\alpha| \leqq 2m-1} = \{z_{(\alpha_1, \ldots, \alpha_n)}\} \in R^s$, where s is the number of such α that $|\alpha| \leqq 2m - 1$.

[9]) "The best constants" means that

$$\frac{1}{c_p} = \sup_{u \in \mathscr{H}} \frac{\|u\|_{\mathscr{H}}}{\|Tu\|_p} \quad \text{and} \quad K_p = \sup_{u \in \mathscr{H}} \frac{\|Tu\|_p}{\|u\|_{\mathscr{H}}} .$$

Lemma 3.1. *Let $\{u_n\}$ be a sequence in $C_B^{2m-1}(\Omega)$* [10]*. If $\{u_n\}$ converges in C_B^{2m-1} to a function $u \in C_B^{2m-1}(\Omega)$, then the functions $\alpha_i(x, \zeta(u_n))$, $\gamma(x, \zeta(u_n))$ converge uniformly to the functions $\alpha_i(x, \zeta(u))$, $\gamma(x, \zeta(u))$.*

Proof. We prove the assertion for γ; the same proceeding will be valid for α_i.

As $\{u_n\}$ converges to u $C_B^{2m-1}(\Omega)$, once $\varepsilon > 0$ is fixed and the corresponding δ_ε is found, there exists $v \in N$ such that $\sum_{|\alpha| \leqq 2m-1} |D^\alpha u_n(x) - D^\alpha u(x)| < \delta_\varepsilon$ for all $n \geqq v$ and $x \in \Omega$. Hence, in virtue of (3.1), it follows that

$$|\gamma(x, \{D^\alpha u_n(x)\}) - \gamma(x, \{D^\alpha u(x)\})| = |\gamma(x, \zeta(u_n)) - \gamma(x, \zeta(u))| < \varepsilon$$
$$\text{for all} \quad n \geqq v \quad \text{and} \quad x \in \Omega\,;$$

the assertion is proved.

Lemma 3.2. *The operator $L: \mathscr{H} \to L^p$ is bounded and weakly closed.*

Proof. Of course L is bounded. For showing that L is weakly closed it is necessary to prove that

$$\left.\begin{matrix} u_n \rightharpoonup_{\mathscr{H}} u \\ Lu_n \rightharpoonup_{L^p} w \end{matrix}\right\} \Rightarrow w = Lu\,. \tag{3.4}$$

Let us start by observing that the sequence $\{u_n\}$ is bounded in $\mathscr{H}$ and so, by a theorem of Rellich-Kondrachov (see [8]) there exists a subsequence $\{u_{n_k}\}$ convergent to u in $C_B^{2m-1}(\Omega)$ (remember that $p > n$). Then, from the previous lemma, we have that the coefficients $\alpha_i(x, \zeta(u_{n_k}))$, $\gamma(x, \zeta(u_{n_k}))$ converge uniformly to $\alpha_i(x, \zeta(u))$, $\gamma(x, \zeta(u))$.

Moreover, since $\{u_n\}$ converges to u in the weak topology, we have

$$D_i^{2m} u_n \rightharpoonup_{L^p} D_i^{2m} u\,, \quad u_n \rightharpoonup_{L^p} u\,. \tag{3.5}$$

Let χ be a function from $L^q(\Omega)$ $(1/p + 1/q = 1)$; by writing $\langle \chi, \varphi \rangle$ instead of $\int_\Omega \chi\varphi$, we obtain

$$\begin{aligned}
&|\langle \chi, Lu_{n_k}\rangle - \langle \chi, Lu\rangle| \leqq \sum_{i=1}^{n} \langle \chi, \alpha_i(x, \zeta(u_{n_k}))\, D_i^{2m} u_{n_k} - \alpha_i(x, \zeta(u))\, D_i^{2m} u\rangle| + \\
&+ |\langle \chi, \gamma(x, \zeta(u_{n_k}))\, u_{n_k} - \gamma(x, \zeta(u))\, u\rangle| \leqq \sum_{i=1}^{n} |\langle \chi, \alpha_i(x, \zeta(u_{n_k}))\, D_i^{2m} u_{n_k} - \\
&- \alpha_i(x, \zeta(u))\, D_i^{2m} u_{n_k}\rangle| + |\langle \chi, \alpha_i(x, \zeta(u))\, D_i^{2m} u_{n_k} - \alpha_i(x, \zeta(u))\, D_i^{2m} u\rangle| + \\
&+ |\langle \chi, \gamma(x, \zeta(u_{n_k}))\, u_{n_k} - \gamma(x, \zeta(u))\, u_{n_k}\rangle| + |\langle \chi, \gamma(x, \zeta(u))\, u_{n_k} - \gamma(x, \zeta(u))\, u\rangle| \leqq \\
&\leqq \sum_{i=1}^{n} \sup_{\Omega} |\alpha_i(x, \zeta(u_{n_k})) - \alpha_i(x, \zeta(u))|\, \|\chi\|_q\, \|D_i^{2m} u_{n_k}\|_p +
\end{aligned}$$

[10]) $C_B^{2m-1}(\Omega) = \{u \in C^{2m-1}(\Omega) : D^\alpha u$ is bounded in Ω for $|\alpha| \leqq 2m - 1\}$ is a Banach space with the norm $\|u\|_{C_B^{2m-1}} = \sum_{|\alpha| \leqq 2m-1} \sup_{\Omega} |D^\alpha u|$ (see [8]).

$$+ \sum_{i=1}^{n} \left|\langle \chi, \alpha_i(x, \zeta(u)) \left[D_i^{2m} u_{n_k} - D_i^{2m} u\right]\rangle\right| + \sup_{\Omega} \left|\gamma(x, \zeta(u_{n_k}) - \gamma(x, \zeta(u))\right| \cdot$$

$$\cdot \|\chi\|_q \|u_{n_k}\|_p + \left|\langle \chi, \gamma(x, \zeta(u)) \left[u_{n_k} - u\right]\rangle\right| \to 0 \quad \text{for} \quad k \to \infty$$

by (3.5) and the uniform convergence of $\alpha_i(x, \zeta(u_{n_k}))$ and $\gamma(x, \zeta(u_{n_k}))$.

Then $Lu_{n_k} \rightharpoonup Lu$ in $L^p(\Omega)$ and hence we have $w = Lu$ in (3.4), as we wanted to show.

At this point we can prove the following existence theorem.

Theorem 3.1. *If $M < m((1 + c_p)/(1 - c_p))$ then L is regularly surjective; so the equation (3.3) has a solution for each $f \in L^p(\Omega)$.*

Proof. Let us put

$$a_i(x, z) = \alpha_i(x, z) - r, \quad c(x, z) = \gamma(x, z) - r \quad i = 1, \ldots, n,$$

where r is equal to $\frac{1}{2}(M + m)$.

Of course the functions a_i and c satisfy again the condition (3.1); we also have

$$\text{(3.6)} \qquad \frac{m - M}{2} = m - r \leqq a_i(x, z) \leqq M - r = \frac{M - m}{2} \quad i = 1, \ldots, n,$$

$$\frac{m - M}{2} = m - r \leqq c(x, z) \leqq M - r = \frac{M - m}{2}.$$

Thus we have

$$Lu = (-1)^m \sum_{i=1}^{n} a_i(x, \zeta(u)) + rD_i^{2m} u + \left[c(x, \zeta(u)) + r\right] u = (rT - S) u,$$

where $Su = -(-1)^m \sum_{i=1}^{n} a_i(x, \zeta(u)) D_i^{2m} u - c(x, \zeta(u)) u$.

By Lemma 3.2 it follows that S (which is of the same type of L) is bounded and weakly closed; it also satisfies the inequality

$$\text{(3.7)} \qquad \|Su\|_p \leqq \frac{M - m}{2} \|u\|_{\mathscr{H}} \quad u \in \mathscr{H}$$

and hence we have

$$\text{(3.8)} \qquad \limsup_{\|u\| \to \infty} \frac{\|S(u)\|_p}{\|u\|_{\mathscr{H}}} \leqq \frac{M - m}{2}.$$

Therefore, by Theorem 2.2, we can say that, for

$$|\lambda| > \frac{M - m}{2} \Big/ c_p,$$

the operator $\lambda T - S$ is regularly surjective.

In particular, since M is less than $m((1 + c_p)/(1 - c_p))$, we have

$$\lambda = r = \frac{M + m}{2} > \frac{M - m}{2} \Big/ c_p$$

and, thus, $rT - S = L$ is regularly surjective. The assertion is proved.

Remark 3.1. If $\alpha_i(x, z) = \alpha_i(x)$ and $\gamma(x, z) = \gamma(x)$ (i.e. if L is linear and therefore 1-homogeneous) the number r of the previous theorem is not an eigenvalue for the couple (T, S) (see Theorem 2.4). This, since $L = rT - S$ is linear, implies that L is one-to-one and hence bijective.

Remark 3.2. In the linear case we could get the same result by making use of the classical perturbation theory.

In fact, with the same notation as in Theorem 3.1 we have

$$Lu = (-1)^m \sum_{i=1}^{n} \alpha_i(x)\, D_i^{2m} u + \gamma(x)\, u = (-1)^m \sum_{i=1}^{n} [a_i(x) + r]\, D_i^{2m} u +$$
$$+ [c(x) + r]\, u = rTu - Su\,.$$

Moreover, because $\|Su\|_p \leqq \frac{1}{2}(M - m)\, \|u\|$, it is obvious that

$$\|S\|_{\mathscr{L}(\mathscr{H}, L^p(\Omega))} = \sup_{u \in \mathscr{H}} \frac{\|Su\|_p}{\|u\|} \leqq \frac{M - m}{2}$$

as well as

$$\|(rT)^{-1}\|_{\mathscr{L}(\mathscr{H}, L^p(\Omega))} = \sup_{u \in \mathscr{H}} \frac{\|u\|}{\|rTu\|_p} = \frac{1}{rc_p} \quad \text{(see footnote 9))}\,.$$

Then, if we suppose $M < ((1 + c_p)/(1 - c_p))$, like in Theorem 3.1, we have

$$\|S\| \leqq \frac{M - m}{2} < \frac{M + m}{2} \Big/ c_p = rc_p = (\|(rT)^{-1}\|)^{-1}$$

and this, via the perturbation theory, yields that $rT - S = L$ is bijective.

Remark 3.3. If, instead of (3.1), the coefficients $\alpha_i(x, z)$, $\gamma(x, z)$ of L satisfy the condition

$$(3.1)' \qquad \alpha_i(x, z) \in C^{2m}(\Omega \times R^s)\,, \quad \gamma(x, z) \in C^0(\Omega \times R^s)$$

(and also (3.2)), then the equation (3.3) can be solved, at least for $m = 1$, by having recourse to a classical proceeding based on the Leray-Schauder Theorem. In fact, if we introduce the mapping

$$T : v \in \mathscr{H} \to T(v) = u_v \in \mathscr{H} \quad \text{such that}$$

$$L_v(u_v) = (-1)^m \sum_{i=1}^{n} \alpha_i(x, \zeta(v))\, D_i^{2m} u_v + \gamma(x, \zeta(v))\, u_v = f \in L^p(\Omega)\,,$$

it is easy to show that this mapping is completely continuous, on the basis of Rellich-Kondrachov's theorem and the inequality

$$\|u\|_{\mathscr{H}} \leqq c_v \|L_v u\|_p \quad \text{for all} \quad u \in \mathscr{H} . \tag{3.9}$$

This last inequality is certainly valid since the coefficients of L_v are regular (of class C^{2m}) for (3.1)'. In (3.9) the constant c_v depends, at least for $m = 1$, on the modulus of continuity of the coefficients (see [11]).

Thus we can find, in a ball "large enough", a fixed point $u_v = v$ of T which, of course, is a solution of (3.3).

Nevertheless, under the assumption (3.1) the coefficients of L_v are only bounded and measurable, and under this conditions, in general, we have not inequalities (in L^p) of the type (3.9) (see [10] for an extensive study of this question).

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