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REPRESENTATION OF SOLUTIONS OF GENERAL LINEAR DIFFERENTIAL SYSTEMS OF THE SECOND ORDER

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1. INTRODUCTION

We consider linear differential systems of the second order

$$(F(x)\,Y')' + G(x)\,Y = 0\,, \tag{1.1}$$

where $F(x)$, $G(x)$ are continuous $n \times n$ matrices and $F(x)$ is nonsingular. In [3] the transformation of (1.1) was investigated under the condition that $F(x)$, $G(x)$ are symmetric, i.e. that (1.1) is self-adjoint. It was shown that the investigated transformation defines a certain equivalence on the class of all self-adjoint linear differential systems of the second order and certain special forms of (1.1) were suggested as the canonical representations of the classes of decomposition defined by this equivalence. The aim of the present paper is to extend these results to nonself-adjoint systems.

The matrix notation is used. E and 0 denote the identity matrix and the zero matrix of any dimension, A^{T} and A^* denote the transpose and the conjugate transpose of the matrix A, respectively. An $n \times n$ matrix A is said to be symmetric if $A^{\mathrm{T}} = A$, antisymmetric if $A^{\mathrm{T}} = -A$ and orthonormal if $A^{\mathrm{T}} = A^{-1}$. $C^m(I)$ denotes the class of m-times differentiable real functions on an interval I, $C^0(I)$ means continuity. If $A(x)$ is an arbitrary matrix of functions, we write $A(x) \in C^m(I)$ if each entry of $A(x)$ belongs to $C^m(I)$.

Throughout the paper the system (1.1) is investigated on an interval I of an arbitrary kind. An $n \times n$ matrix $Y(x)$ is a solution of (1.1) if $Y(x) \in C^1(I)$, $F(x)\,Y'(x) \in C^1(I)$ and (1.1) is identically satisfied on I.

2. EQUIVALENT DIFFERENTIAL SYSTEMS

We shall investigate the following transformation of systems (1.1).

Theorem 1. *Let $H_1(x), H_2(x) \in C^1(I)$ be nonsingular $n \times n$ matrices for which*

$$H_2^{\mathrm{T}\prime}(x)\,F(x)\,H_1(x) - H_2^{\mathrm{T}}(x)\,F(x)\,H_1'(x) = 0\,. \tag{2.1}$$

Then the transformation $Y(x) = H_1(x)\, U(x)$ *gives*

$$H_2^{\mathrm{T}}(x)\left[(F(x)\, Y')' + G(x)\, Y\right] = (F_1(x)\, U')' + G_1(x)\, U\,, \tag{2.2}$$

where

$$F_1(x) = H_2^{\mathrm{T}}(x)\, F(x)\, H_1(x) \tag{2.3}$$

$$G_1(x) = H_2^{\mathrm{T}}(x)\left[(F(x)\, H_1'(x))' + G(x)\, H_1(x)\right].$$

Proof. $H_2^{\mathrm{T}}(FY')' + H_2^{\mathrm{T}}GY = H_2^{\mathrm{T}}(FH_1'U + FH_1U')' + H_2^{\mathrm{T}}GH_1U =$
$= (H_2^{\mathrm{T}}FH_1'U + H_2^{\mathrm{T}}FH_1U')' - H_2^{\mathrm{T}\prime}(FH_1'U + FH_1U') + H_2^{\mathrm{T}}GH_1U =$
$= H_2^{\mathrm{T}\prime}FH_1'U + H_2^{\mathrm{T}}(FH_1')'U + H_2^{\mathrm{T}}FH_1'U' + (H_2^{\mathrm{T}}FH_1U')' - H_2^{\mathrm{T}\prime}FH_1'U -$
$- H_2^{\mathrm{T}\prime}FH_1U' + H_2^{\mathrm{T}}GH_1U = (H_2^{\mathrm{T}}FH_1U')' + H_2^{\mathrm{T}}[(FH_1')' + GH_1]\, U = (F_1U')' + G_1U.$

Remark 1. From Theorem 1 we see that a matrix $Y(x)$ is a solution of (1.1) if and only if the matrix $U(x) = H_1^{-1}(x)\, Y(x)$ is a solution of $(F_1(x)\, U')' + G_1(x)\, U = 0$. It is also seen that if $Z(x) = H_2(x)\, V(x)$ then

$$H_1^{\mathrm{T}}(x)\left[(F^{\mathrm{T}}(x)\, Z')' + G^{\mathrm{T}}(x)\, Z\right] = (F_1^{\mathrm{T}}(x)\, V')' + G_1^{\mathrm{T}}(x)\, V.$$

Indeed, $H_1^{\mathrm{T}\prime}F^{\mathrm{T}}H_2 - H_1^{\mathrm{T}}F^{\mathrm{T}}H_2' = 0$, $F_1^{\mathrm{T}} = H_1^{\mathrm{T}}F^{\mathrm{T}}H_2$ and $G_1^{\mathrm{T}} = (H_1^{\mathrm{T}\prime}F^{\mathrm{T}})'\, H_2 +$
$+ H_1^{\mathrm{T}}G^{\mathrm{T}}H_2 = (H_1^{\mathrm{T}\prime}F^{\mathrm{T}}H_2)' - H_1^{\mathrm{T}\prime}F^{\mathrm{T}}H_2' + H_1^{\mathrm{T}}G^{\mathrm{T}}H_2 = (H_1^{\mathrm{T}}F^{\mathrm{T}}H_2')' - H_1^{\mathrm{T}\prime}F^{\mathrm{T}}H_2' +$
$+ H_1^{\mathrm{T}}G_1^{\mathrm{T}}H_2 = H_1^{\mathrm{T}}(F^{\mathrm{T}}H_2')' + H_1^{\mathrm{T}\prime}F^{\mathrm{T}}H_2' - H_1^{\mathrm{T}\prime}F^{\mathrm{T}}H_2' + H_1^{\mathrm{T}}G^{\mathrm{T}}H_2 =$
$= H_1^{\mathrm{T}}[(F^{\mathrm{T}}H_2')' + G^{\mathrm{T}}H_2]$.

Now, let us suppose, for a moment, the system (1.1) to be self-adjoint and let the transformation described in Theorem 1 transform this system into the system

$$(F_1(x)\, U')' + G_1(x)\, U = 0$$

which is also self-adjoint. From (2.1) and (2.3) it follows that $F_1' = H_2^{\mathrm{T}\prime}FH_1 +$
$+ H_2F'H_1 + H_2FH_1' = 2H_2^{\mathrm{T}\prime}FH_1 + H_2^{\mathrm{T}}F'H_1 = 2H_2^{\mathrm{T}\prime}H_2^{\mathrm{T}-1}H_2^{\mathrm{T}}FH_1 + H_2^{\mathrm{T}}F'H_1$,
hence $H_2^{\mathrm{T}\prime} = \frac{1}{2}(F_1'F_1^{-1}H_2^{\mathrm{T}} - H_2^{\mathrm{T}}F'F^{-1})$ and thus

$$H_2' = \tfrac{1}{2}(H_2F_1^{-1}F_1' - F^{-1}F'H_2)\,, \tag{2.4$_2$}$$

where the fact that $F(x)$ and $F_1(x)$ are symmetric has been used. Similarly

$$H_1' = \tfrac{1}{2}(H_1F_1^{-1}F_1' - F^{-1}F'H_1)\,. \tag{2.4$_1$}$$

Therefore $H_1(x)$ and $H_2(x)$ satisfy the same differential system. Let $M(x)$, $N(x)$ be matrix solutions of the differential systems

$$M' = -\tfrac{1}{2}F_1^{-1}(x)\, F_1'(x)\, M\,,$$

$$N' = -\tfrac{1}{2}F^{-1}(x)\, F'(x)\, N\,.$$

Then $H_1(x) = N(x)\, C_1M^{-1}(x)$, $H_2(x) = N(x)\, C_2M^{-1}(x)$, where C_1, C_2 are constant nonsingular $n \times n$ matrices. As the matrices $F(x)$ and $F_1(x)$ are symmetric, the first relation of (2.3) implies that the matrices $C_2^{\mathrm{T}}N^{\mathrm{T}}(x)\, F(x)\, N(x)\, C_1$ and

$C_2^{T-1} M^T(x) F_1(x) M(x) C_1^{-1}$ are also symmetric. Thus, if we denote $F_2(x) = N^T(x) F(x) N(x)$ and $F_3(x) = M^T(x) F_1(x) M(x)$, we have

$$C_1^{T-1} C_2^T F_2(x) - F_2(x) C_2 C_1^{-1} = 0$$
$$C_1^T C_2^{T-1} F_3(x) - F_3(x) C_2^{-1} C_1 = 0$$

for $x \in I$. From the second relation of (2.3) we obtain conditions of the similar kind involving matrices $G(x)$ and $G_1(x)$. Note that only in a very special cases these conditions are fulfilled by a pair of nonsingular matrices C_1, C_2 for which $C_2 C_1^{-1} \neq kE$, i.e. $C_2 \neq kC_1$, where k is a real constant. Thus we see that if we consider only self-adjoint systems (1.1), then the transformation from Theorem 1 is essentially the same as the transformation investigated in [3].

Definition. Two differential systems

$$(F_i(x)\, Y')' + G_i(x)\, Y = 0\,, \quad i = 1, 2 \tag{2.5$_i$}$$

are said to be *equivalent* if there exist nonsingular matrices $H_1(x), H_2(x) \in C^1(I)$ satisfying

$$H_2^{T\prime}(x) F_1(x) H_1(x) - H_2^T(x) F_1(x) H_1'(x) = 0 \tag{2.6}$$

and

$$F_2(x) = H_2^T(x) F_1(x) H_1(x) \tag{2.7}$$
$$G_2(x) = H_2^T(x) \left[(F_1(x) H_1'(x))' + G_1(x) H_1(x)\right].$$

If $H_1(x)$, $H_2(x)$ satisfy (2.6) and (2.7), we shall say that *the transformation* $\{H_2(x), H_1(x)\}$ *transforms* $(2.5)_1$ into $(2.5)_2$.

It is to prove that the relation "$(F_1(x)\, Y')' + G_1(x)\, Y = 0$ can be transformed into $(F_2(x)\, Y')' + G_2(x)\, Y = 0$" is really an equivalence. It is obvious that the transformation $\{E, E\}$ transforms every system into itself. Let $\{H_2(x), H_1(x)\}$ transform $(2.5)_1$ into $(2.5)_2$, then $\{H_2^{-1}(x), H_1^{-1}(x)\}$ transforms $(2.5)_2$ into $(2.5)_1$. In fact, $F_1 = H_2^{T-1} F_2 H_1^{-1}$, $(H_2^{T-1})' F_2 H_1^{-1} - H_2^{T-1} F_2 (H_1^{-1})' = -H_2^{T-1} H_2^{T\prime} H_2^{T-1} F_2 H_1^{-1} + H_2^{T-1} F_2 H_1^{-1} H_1' H_1^{-1} = H_2^{T-1} H_2^{T\prime} F_1 - F_1 H_1' H_1^{-1} = H_2^{T-1}(H_2^{T\prime} F_1 H_1 - H_2^T F_1 H_1') H_1^{-1} = 0$ and $G_1 = H_2^{T-1} G_2 H_1^{-1} - (F_1 H_1')' H_1^{-1} = H_2^{T-1} G_2 H_1^{-1} - (H_2^{T-1} F_2 H_1^{-1} H_1')' H_1^{-1} = H_2^{T-1} G_2 H_1^{-1} - (H_2^{T-1} F_2 H_1^{-1} H_1' H_1^{-1})' + H_2^{T-1} F_2 H_1^{-1} H_1' (H_1^{-1})' = H_2^{T-1} G_2 H_1^{-1} + (H_2^{T-1} F_2 (H_1^{-1})')' - H_2^{T-1} F_2 H_1^{-1} H_1' H_1^{-1} H_1' H_1^{-1} = H_2^{T-1} G_2 H_1^{-1} + H_2^{T-1}(F_2 (H_1^{-1})')' + H_2^{T-1} H_2^{T\prime} H_2^{T-1} F_2 H_1^{-1} H_1' H_1^{-1} - H_2^{T-1} F_2 H_1^{-1} H_1' H_1^{-1} H_1' H_1^{-1} = H_2^{T-1}[(F_2 (H_1^{-1})')' + G_2 H_1^{-1}] + H_2^{T-1}(H_2^{T\prime} F_1 H_1 - H_2^T F_1 H_1') H_1^{-1} H_1' H_1^{-1} = H_2^{T-1}[(F_2 (H_1^{-1})')' + G_2 H_1^{-1}]$.

Now, let $\{H_2(x), H_1(x)\}$ transform $(2.5)_1$ into $(2.5)_2$ and let $\{H_4(x), H_3(x)\}$ transform $(2.5)_2$ into

$$(F_3(x)\, Y')' + G_3(x)\, Y = 0\,. \tag{2.5$_3$}$$

Then $F_3 = H_4^T F_2 H_3 = (H_2 H_4)^T F_1 H_1 H_3$, $(H_2 H_4)^{T\prime} F_1 H_1 H_3 - (H_2 H_4)^T F_1 (H_1 H_3)' =$
$= H_4^{T\prime} H_2^T F_1 H_1 H_3 + H_4^T H_2^{T\prime} F_1 H_1 H_3 - H_4^T H_2^T F_1 H_1' H_3 - H_4^T H_2^T F_1 H_1 H_3' =$
$= H_4^{T\prime} F_2 H_3 + H_4^T (H_2^{T\prime} F_1 H_1 - H_2^T F_1 H_1') H_3 - H_4^T F_3 H_3' = 0$ and $G_3 =$
$= H_4^T (F_2 H_3')' + H_4^T G_2 H_3 = H_4^T (H_2^T F_1 H_1 H_3')' + H_4^T H_2^T (F_1 H_1')' H_3 +$
$+ H_4^T H_2^T G_1 H_1 H_3 = H_4^T H_2^T (F_1 H_1 H_3')' + H_4^T H_2^{T\prime} F_1 H_1 H_3' + H_4^T H_2^T (F_1 H_1')' H_3 +$
$+ (H_2 H_4)^T G_1 H_1 H_3 = (H_2 H_4)^T (F_1 (H_1 H_3)')' - H_4^T H_2^T (F_1 H_1' H_3)' +$
$+ H_4^T H_2^{T\prime} F_1 H_1 H_3' + H_4^T H_2^T (F_1 H_1')' H_3 + (H_2 H_4)^T G_1 H_1 H_3 =$
$= (H_2 H_4)^T (F_1 (H_1 H_3)')' - H_4^T H_2^T (F_1 H_1')' H_3 - H_4^T H_2^T F_1 H_1' H_3' + H_4^T H_2^{T\prime} F_1 H_1 H_3' +$
$+ H_4^T H_2^T (F_1 H_1')' H_3 + (H_2 H_4)^T G_1 H_1 H_3 = (H_2 H_4)^T [(F_1 (H_1 H_3)')' + G_1 H_1 H_3]$.
Thus $\{H_2(x) H_4(x), H_1(x) H_3(x)\}$ transforms $(2.5)_1$ into $(2.5)_3$.

Theorem 2. *System* (1.1) *can always be transformed into the system*

$$Y'' + P(x) Y = 0 \,, \tag{2.8}$$

i.e. there exist nonsingular $n \times n$ matrices $H_1(x), H_2(x) \in C^1(I)$ satisfying (2.1) *for which $H_2^T(x) F(x) H_1(x) = E$. The matrix $P(x)$ is determined by the relation*

$$P(x) = H_2^T(x) [(F(x) H_1'(x))' + G(x) H_1(x)] \,. \tag{2.9}$$

Proof. Let $H_1(x)$ and $H_2(x)$ be solutions of differential systems

$$H_1' = -\tfrac{1}{2} F^{-1}(x) F'(x) H_1 \,,$$
$$H_2' = -\tfrac{1}{2} F^{T-1}(x) F^{T\prime}(x) H_2$$

such that for some real a, $H_2^T(a) F(a) H_1(a) = E$. Then $(H_2^T F H_1)' = H_2^{T\prime} F H_1 + H_2^T F' H_1 + H_2^T F H_1' = -\tfrac{1}{2} H_2^T F' F^{-1} F H_1 + H_2^T F' H_1 - \tfrac{1}{2} H_2^T F F^{-1} F' H_1 = 0$ and from the initial condition at $x = a$ we have $H_2^T(x) F(x) H_1(x) = E$. Further, $H_2^{T\prime} F H_1 - H_2^T F H_1' = -\tfrac{1}{2} H_2^T F' F^{-1} F H_1 + \tfrac{1}{2} H_2^T F F^{-1} F' H_1 = 0$. This completes the proof since (2.9) follows from (2.3).

Remark 2. If $\{H_2(x), H_1(x)\}$ transforms (1.1) into (2.8) then $\{H_2(x) C_2, H_1(x) C_1\}$, where C_1, C_2 are constant $n \times n$ matrices for which $C_2^T C_1 = E$, transforms (1.1) into $Y'' + C_2^T P(x) C_1 Y = 0$. Conversely, if $\{H_2(x), H_1(x)\}$ and $\{H_4(x), H_3(x)\}$ transform (1.1) into $Y'' + P(x) Y = 0$ and $Y'' + P_1(x) Y = 0$, respectively, then there exists a constant $n \times n$ matrix C_1 such that $H_4(x) = H_2(x) C_1^{T-1}$, $H_3(x) = H_1(x) C_1$ and $P_1(x) = C_1^{-1} P(x) C_1$. Therefore, we see that the transformation of (1.1) into (2.8) is unique up to a right and a left multiple of $P(x)$ by a constant nonsingular matrix. For this reason we suggest system (2.8) to be the canonical representation of the class of all equivalent differential systems of the second order.

A natural question which arises when investigating transformations of nonself-adjoint systems is whether or not a nonself-adjoint system can be transformed into a self-adjoint one. According to Theorem 2 we can suppose that the nonself-adjoint system is in the form (2.8). Let the transformation $\{H_2(x), H_1(x)\}$ transform this system into system (1.1), where $F(x)$ and $G(x)$ are symmetric. Without loss of generality we can suppose, in addition, that in (1.1) $F(x) = F$, where F is a constant matrix, see [3, Theorem 2]. Then (2.4) implies $H_1'(x) = 0$, $H_2'(x) = 0$, hence $H_i(x) = H_i$, $i = 1, 2$, are constant matrices. From (2.3) we have $H_2^T H_1 = F$ and $G(x) = H_2^T H_1'' + H_2^T P(x) H_1 = H_2^T P(x) H_1$. Since the matrix $G(x)$ is symmetric, we proved the following statement.

Theorem 3. *The nonself-adjoint system (2.8) can be transformed into a self-adjoint system if and only if $P(x) = H_2^T P_0(x) H_1$, where $P_0(x)$ is a symmetric matrix and H_1, H_2 are constant $n \times n$ matrices such that the matrix $H_1 H_2^T$ is symmetric.*

3. MAIN THEOREM

Let us suppose that the system

$$Y'' + P(x) Y = 0 \tag{3.1}$$

is self-adjoint (i.e. $P^T(x) = P(x)$). In [2] it was shown that there exists a nonsingular $n \times n$ matrix $R(x) \in C^1(I)$ satisfying

$$R^{T\prime}(x) R(x) - R^T(x) R'(x) = 0$$

which transforms system (3.1) into the system

$$(Q^{-1}(x) S')' + Q(x) S = 0\,, \tag{3.2}$$

where $Q(x) = (R^T(x) R(x))^{-1}$, i.e. if $Y(x) = R(x) S(x)$ then

$$R^T(x) [Y'' + P(x) Y] = (Q^{-1}(x) S')' + Q(x) S\,.$$

This transformation enables us to investigate systems (3.1) through systems (3.2) since the solutions of (3.2) have many nice properties, see [4] and [5].

In this section we shall show that a similar transformation is possible also in the case when (3.1) is nonself-adjoint. First we recall several results concerning self-adjoint systems which we shall use in the sequel.

Let $Y_1(x)$, $Y_2(x)$ be solutions of the self-adjoint system (1.1). Then

$$Y_1^{T\prime}(x) F(x) Y_2(x) - Y_1^T(x) F(x) Y_2'(x) = K\,,$$

where K is a constant $n \times n$ matrix. If $Y_1(x) = Y_2(x)$ and $K = 0$ then this solution is said to be *isotropic*. $Y_1(x)$, $Y_2(x)$ are said to be *independent* if every solution $Z(x)$ of (1.1) can be expressed in the form $Z(x) = Y_1(x) C_1 + Y_2(x) C_2$, where C_1, C_2 are constant $n \times n$ matrices.

Theorem 4. *Let $P(x) \in C^0(I)$ be an arbitrary matrix. There exists a transformation $\{R_2(x), R_1(x)\}$ which transforms the system*

$$Y'' + P(x) Y = 0 \tag{3.3}$$

into the system

$$(Q^{-1}(x)\,S')' + Q^{\mathrm{T}}(x)\,S = 0\,, \tag{3.4}$$

where $Q(x) = (R_2^{\mathrm{T}}(x)\,R_1(x))^{-1}$, *i.e. there exist nonsingular* $n \times n$ *matrices* $R_i(x) \in C^1(I)$, $i = 1, 2$, *for which*

$$R_2^{\mathrm{T}\prime}(x)\,R_1(x) - R_2^{\mathrm{T}}(x)\,R_1'(x) = 0\,, \tag{3.5}$$

such that the transformation $Y(x) = R_1(x)\,S(x)$ *gives*

$$R_2^{\mathrm{T}}(x)\left[Y'' + P(x)\,Y\right] = (Q^{-1}(x)\,S')' + Q^{\mathrm{T}}(x)\,S\,.$$

Proof. Let $U_1(x)$, $V_1(x)$ be solutions of (3.3), let $U_2(x)$, $V_2(x)$ be solutions of its adjoint

$$Y'' + P^{\mathrm{T}}(x)\,Y = 0 \tag{3.6}$$

for which $U_i(a) = 0$, $U_i'(a) = E$, $V_i(a) = E$, $V_i'(a) = 0$, $i = 1, 2$, $a \in I$. Let us set

$$U(x) = \begin{bmatrix} 0 & U_1(x) \\ U_2(x) & 0 \end{bmatrix} \qquad V(x) = \begin{bmatrix} V_1(x) & 0 \\ 0 & V_2(x) \end{bmatrix} \tag{3.7}$$

$$\mathscr{E} = \begin{bmatrix} 0 & E \\ E & 0 \end{bmatrix} \qquad \mathscr{P}(x) = \begin{bmatrix} 0 & P^{\mathrm{T}}(x) \\ P(x) & 0 \end{bmatrix}.$$

Then $U(x)$, $V(x)$ are isotropic solutions of the $2n$-dimensional elf-adjoint system

$$(\mathscr{E}\,Y')' + \mathscr{P}(x)\,Y = 0 \tag{3.8}$$

for which $U(a) = 0$, $\mathscr{E}\,U'(a) = E$, $V(a) = E$, $\mathscr{E}\,V'(a) = 0$. Thus we have

$$U^{\mathrm{T}\prime}(x)\,\mathscr{E}\,V(x) - U^{\mathrm{T}}(x)\,\mathscr{E}\,V'(x) = E\,. \tag{3.9}$$

This, together with the fact that $U(x)$, $V(x)$ are isotropic, implies

$$\begin{bmatrix} U^{\mathrm{T}}(x) & U^{\mathrm{T}\prime}(x) \\ V^{\mathrm{T}}(x) & V^{\mathrm{T}\prime}(x) \end{bmatrix} \begin{bmatrix} \mathscr{E} & 0 \\ 0 & \mathscr{E} \end{bmatrix} \begin{bmatrix} -V'(x) & U'(x) \\ V(x) & -U(x) \end{bmatrix} = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}.$$

hence

$$\begin{bmatrix} -V'(x) & U'(x) \\ V(x) & -U(x) \end{bmatrix} \begin{bmatrix} U^{\mathrm{T}}(x) & U^{\mathrm{T}\prime}(x) \\ V^{\mathrm{T}}(x) & V^{\mathrm{T}\prime}(x) \end{bmatrix} = \begin{bmatrix} \mathscr{E} & 0 \\ 0 & \mathscr{E} \end{bmatrix},$$

and thus

$$\begin{aligned} U'V^{\mathrm{T}} - V'U^{\mathrm{T}} &= \mathscr{E} & \qquad VU^{\mathrm{T}} - UV^{\mathrm{T}} &= 0 \\ U'V^{\mathrm{T}\prime} - V'U^{\mathrm{T}\prime} &= 0 & \qquad VU^{\mathrm{T}\prime} - UV^{\mathrm{T}\prime} &= \mathscr{E} \end{aligned} \tag{3.10}$$

which by virtue of (3.7) yields

$$\begin{aligned} U_1^{\mathrm{T}\prime}V_2 - U_1^{\mathrm{T}}V_2' &= E & \qquad V_2U_1^{\mathrm{T}\prime} - U_2V_1^{\mathrm{T}\prime} &= E \\ U_1^{\mathrm{T}\prime}U_2 - U_1^{\mathrm{T}}U_2' &= 0 & \qquad V_2U_1^{\mathrm{T}} - U_2V_1^{\mathrm{T}} &= 0 \\ V_1^{\mathrm{T}\prime}V_2 - V_1^{\mathrm{T}}V_2' &= 0 & \qquad V_2'U_1^{\mathrm{T}\prime} - U_2'V_1^{\mathrm{T}\prime} &= 0 \\ U_2^{\mathrm{T}\prime}V_1 - U_2^{\mathrm{T}}V_1' &= E & \qquad V_1U_2^{\mathrm{T}\prime} - U_1V_2^{\mathrm{T}\prime} &= E\,. \end{aligned}$$

Let $H(x) = U(x)\,U^{\mathrm{T}}(x) + V(x)\,V^{\mathrm{T}}(x)$. The matrix $H(x)$ is nonsingular on I. In fact, $H(x) = (U(x) + \mathrm{i}V(x)\,(U^{\mathrm{T}}(x) - \mathrm{i}V^{\mathrm{T}}(x)) = (U(x) + \mathrm{i}V(x))\,(U(x) + \mathrm{i}V(x))^*$. Therefore $H(x)$ is nonsingular if and only if the complex matrix $U(x) + \mathrm{i}V(x)$ is nonsingular. Let $c = c_1 + \mathrm{i}c_2$ be a constant complex $2n$-dimensional vector for which $(U + \mathrm{i}V)(c_1 + \mathrm{i}c_2) = 0$. Then $Uc_1 - Vc_2 = 0$ and $Uc_2 + Vc_1 = 0$. As $U(x)$, $V(x)$ are independent solutions of (3.8), the last equality yields $c_1 = 0 = c_2$, hence $U(x) + \mathrm{i}V(x)$ is nonsingular. Since the matrix $H(x)$ is of the form

$$H(x) = \begin{bmatrix} U_1(x)\,U_1^{\mathrm{T}}(x) + V_1(x)\,V_1^{\mathrm{T}}(x) & 0 \\ 0 & U_2(x)\,U_2^{\mathrm{T}}(x) + V_2(x)\,V_2^{\mathrm{T}}(x) \end{bmatrix},$$

the matrices $H_i(x) = U_i(x)\,U_i^{\mathrm{T}}(x) + V_i(x)\,V_i^{\mathrm{T}}(x)$, $i = 1, 2$, are also nonsingular. Let us denote $X = U_1'U_1^{\mathrm{T}} + V_1'V_1^{\mathrm{T}}$, $Y = U_2'U_2^{\mathrm{T}} + V_2'V_2^{\mathrm{T}}$, $W = U_2U_2^{\mathrm{T}\prime} + V_2V_2^{\mathrm{T}\prime}$, $Z = U_1U_1^{\mathrm{T}\prime} + V_1V_1^{\mathrm{T}\prime}$. Then $H_2X - WH_1 = (U_2U_2^{\mathrm{T}} + V_2V_2^{\mathrm{T}})(U_1'U_1^{\mathrm{T}} + V_1'V_1^{\mathrm{T}}) - (U_2U_2^{\mathrm{T}\prime} + V_2V_2^{\mathrm{T}\prime})(U_1U_1^{\mathrm{T}} + V_1V_1^{\mathrm{T}}) = U_2U_2^{\mathrm{T}}U_1'U_1^{\mathrm{T}} + U_2U_2^{\mathrm{T}}V_1'V_1^{\mathrm{T}} + V_2V_2^{\mathrm{T}}U_1'U_1^{\mathrm{T}} + V_2V_2^{\mathrm{T}}V_1'V_1^{\mathrm{T}} - U_2U_2^{\mathrm{T}\prime}U_1U_1^{\mathrm{T}} - U_2U_2^{\mathrm{T}\prime}V_1V_1^{\mathrm{T}} - V_2V_2^{\mathrm{T}\prime}U_1U_1^{\mathrm{T}} - V_2V_2^{\mathrm{T}\prime}V_1V_1^{\mathrm{T}} = U_2(U_2^{\mathrm{T}}V_1' - U_2^{\mathrm{T}\prime}V_1)\,V_1^{\mathrm{T}} + V_2(V_2^{\mathrm{T}}U_1' - V_2^{\mathrm{T}\prime}U_1)\,U_1^{\mathrm{T}} = -U_2V_1^{\mathrm{T}} + V_2U_1^{\mathrm{T}} = 0$. Similarly $H_1Y - ZH_2 = 0$. Therefore $\{X, Y, W, Z\}$ is a solution of the matrix system

$$H_2X - WH_1 = 0\,, \qquad X + Z = H_1'\,,$$
$$H_1Y - ZH_2 = 0\,, \qquad Y + W = H_2'\,.$$

The elimination of Y and Z gives

$$\begin{aligned} H_1H_2X - XH_2H_1 &= H_1H_2'H_1 - H_1'H_2H_1\,, \\ H_2H_1Y - YH_1H_2 &= H_2H_1'H_2 - H_2'H_1H_2\,. \end{aligned} \tag{3.11}$$

By multiplying the second equation of (3.11) from the left by H_1 and from the right by H_2^{-1}, we obtain

$$H_1H_2(H_1YH_2^{-1}) - (H_1YH_2^{-1})\,H_2H_1 = H_1H_2H_1' - H_1H_2'H_1\,. \tag{3.12}$$

By subtracting this equation from the first equation of (3.11) we see that the matrix $G(x) = \frac{1}{2}(X(x) - H_1(x)\,Y(x)\,H_2^{-1}(x))$ is the solution of

$$H_1H_2G - GH_2H_1 = H_1H_2'H_1 - \tfrac{1}{2}(H_1'H_2H_1 + H_1H_2H_1')\,. \tag{3.13}$$

Further, we have $G = \frac{1}{2}(X - H_1YH_2^{-1}) = \frac{1}{2}(X - ZH_2H_2^{-1}) = \frac{1}{2}(U_1'U_1^{\mathrm{T}} + V_1'V_1^{\mathrm{T}} - U_1U_1^{\mathrm{T}\prime} - V_1V_1^{\mathrm{T}\prime})$, hence the matrix $G(x)$ is antisymmetric.

Now, let $T(x)$ be the solution of the matrix differential system

$$T' = \tfrac{1}{2}(D_1'(x)\,D_1^{-1}(x) - D_1^{-1}(x)\,D_1'(x))\,T\,, \quad T(a) = E\,, \tag{3.14}$$

where $D_1(x)$ is the symmetric positive definite matrix for which $D_1^2(x) = H_1(x)$. We set

$$B_1(x) = D_1(x)\,T(x)\,.$$

Since the matrix $D_1' D_1^{-1} - D_1^{-1} D_1'$ is antisymmetric, the matrix $T(x)$ is orthonormal and by a direct calculation we can verify that

$$(3.15) \qquad B_1(x)\, B_1^{\mathrm{T}}(x) = H_1(x)\,,$$
$$B_1(x)\, B_1^{\mathrm{T}\prime}(x) - B_1'(x)\, B_1^{\mathrm{T}}(x) = 0\,.$$

Using (3.15) we can rewrite the right hand side of (3.13) in this way: $H_1 H_2' H_1 - \frac{1}{2}(H_1' H_2 H_1 + H_1 H_2 H_1') = H_1 H_2' H_1 - \frac{1}{2}(B_1 B_1^{\mathrm{T}\prime} + B_1' B_1^{\mathrm{T}})\, H_2 H_1 - \frac{1}{2} H_1 H_2 (B_1 B_1^{\mathrm{T}\prime} + B_1' B_1^{\mathrm{T}}) = H_1 H_2' H_1 - B_1 B_1^{\mathrm{T}\prime} H_2 H_1 - H_1 H_2 B' B_1^{\mathrm{T}}$. Thus (3.13) is of the form

$$(3.16) \qquad H_1 H_2 G - G H_2 H_1 = H_1 H_2' H_1 - B_1 B_1^{\mathrm{T}\prime} H_2 H_1 - H_1 H_2 B_1' B_1^{\mathrm{T}}\,.$$

Let $G_1 = B_1^{-1} G B_1^{\mathrm{T}-1}$. Then G_1 is obviously antisymmetric (since G is antisymmetric) and it is the solution of

$$B_1^{\mathrm{T}} H_2 B_1 G_1 - G_1 B_1^{\mathrm{T}} H_2 B_1 = B_1^{\mathrm{T}} H_2' B_1 - B_1^{\mathrm{T}\prime} H_2 B_1 - B_1^{\mathrm{T}} H_2 B_1'\,.$$

Further, let $B_2(x)$ be an arbitrary matrix for which $B_2(x)\, B_2^{\mathrm{T}}(x) = H_2(x)$. We set $G_2 = (B_2^{\mathrm{T}} B_1' - B_2^{\mathrm{T}\prime} B_1)(B_2^{\mathrm{T}} B_1)^{-1} + B_2^{\mathrm{T}} B_1 G_1 (B_2^{\mathrm{T}} B_1)^{-1}$. Then $G_2 + G_2^{\mathrm{T}} = (B_2^{\mathrm{T}} B_1' - B_2^{\mathrm{T}\prime} B_1)(B_2^{\mathrm{T}} B_1)^{-1} + B_2^{\mathrm{T}} B_1 G_1 (B_2^{\mathrm{T}} B_1)^{-1} + (B_1^{\mathrm{T}} B_2)^{-1} (B_1^{\mathrm{T}\prime} B_2 - B_1^{\mathrm{T}} B_2') - (B_1^{\mathrm{T}} B_2)^{-1} G_1 B_1^{\mathrm{T}} B_2 = (B_1^{\mathrm{T}} B_2)^{-1} [B_1^{\mathrm{T}} B_2 (B_2^{\mathrm{T}} B_1' - B_2^{\mathrm{T}\prime} B_1) + B_1^{\mathrm{T}} B_2 B_2^{\mathrm{T}} B_1 G_1 + (B_1^{\mathrm{T}\prime} B_2 - B_1^{\mathrm{T}} B_2')(B_2^{\mathrm{T}} B_1) - G_1 B_1^{\mathrm{T}} B_2 B_2^{\mathrm{T}} B_1] (B_2^{\mathrm{T}} B_1)^{-1} = (B_1^{\mathrm{T}} B_2)^{-1} [B_1^{\mathrm{T}} H_2 B_1' - B_1^{\mathrm{T}} (B_2 B_2^{\mathrm{T}\prime} + B_2' B_2^{\mathrm{T}})\, B_1 + B_1^{\mathrm{T}\prime} H_2 B_1 + B_1^{\mathrm{T}} H_2 B_1 G_1 - G_1 B_1^{\mathrm{T}} H_2 B_1] (B_2^{\mathrm{T}} B_1)^{-1} = 0$, since the term in the square brackets is (3.16).

Now, let $T_1(x)$, $T_2(x)$ be the solutions of

$$T_1' = G_1(x)\, T_1\,, \quad T_1(a) = E\,,$$
$$T_2' = -G_2(x)\, T_2\,, \quad T_2(a) = E\,.$$

As the matrices $G_1(x)$, $G_2(x)$ are antisymmetric, the matrices $T_1(x)$, $T_2(x)$ are orthonormal and the matrices

$$R_1(x) = B_1(x)\, T_1(x)$$
$$R_2(x) = B_2(x)\, T_2(x)$$

fulfil $R_2^{\mathrm{T}\prime} R_1 - R_2^{\mathrm{T}} R_1' = (T_2^{\mathrm{T}\prime} B_2^{\mathrm{T}} + T_2^{\mathrm{T}} B_2^{\mathrm{T}\prime})\, B_1 T_1 - T_2^{\mathrm{T}} B_2^{\mathrm{T}} (B_1' T_1 + B_1 T_1') = T_2^{\mathrm{T}\prime} B_2^{\mathrm{T}} B_1 T_1 + T_2^{\mathrm{T}} B_2^{\mathrm{T}\prime} B_1 T_1 - T_2^{\mathrm{T}} B_2^{\mathrm{T}} B_1' T_1 - T_2^{\mathrm{T}} B_2^{\mathrm{T}} B_1 T_1' = T_2^{\mathrm{T}} (G_2 B_2^{\mathrm{T}} B_1 + B_2^{\mathrm{T}\prime} B_1 - B_2^{\mathrm{T}} B_1' - B_2^{\mathrm{T}} B_1 G_1)\, T_1 = T_2^{\mathrm{T}} [(B_2^{\mathrm{T}} B_1' - B_2^{\mathrm{T}\prime} B_1)(B_2^{\mathrm{T}} B_1)^{-1} B_2^{\mathrm{T}} B_1 + B_2^{\mathrm{T}} B_1 G_1 (B_2^{\mathrm{T}} B_1)^{-1} B_2^{\mathrm{T}} B_1 + B_2^{\mathrm{T}\prime} B_1 - B_2^{\mathrm{T}} B_1' - B_2^{\mathrm{T}} B_1 G_1]\, T_1 = T_2^{\mathrm{T}} (B_2^{\mathrm{T}} B_1' - B_2^{\mathrm{T}\prime} B_1 + B_2^{\mathrm{T}} B_1 G_1 + B_2^{\mathrm{T}\prime} B_1 - B_2^{\mathrm{T}} B_1' - B_2^{\mathrm{T}} B_1 G)\, T_1 = 0$.

Let $Q(x) = (R_2^{\mathrm{T}}(x)\, R_1(x))^{-1}$. To complete the proof, according to Theorem 1, it suffices to verify that $R_2^{\mathrm{T}}(x)\, R_1''(x) + R_2^{\mathrm{T}}(x)\, P(x)\, R_1(x) = Q^{\mathrm{T}}(x)$. Denote

$$(3.17) \qquad R(x) = \begin{bmatrix} R_1(x) & 0 \\ 0 & R_2(x) \end{bmatrix}, \quad Q_0(x) = \begin{bmatrix} 0 & Q(x) \\ Q^{\mathrm{T}}(x) & 0 \end{bmatrix}.$$

According to (3.5) and (3.17), $Q_0(x) = (R^{\mathrm{T}}(x)\,\mathscr{E}\,R(x))^{-1}$ and $R^{\mathrm{T}\prime}(x)\,\mathscr{E}\,R(x) - R^{\mathrm{T}}(x)\,\mathscr{E}\,R'(x) = 0$. By multiplying the last equality from the left by $R(x)$ and from the right by $R^{\mathrm{T}}(x)$ we obtain $RR^{\mathrm{T}\prime}\mathscr{E}RR^{\mathrm{T}} - RR^{\mathrm{T}}\mathscr{E}R'R^{\mathrm{T}} = 0$. Further, using (3.9) and (3.10), we obtain $RR^{\mathrm{T}}\mathscr{E}(U'U^{\mathrm{T}} + V'V^{\mathrm{T}}) = (UU^{\mathrm{T}} + VV^{\mathrm{T}})\,\mathscr{E}(U'U^{\mathrm{T}} + V'V^{\mathrm{T}}) = UU^{\mathrm{T}}\mathscr{E}U'U^{\mathrm{T}} + UU^{\mathrm{T}}\mathscr{E}V'V^{\mathrm{T}} + VV^{\mathrm{T}}\mathscr{E}U'U^{\mathrm{T}} + VV^{\mathrm{T}}\mathscr{E}V'V^{\mathrm{T}} = UU^{\mathrm{T}\prime}\mathscr{E}UU^{\mathrm{T}} + U(-E + U^{\mathrm{T}\prime}\mathscr{E}V)\,V^{\mathrm{T}} + V(E + V^{\mathrm{T}\prime}\mathscr{E}U)\,U^{\mathrm{T}} + VV^{\mathrm{T}\prime}\mathscr{E}VV^{\mathrm{T}} = (UU^{\mathrm{T}\prime} + VV^{\mathrm{T}\prime})\,\mathscr{E}(UU^{\mathrm{T}} + VV^{\mathrm{T}}) = (UU^{\mathrm{T}\prime} + VV^{\mathrm{T}\prime})\,\mathscr{E}RR^{\mathrm{T}}$. Let us denote $X_1 = \mathscr{E}R'R^{\mathrm{T}}$, $Y_1 = RR^{\mathrm{T}\prime}\mathscr{E}$, $X_2 = \mathscr{E}(U'U^{\mathrm{T}} + V'V^{\mathrm{T}})$, $Y_2 = (UU^{\mathrm{T}\prime} + VV^{\mathrm{T}\prime})\,\mathscr{E}$. Then

$$RR^{\mathrm{T}}X_i - Y_iRR^{\mathrm{T}} = 0\,,$$

$$X_i + Y_i = (RR^{\mathrm{T}})'\,,\quad i = 1, 2\,,$$

hence

$$\begin{aligned} RR^{\mathrm{T}}X_i + X_iRR^{\mathrm{T}} &= (RR^{\mathrm{T}})'\,RR^{\mathrm{T}}\,, \\ RR^{\mathrm{T}}Y_i + Y_iRR^{\mathrm{T}} &= RR^{\mathrm{T}}(RR^{\mathrm{T}})'\,,\quad i = 1, 2\,. \end{aligned} \tag{3.18}$$

Since the matrix RR^{T} is positive definite, it is known that the both matrix equations (3.18) have unique solutions, hence $X_1 = X_2$, $Y_1 = Y_2$, i.e. $R'R^{\mathrm{T}} = U'U^{\mathrm{T}} + V'V^{\mathrm{T}}$ and $RR^{\mathrm{T}\prime} = UU^{\mathrm{T}\prime} + VV^{\mathrm{T}\prime}$. Further, using (3.10) we obtain $R^{\mathrm{T}}(\mathscr{E}R')' + R^{\mathrm{T}}\mathscr{P}R = R^{-1}(RR^{\mathrm{T}}\mathscr{E}R''R^{\mathrm{T}} + RR^{\mathrm{T}}\mathscr{P}RR^{\mathrm{T}})\,R^{\mathrm{T}-1} = R^{-1}[RR^{\mathrm{T}}\mathscr{E}(R'R^{\mathrm{T}})' - RR^{\mathrm{T}}\mathscr{E}R'R^{\mathrm{T}\prime} + RR^{\mathrm{T}}\mathscr{P}RR^{\mathrm{T}}]\,R^{\mathrm{T}-1} = R^{-1}[RR^{\mathrm{T}}\mathscr{E}(U'U^{\mathrm{T}} + V'V^{\mathrm{T}})' - RR^{\mathrm{T}}\mathscr{E}R'R^{\mathrm{T}\prime} + RR^{\mathrm{T}}\mathscr{P}RR^{\mathrm{T}}]\,R^{\mathrm{T}-1} = R^{-1}[RR^{\mathrm{T}}\mathscr{E}(U''U^{\mathrm{T}} + V''V^{\mathrm{T}}) + (UU^{\mathrm{T}} + VV^{\mathrm{T}})\,\mathscr{E}(U'U^{\mathrm{T}\prime} + V'V^{\mathrm{T}\prime}) - RR^{\mathrm{T}}\mathscr{E}R'R^{\mathrm{T}\prime} + RR^{\mathrm{T}}\mathscr{P}RR^{\mathrm{T}}]\,R^{\mathrm{T}-1} = R^{-1}[-RR^{\mathrm{T}}\mathscr{E}\mathscr{E}\mathscr{P}(UU^{\mathrm{T}} + VV^{\mathrm{T}}) + UU^{\mathrm{T}}\mathscr{E}U'U^{\mathrm{T}\prime} + VV^{\mathrm{T}}\mathscr{E}U'U^{\mathrm{T}\prime} + UU^{\mathrm{T}}\mathscr{E}V'V^{\mathrm{T}\prime} + VV^{\mathrm{T}}\mathscr{E}V'V^{\mathrm{T}\prime} - RR^{\mathrm{T}\prime}\mathscr{E}RR^{\mathrm{T}\prime} + RR^{\mathrm{T}}\mathscr{P}RR^{\mathrm{T}}]\,R^{\mathrm{T}-1} = R^{-1}[-RR^{\mathrm{T}}\mathscr{P}RR^{\mathrm{T}} + UU^{\mathrm{T}\prime}\mathscr{E}UU^{\mathrm{T}\prime} + V(E + V^{\mathrm{T}\prime}\mathscr{E}U)\,U^{\mathrm{T}\prime} + U(-E + U^{\mathrm{T}\prime}\mathscr{E}V)\,V^{\mathrm{T}\prime} + VV^{\mathrm{T}}\mathscr{E}VV^{\mathrm{T}\prime} - RR^{\mathrm{T}\prime}\mathscr{E}RR^{\mathrm{T}\prime} + RR^{\mathrm{T}}\mathscr{P}RR^{\mathrm{T}}]\,R^{\mathrm{T}-1} = R^{-1}[(UU^{\mathrm{T}\prime} + VV^{\mathrm{T}\prime})\,\mathscr{E}(UU^{\mathrm{T}\prime} + VV^{\mathrm{T}\prime}) + VU^{\mathrm{T}\prime} - UV^{\mathrm{T}\prime} - RR^{\mathrm{T}\prime}\mathscr{E}RR^{\mathrm{T}\prime}]\,R^{\mathrm{T}-1} = R^{-1}(RR^{\mathrm{T}\prime}\mathscr{E}RR^{\mathrm{T}\prime} + \mathscr{E} - RR^{\mathrm{T}\prime}\mathscr{E}RR^{\mathrm{T}})\,R^{\mathrm{T}-1} = (R^{\mathrm{T}}\mathscr{E}R)^{-1} = Q_0$. Thus $R^{\mathrm{T}}\mathscr{E}R'' = + R^{\mathrm{T}}\mathscr{P}R = Q_0$ which in virtue of (3.17) gives

$$\begin{bmatrix} R_1^{\mathrm{T}} & 0 \\ 0 & R_2^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} 0 & E \\ E & 0 \end{bmatrix}\begin{bmatrix} R_1'' & 0 \\ 0 & R_2'' \end{bmatrix} + \begin{bmatrix} R_1^{\mathrm{T}} & 0 \\ 0 & R_2^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} 0 & P^{\mathrm{T}} \\ P & 0 \end{bmatrix}\begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & R_1^{\mathrm{T}}R_2'' \\ R_2^{\mathrm{T}}R_1'' & 0 \end{bmatrix} + \begin{bmatrix} 0 & R_1^{\mathrm{T}}P^{\mathrm{T}}R_2 \\ R_2^{\mathrm{T}}PR_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & Q \\ Q^{\mathrm{T}} & 0 \end{bmatrix},$$

hence $R_2^{\mathrm{T}}R_1'' + R_2^{\mathrm{T}}PR_1 = Q^{\mathrm{T}}$, which was to prove.

Remark 3. We also proved that the transformation $\{R_1(x), R_2(x)\}$ transforms (3.6) into the system

$$(Q^{\mathrm{T}-1}(x)\,S')' + Q(x)\,S = 0\,. \tag{3.19}$$

In this section we shall study relations between the conjugate points relative to (3.3) and the conjugate points relative to its adjoint system (3.6). Recall that a point $x_2 > x_1$ ($x_2 < x_1$) is called *the right (left) conjugate point of* x_1 *relative to* (3.3) if there exists a solution $Y(x)$ of (3.3) for which $Y(x_1) = 0$, $Y'(x_1) = E$ and $Y(x_2)\,c = 0$ for some n-dimensional constant vector $c \neq 0$. Further, we say that x_2 is *a conjugate point of multiplicity* k, $1 \leqq k \leqq n$, if there exist k linearly independent vectors $c_1, \ldots, c_k$ for which $Y(x_2)\,c_i = 0$, $i = 1, \ldots, k$.

The following theorem extends the result of Ahmed and Lazer [1].

Theorem 5. *Let* $a \in I$ *and let* $a < r_1 \leqq r_2 \leqq \ldots$, $a < \bar{r}_1 \leqq \bar{r}_2 \leqq \ldots$, ($a > l_1 \geqq l_2 \geqq \ldots$, $a > \bar{l}_1 \geqq \bar{l}_2 \geqq \ldots$) *be the sequences of its right (left) conjugate points relative to* (3.3) *and* (3.6) *respectively, every point repeated the number of times equal to its multiplicity. Then* $\bar{r}_i = r_i$ ($\bar{l}_i = l_i$).

Proof. We shall need the following auxiliary statement. Its proof can be found e.g. in [6].

Lemma 1. *Let* $S_1(x)$, $C_1(x)$ *and* $S_2(x)$, $C_2(x)$ *be solutions of* (3.4) *and* (3.19), *respectively, for which* $S_1(a) = 0$, $Q^{-1}(a)\,S_1'(a) = E$, $S_2(a) = 0$, $Q^{\mathrm{T}-1}(a)\,S_2'(a) = E$, $C_1(a) = E$, $Q^{-1}(a)\,C_1'(a) = 0$, $C_2(a) = E$, $Q^{\mathrm{T}-1}(a)\,C_2'(a) = 0$. *Then the following identities hold*:

$$(4.1)\qquad S_2^{\mathrm{T}}S_2 + C_1^{\mathrm{T}}C_1 = E\,,\quad S_1S_1^{\mathrm{T}} + C_1C_1^{\mathrm{T}} = E\,,$$
$$S_1^{\mathrm{T}}S_1 + C_2^{\mathrm{T}}C_2 = E\,,\quad S_2S_2^{\mathrm{T}} + C_2C_2^{\mathrm{T}} = E\,.$$

Now, let $U_1(x)$, $V_1(x)$ and $U_2(x)$, $V_2(x)$ be solutions of (3.3) and (3.6), respectively, for which $U_i(a) = 0$, $U_i'(a) = E$, $V_i(a) = E$, $V_i'(a) = 0$, $i = 1, 2$. Then by Theorem 4 and Remark 3 there exist nonsingular matrices $R_1(x)$, $R_2(x)$ such that

$$(4.2)\qquad S_i(x) = R_i^{-1}(x)\,U_i(x)\,,\quad C_i(x) = R_i^{-1}(x)\,V_i(x)\,,\quad i = 1, 2$$

are solutions of (3.4) and (3.19) for which $S_i(a) = 0$, $C_i'(a) = 0$, $C_i(a) = E$, $i = 1, 2$, and $Q^{-1}(a)\,S_1'(a) = E$, $Q^{\mathrm{T}-1}(a)\,S_2'(a) = E$. Let x_0 be a k-multiple (left or right) conjugate point of a relative to (3.3). Then by (4.2) there exist k linearly independent unit vectors $c_1, \ldots, c_k$ such that $S_1(x_0)\,c_i = 0$, $i = 1, \ldots, k$. According to (4.1), $c_i^{\mathrm{T}}C_2^{\mathrm{T}}(x_0)\,C_2(x_0)\,c_i = 1$, and it follows that $d_i^{\mathrm{T}}C_2(x_0)\,C_2^{\mathrm{T}}(x_0)\,d_i = 1$ for some independent unit vectors $d_1, \ldots, d_k$. Hence $d_i^{\mathrm{T}}S_2(x_0)\,S_2^{\mathrm{T}}(x_0)\,d_i = 0$, and it follows that $S_2^{\mathrm{T}}(x_0)\,d_i = 0$, $i = 1, \ldots, k$, and thus $S_2(x_0)\,e_i = 0$ for some independent unit vectors $e_1, \ldots, e_k$. Therefore x_0 is a k-multiple conjugate point of a relative to (3.6). In the same way we prove the converse, i.e. that every k-multiple conjugate point of a relative to (3.6) is also a k-multiple conjugate point of a relative to (3.3). The proof is complete.

Remark 4. Let $a < r_1 \leqq r_2 \leqq \ldots$, $a < \bar{r}_1 \leqq \bar{r}_2 \leqq \ldots$, ($a > l_1 \geqq l_2 \geqq \ldots$,

$a > \bar{l}_1 \geqq \bar{l}_2 \geqq \ldots,$) be the sequences of right (left) conjugate points of a relative to (1.1) and $(F^T(x)\, Y')' + G^T(x)\, Y = 0$, respectively, every point repeated the number of times equal to its multiplicity. Then, according to the preceding theorem, Theorem 2 and Remark 1, $r_i = \bar{r}_i$ ($l_i = \bar{l}_i$).

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