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FORM OF GENERAL POINTWISE TRANSFORMATIONS OF LINEAR DIFFERENTIAL EQUATIONS

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1. In [6] Stäckel found the form of the most general pointwise transformation

$$(1) \qquad t = T_1(x, y), \quad z = T_2(x, y)$$

converting any linear homogeneous differential equation of the n-th order

$$(2) \qquad y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_0(x)\, y = 0, \quad n \geqq 2$$

into an equation of the same kind in variables z and t (see also [7]). He considered only diffeomorphisms of the class C^n. The aim of this paper is to give the proof of the same result without any assumptions of differentiability of T (see Theorem).

Let I and J be open intervals. The equation of the form (2) on the interval I will be denoted by $Q(\mathbf{p}, I)$ where $\mathbf{p} = (p_0, p_1, \ldots, p_{n-1})$. We shall take into account only the equations with continuous coefficients. If $y \in C^n(I, \mathbf{R})$ we define the vector function $\bar{y} = (y, y', y'', \ldots, y^{(n-1)})$. For every vector function $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in C^n(I, \mathbf{R}^n)$ the symbol $W[\mathbf{y}](x)$ will denote the Wronski determinant of $\mathbf{y}$ at $x \in I$.

Consider the transformation (1) satisfying the following conditions:

(**A**) T is a homeomorphism of $I \times \mathbf{R}$ onto $J \times \mathbf{R}$.

(**B**) For every equation $Q(\mathbf{p}, I)$ there is an equation $Q(\mathbf{q}, J)$ such that

(i) if $y(x)$ is a solution of the equation $Q(\mathbf{p}, I)$ and if

$$t = T_1(x, y(x)), \quad z = T_2(x, y(x))$$

then z is a function of t of the class $C^n(J, \mathbf{R})$,

(ii) the functions $z_1(t), z_2(t), \ldots, z_n(t)$ obtained by the transformation T from an arbitrary fundamental system $y_1(x), y_2(x), \ldots, y_n(x)$ of solutions of $Q(\mathbf{p}, I)$ form a fundamental system of $Q(\mathbf{q}, J)$.

Theorem. *Under the assumptions* (**A**), (**B**) *and* $n \geqq 2$ *every transformation* T *has the form*

$$(3) \qquad t = g(x), \quad z = k(t)\, y,$$

where $k \in C^n(J, \mathbf{R})$, $k(t) \neq 0$ *for every* $t \in J$ *and* g *is a* C^n*-diffeomorphism of* I *onto* J.

Remark 1. It is known (see [5]) that every transformation of the form (3) has the properties (**A**) and (**B**). Moreover, for $\mathbf{y} \in C^n(I, \mathbf{R}^n)$ and $\mathbf{z}(t) = k(t)\,\mathbf{y}(h(t))$, $k \in C^n(J, \mathbf{R})$, $h \in C^n(J, I)$ the formula

$$W[\mathbf{z}](t) = (k(t))^n\, W[\mathbf{y}](h(t))\,(h'(t))^{n(n-1)/2}$$

is fulfilled.

Remark 2. The transformation (1) which converts every C^n-function into some C^n-function is not necessarily C^n-differentiable. According to Theorem 10 in [2] an appropriate counterexample can be constructed for every $n \geqq 1$.

2. The proof of the theorem is based on two lemmas given in this section.

Lemma 1. *Let $y \in C^n(I, \mathbf{R})$, $\bar{y}(x) \neq \mathbf{0}$ for every $x \in I$. There is a continuous function $\mathbf{p} = (p_0, p_1, \ldots, p_{n-1})$ on I such that y is a solution of the equation $Q(\mathbf{p}, I)$.*

Proof. We can choose a sequence $\{U_m\}_{m=-\infty}^{\infty}$ of open intervals having the following properties: $U_m \cap U_{m+1} \neq \emptyset$, $U_m \cap U_{m+j} = \emptyset$ for $j \geqq 2$, $I = \bigcup_{m=-\infty}^{\infty} U_m$ and for every m there is $i_m \in \{0, 1, \ldots, n-1\}$ such that $y^{(i_m)}(x) \neq 0$ on the closure $\mathrm{cl}(U_m)$ of the interval U_m. For $i \in \{0, 1, \ldots, n-1\} \setminus \{i_1\}$ let p_i be an arbitrary continuous function on $\mathrm{cl}(U_1)$. Define

$$p_{i_1}(x) := -\frac{1}{y^{(i_1)}(x)}\left[y^{(n)}(x) + \sum_{\substack{i=0\\ i \neq i_1}}^{n-1} p_i(x)\, y^{(i)}(x)\right].$$

Then the function y is a solution of the equation $Q(\mathbf{p}, U_1)$. For $i \in \{0, 1, \ldots, n-1\} \setminus \{i_2\}$ extend the functions p_i continuously to the interval $\mathrm{cl}(U_1 \cup U_2)$. On $\mathrm{cl}(U_2)$ put

$$q_{i_2}(x) := -\frac{1}{y^{(i_2)}(x)}\left[y^{(n)}(x) + \sum_{\substack{i=0\\ i \neq i_2}}^{n-1} p_i(x)\, y^{(i)}(x)\right].$$

Since $y^{(i_2)}(x) \neq 0$ on $\mathrm{cl}(U_1 \cap U_2)$ we get $p_{i_2} = q_{i_2}$ on this interval. Putting $p_{i_2} := q_{i_2}$ on $\mathrm{cl}(U_2)$ we extend the function p_{i_2} to the interval $\mathrm{cl}(U_1 \cup U_2)$. Hence y is a solution of the equation $Q(\mathbf{p}, U_1 \cup U_2)$. By induction for $m = 1, 2, \ldots$ and for $m = 1, 0, -1, \ldots$ we can construct a continuous function $\mathbf{p}$ on the whole interval I.

Lemma 2. *Let $a < b$, $n \geqq 1$ and let $(u_0, u_1, \ldots, u_n)$, $(v_0, v_1, \ldots, v_n)$ be arbitrary vectors in $\mathbf{R}^{n+1}$ such that $(u_0, u_1, \ldots, u_{n-1})$ and $(v_0, v_1, \ldots, v_{n-1})$ are nonzero in $\mathbf{R}^n$ and $u_0 v_0 > 0$ if $n = 1$. Then there is a function $y \in C^n([a, b])$ satisfying $\bar{y}(x) \neq \mathbf{0}$ for every $x \in (a, b)$ and*

$$y_+^{(i)}(a) = u_i\,, \quad y_-^{(i)}(b) = v_i$$

for $i = 0, 1, \ldots, n$.

Proof. We shall distinguish the following cases.

(i) Let $u_0 > 0$, $v_0 > 0$. According to Borel's theorem for fixed $x_0 \in \mathbf{R}$ there is an infinitely smooth function (i.e. from $C^\infty(\mathbf{R}, \mathbf{R})$) having arbitrarily prescribed value and derivatives at x_0. Hence there are such infinitely smooth functions φ_1, φ_2 that

$$\varphi_1(a) = u_0 - \delta\,, \quad \varphi_1^{(i)}(a) = u_i\,,$$

$$\varphi_2(b) = v_0 - \delta\,, \quad \varphi_2^{(i)}(b) = v_i$$

for $0 < \delta < \min(u_0, v_0)$ and $i = 1, 2, \ldots, n$. Choose $\varepsilon \in (0, \frac{1}{2}(b - a))$ such that $\varphi_1 > 0$, $\varphi_2 > 0$ on the intervals $(a - \varepsilon, a + \varepsilon)$, $(b - \varepsilon, b + \varepsilon)$, respectively. There exist nonnegative smooth functions α_1, α_2 defined on $\mathbf{R}$ such that $\alpha_1 = 1$, $\alpha_2 = 1$ on $(a - \frac{1}{2}\varepsilon, a + \frac{1}{2}\varepsilon)$, $(b - \frac{1}{2}\varepsilon, b + \frac{1}{2}\varepsilon)$, respectively and $\alpha_1 = 0$, $\alpha_2 = 0$ outside of $(a - \varepsilon, a + \varepsilon)$, $(b - \varepsilon, b + \varepsilon)$, respectively. The function

$$y(x) := \alpha_1(x)\,\varphi_1(x) + \alpha_2(x)\,\varphi_2(x) + \delta$$

has the required properties, in particular $y(x) > 0$ and hence $\bar{y}(x) \neq \mathbf{0}$ for every $x \in (a, b)$.

(ii) The case $u_0 < 0$, $v_0 < 0$ is converted into (i) if $-u$, $-v$ are considered instead of u, v.

(iii) Let $u_0 < 0$, $v_0 > 0$. Put $\varepsilon = \frac{1}{4}(b - a)$ and define

$$y(x) := x - a - 2\varepsilon$$

on the interval $(a + \varepsilon, b - \varepsilon)$. According to (i) we can find a function y on $[b - \varepsilon, b]$ such that $y(x) > 0$ on this interval and

$$y(b - \varepsilon) = \varepsilon\,, \quad y'_+(b - \varepsilon) = 1\,,$$

$$y_+^{(i)}(b - \varepsilon) = 0\,, \quad i = 2, 3, \ldots, n\,,$$

$$y_-^{(i)}(b) = v_i\,, \quad i = 0, 1, \ldots, n\,.$$

Similar construction can be carried out on the interval $[a, a + \varepsilon]$. The function constructed in this way satisfies all requirements of the lemma.

(iv) The construction for the case $u_0 > 0$, $v_0 < 0$ follows from (iii).

(v) Let $u_0 = 0$ or $v_0 = 0$. There are smooth functions φ_1, φ_2 such that

$$\varphi_1^{(i)}(a) = u_i\,, \quad \varphi_2^{(i)}(b) = v_i$$

for $i = 0, 1, \ldots, n$. Since $(u_0, u_1, \ldots, u_{n-1}) \neq \mathbf{0}$ and $(v_0, v_1, \ldots, v_{n-1}) \neq \mathbf{0}$ there is $\varepsilon \in (0, b - a)$ such that $\varphi_1 \neq 0$ on $(a, a + \varepsilon]$ and $\varphi_2 \neq 0$ on $[b - \varepsilon, b)$. Otherwise, all the derivatives of φ_1 at a and φ_2 at b would be zeros, which is excluded. Now it is sufficient to construct the required function y on the interval $[a + \varepsilon, b - \varepsilon]$ under the conditions $y(a + \varepsilon) \neq 0$ and $y(b - \varepsilon) \neq 0$. This completes the proof.

3. The proof of Theorem will be given in several steps. We shall always suppose $n \geqq 2$.

(i) Denote the inverse transformation of T by P. Then

$$x = P_1(t, z)\,, \quad y = P_2(t, z)\,.$$

We shall show that P_1 is independent of the second variable. On the contrary, suppose there are $t_0 \in J$, $\zeta_1 \in \mathbf{R}$. $\zeta_2 \in \mathbf{R}$ such that

$$x_1 = P_1(t_0, \zeta_1) \neq P_1(t_0, \zeta_2) = x_2$$

and put

$$\eta_i = P_2(t_0, \zeta_i)\,, \quad i = 1, 2\,.$$

Due to Lemma 2 one can show that there is a function $y \in C^n(I, \mathbf{R})$, $\bar{y} \neq 0$ on I and $y(x_i) = \eta_i$ for $i = 1, 2$. By assumption ($\mathbf{B}$) the transformation T converts the function $y(x)$ into a function $z(t)$ on J. But this is a contradiction with

$$T(x_1, \eta_1) = (t_0, \zeta_1)\,, \quad T(x_2, \eta_2) = (t_0, \zeta_2)\,.$$

Thus

$$x = P_1(t)\,.$$

For every fixed $t \in J$ the mapping P maps the line $\{t\} \times \mathbf{R}$ on to the set $\{P_1(t)\} \times K_t$, where K_t is an open interval. Since P is a homeomorphism between $J \times \mathbf{R}$ and $I \times \mathbf{R}$, we get $K_t = \mathbf{R}$. Hence P_1 is a homeomorphism between J and I.

Put $h := P_1$, $f(t, y) := T_2(h(t), y)$ and instead of $z = T_2(x, y)$ write

$$z(t) = f(t, y(h(t)))\,. \tag{4}$$

(ii) **Lemma 3.** *Let r, s be fixed real numbers. If for some $y \in C^n(I, \mathbf{R})$, $\bar{y}(x) \neq \mathbf{0}$ for $x \in I$,*

$$f(t, ry(h(t))) = sf(t, y(h(t)))\,, \quad t \in J \tag{5}$$

then (5) is fulfilled for every $y \in C^n(I, \mathbf{R})$, $\bar{y}(x) \neq \mathbf{0}$ for $x \in I$.

Proof. Let $y_1 \in C^n(I, \mathbf{R})$, $\bar{y}_1(x) \neq \mathbf{0}$ on I, satisfy (5) and let $y_2 \in C^n(I, \mathbf{R})$, $\bar{y}_2(x) \neq \mathbf{0}$ on I, be arbitrary. Choose $a, b \in I$, $a < b$. Then $I = I_1 \cup [a, b] \cup I_2$, where I_1, I_2 are open intervals. Lemma 2 implies the existence of a function $y \in C^n(I, R)$, $\bar{y}(x) \neq \mathbf{0}$ on I such that $y = y_i$ on I_i, $i = 1, 2$. According to Lemma 1 the functions y and ry are solutions of an equation $Q(\mathbf{p}, I)$. This equation is transformed into an equation $Q(\mathbf{q}, J)$ which has $z_1(t) := f(t, y(h(t)))$ and $z_2(t) := f(t, ry(h(t)))$ as its solutions. On the open interval $h^{-1}(I_1)$ we have

$$z_2(t) = f(t, ry_1(h(t))) = sf(t, y_1(h(t))) = s\, z_1(t)\,.$$

Since z_1, z_2 are solutions of the same equation, $z_2(t) = s\, z_1(t)$ on J and (5) is satisfied for y as well as for y_2 on the whole interval J.

(iii) Let $Q(\mathbf{p}, I)$ be an arbitrary equation. Consider a fixed fundamental system $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ of this equation. Put

$$z_i(t) := f(t, y_i(h(t)))\,.$$

Due to ($\mathbf{B}$), $\mathbf{z} = (z_1, z_2, \ldots, z_n)$ is a fundamental system of an equation $Q(\mathbf{q}, J)$ and for every vector $\mathbf{c} = (c_1, c_2, \ldots, c_n) \in \mathbf{R}^n$ there is just one vector $\mathbf{d} = (d_1, d_2, \ldots, d_n)$

such that

$$f(t, \sum_{i=1}^{n} c_i y_i(h(t))) = \sum_{i=1}^{n} d_i z_i(t) .$$

Put $G(\mathbf{c}) := \mathbf{d}$.

Lemma 4. *The mapping* $G: \mathbf{R}^n \to \mathbf{R}^n$ *has the following properties.*

(*a*) *For* $1 \leqq k \leqq n$ *it maps every k-tuple of linearly independent vectors on a k-tuple of linearly independent ones.*

(b) *G is a homeomorphism of* $\mathbf{R}^n$ *into* $\mathbf{R}^n$.

(*c*) *For* $1 \leqq k \leqq n$ *the inverse mapping* $G^{-1}: G(\mathbf{R}^n) \to \mathbf{R}^n$ *maps every k-tuple of linearly dependent vectors on a k-tuple of linearly dependent ones.*

(*d*) *Let* $r, s \in \mathbf{R}$ *be fixed. If for some* $\mathbf{c} \in \mathbf{R}^n \setminus \{\mathbf{0}\}$

$$G(r\mathbf{c}) = s\, G(\mathbf{c}) \tag{6}$$

then (6) *is satisfied for every* $\mathbf{c} \in \mathbf{R}^n$.

Proof. (a) follows from the fact that the transformation T converts every fundamental system of solutions of the equation $Q(\mathbf{p}, I)$ on a fundamental system of solutions of the equation $Q(\mathbf{q}, J)$.

(b) Let $\mathbf{c}^1, \mathbf{c}^2 \in \mathbf{R}^n$, $\mathbf{c}^1 \neq \mathbf{c}^2$. Define $\mathbf{d}^i := G(\mathbf{c}^i)$ for $i = 1, 2$. There is $x_0 \in I$ such that

$$\sum_{i=1}^{n} c_i^1\, y_i(x_0) \neq \sum_{i=1}^{n} c_i^2\, y_i(x_0) .$$

Since T is a homeomorphism, we have

$$\sum_{i=1}^{n} d_i^1 z_i(h^{-1}(x_0)) \neq \sum_{i=1}^{n} d_i^2 z_i(h^{-1}(x_0)) .$$

That is why $\mathbf{d}^1 \neq \mathbf{d}^2$ and G is injective.

We shall prove the continuity of G. Let $\mathbf{c}^k \in \mathbf{R}^n$ and $\lim_{k\to\infty} \mathbf{c}^k = \mathbf{c}$. Put $\mathbf{d}^k := G(\mathbf{c}^k)$ and $\mathbf{d} := G(\mathbf{c})$. Then

$$\lim_{k\to\infty} \sum_{i=1}^{n} d_i^k z_i(t) = \lim_{k\to\infty} f(t, \sum_{i=1}^{n} c_i^k y_i(h(t))) = = f(t, \sum_{i=1}^{n} c_i y_i(h(t))) = \sum_{i=1}^{n} d_i z_i(t) \tag{7}$$

for every $t \in J$. If the sequence $\{\mathbf{d}^k\}_{k=1}^{\infty}$ is bounded then from any subsequence we can choose such a subsequence $\{\mathbf{d}^{k_j}\}_{j=1}^{\infty}$ that $\lim_{j\to\infty} \mathbf{d}^{k_j}$ exists. From (7),

$$\sum_{i=1}^{n} (\lim_{j\to\infty} d_i^{k_j} - d_i)\, z_i(t) = 0 .$$

Due to $W[\mathbf{z}](t) \neq 0$ we have $\lim_{j\to\infty} d_i^{k_j} = d_i$ and also $\lim_{k\to\infty} d_i^k = d_i$ for $i = 1, 2, \ldots, n$. Now it is sufficient to prove that the sequence $\{\mathbf{d}^k\}_{k=1}^{\infty}$ is bounded. Passing to a subsequence if necessary, we may suppose without loss of generality that there is an index j such that $\lim_{k\to\infty} d_j^k = \pm\infty$, $d_j^k \neq 0$ and $|d_j^k| \geqq |d_i^k|$ for every positive integer k

and $i \in \{1, 2, \ldots, n\} \setminus \{j\}$. From (7)

$$\lim_{k \to \infty} \left[\sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{d_i^k}{d_j^k} z_i(t) + z_j(t) \right] = \frac{\lim\limits_{k \to \infty} \sum\limits_{i=1}^{n} d_i^k z_i(t)}{\lim\limits_{k \to \infty} d_j^k} = \frac{\sum\limits_{i=1}^{n} d_i z_i(t)}{\lim\limits_{k \to \infty} d_j^k} = 0 = \sum_{i=1}^{n} 0 \, . \, z_i(t)$$

for every $t \in J$. Since the sequences $\{d_i^k / d_j^k\}_{k=1}^{\infty}$ are bounded for $i = 1, 2, \ldots, n$, the above considerations imply $\lim\limits_{k \to \infty} (d_i^k / d_j^k) = 0$, in particular $1 = 0$ for $i = j$, which is a contradiction. By using the continuity of the inverse transformation of T we can similarly prove the continuity of G^{-1}.

(c) follows immediately from (a).

(d) If (6) holds for some $\mathbf{c} \in \mathbf{R}^n \setminus \{\mathbf{0}\}$ the function $y = \sum\limits_{i=1}^{n} c_i y_i$ satisfies (5) and we can apply Lemma 3 to get (6) for every $\mathbf{c} \in \mathbf{R}^n \setminus \{\mathbf{0}\}$. ($c$) implies that (6) holds for $\mathbf{c} = \mathbf{0}$ as well.

(iv) **Lemma 5.** *For every $r \in \mathbf{R}$ there is a unique $s \in \mathbf{R}$ such that*

$$G(r\mathbf{c}) = s\, G(\mathbf{c}) \tag{8}$$

holds for arbitrary $\mathbf{c} \in \mathbf{R}^n$. Moreover, the function $r \mapsto s(r)$ is a homeomorphism of $\mathbf{R}$ into $\mathbf{R}$.

Proof. Let $\mathbf{c} \in \mathbf{R}^n \setminus \{\mathbf{0}\}$ be fixed. Let $M = \{r \in \mathbf{R}$, there is $s \in \mathbf{R}$ such that $G(r\mathbf{c}) = s\, G(\mathbf{c})\}$. From the continuity of G it follows that M is closed. Since G is a homeomorphism of $\mathbf{R}^n$ into $\mathbf{R}^n$, its range $G(\mathbf{R}^n)$ is open in $\mathbf{R}^n$ (see [3]). Hence the set $S = \{s \in \mathbf{R},\ s\, G(\mathbf{c}) \in G(\mathbf{R}^n)\}$ is open. Further, S is not empty because $1 \in S$. For every $s \in S$ there is $\mathbf{d} \in \mathbf{R}^n$ such that

$$G(\mathbf{d}) = s\, G(\mathbf{c}) \, .$$

G^{-1} preserves linear dependence, hence $\mathbf{d} = r\mathbf{c}$ for some $r \in \mathbf{R}$. The function $s \mapsto r$ is a homeomorphism of S into $\mathbf{R}$ as it is defined with help of the homeomorphism G^{-1} restricted to the set $\{s\, G(\mathbf{c}),\ s \in S\}$. S is open, hence the range of this homeomorphism is also open in $\mathbf{R}$. Simultaneously, this range is equal to the set M which is closed in $\mathbf{R}$. That is why $M = \mathbf{R}$ and the inverse function $r \mapsto s(r)$ is a homeomorphism of $\mathbf{R}$ into $\mathbf{R}$. Lemma 4(d) implies that the statement (8) holds for every $\mathbf{c} \in \mathbf{R}^n$.

(v) Lemma 5 and Lemma 3 imply that for every $r \in \mathbf{R}$ there is $s(r) \in \mathbf{R}$ such that

$$f(t, ry(h(t))) = s(r) f(t, y(h(t))) \tag{9}$$

for arbitrary $y \in C^n(I, \mathbf{R})$, $\bar{y}(x) \neq \mathbf{0}$ on I. Double use of (9) yields the formula

$$s(r_1 r_2) = s(r_1)\, s(r_2) \tag{10}$$

for arbitrary $r_1, r_2 \in \mathbf{R}$. All the homeomorphisms on $\mathbf{R}$ satisfying (10) are

$$s(r) = \operatorname{sign}(r)\, |r|^{\lambda} , \quad \lambda > 0 \tag{11}$$

(see [1]). By substituting $y(x) \equiv 1$ in (9) we get

$$f(t, r) = s(r) f(t, 1) .$$

Thus, the transformation (4) has the form

$$z(t) = k(t)\, s(y(h(t))) , \tag{12}$$

where $k(t) := f(t, 1)$ and s satisfies (11).

(vi) In this part we shall show that $k \in C^n(J, \mathbf{R})$, $h \in C^n(J, I)$, $k(t) \neq 0$ and $h'(t) \neq 0$ for $t \in J$. For similar situation see [4].

Putting $y(x) \equiv 1$ on I we get from (12) that $k \in C^n(J, \mathbf{R})$. Let $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ and $\mathbf{z} = (z_1, z_2, \ldots, z_n)$ form two fundamental systems of some equations such that

$$z_i(t) = k(t)\, s(y_i(h(t)))$$

for $i = 1, 2, \ldots, n$. For every $t \in J$ there is $i \in \{1, 2, \ldots, n\}$ such that $z_i(t) \neq 0$. This implies $k(t) \neq 0$ for $t \in J$.

Now we can write

$$\frac{z}{k} = s \circ y \circ h .$$

Choose $y(x) = e^x$ on I. Then z/k is a positive function of the class $C^n(J, \mathbf{R})$, the function s has an inverse function of the class C^n on $(0, \infty)$ and y has an inverse function of the class C^n on I. Thus

$$h = y^{-1} \circ s^{-1} \circ \frac{z}{k}$$

is a function of the class $C^n(J, I)$. It remains to prove $h'(t) \neq 0$ for $t \in J$. On the contrary, suppose $h'(t_0) = 0$ for some $t_0 \in J$. Let $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be a fundamental system of some equation with $y_i(h(t_0)) > 0$ for $i = 1, 2, \ldots, n$. The transformation T converts these functions into a system $\mathbf{z} = (z_1, z_2, \ldots, z_n)$ with a nonzero Wronski determinant $W[\mathbf{z}]$ on J. Thus, according to Remark 1

$$W[\mathbf{z}/k]\,(t_0) = \left[\frac{1}{k(t_0)}\right]^n W[\mathbf{z}]\,(t_0) \neq 0 .$$

Since the function s is differentiable on $(0, \infty)$, for $h'(t_0) = 0$ we get

$$\left(\frac{z_i}{k}\right)'(t_0) = s'(y_i(h(t_0)))\, y_i'(h(t_0))\, h'(t_0) = 0$$

for $i = 1, 2, \ldots, n$. This is a contradiction and hence $h'(t) \neq 0$ for $t \in J$.

(vii) To complete the proof of Theorem it remains to show that $s(y) = y$. Suppose the transformation (12) to convert every fundamental system of solutions into a fundamental system. According to Remark 1 the transformation

$$w(x) = \frac{1}{k(h^{-1}(x))}\, z(h^{-1}(x)) \quad \text{on} \quad I$$

has the same property. By composition of these two transformations we get the transformation

(13) $$w(x) = s(y(x)) \quad \text{on} \quad I$$

which ought to have this property as well. It is sufficient to prove that (13) does not fulfil the required property if $s(y)$ is not identity.

$\lambda \in (0, 1)$ is not possible in (11) since for the function $y(x) = x - x_0$, $x_0 \in I$, we get

$$w'_+(x_0) = \lambda \lim_{x \to x_0+} (x - x_0)^{\lambda - 1} = \infty .$$

For $\lambda > 1$ and $n \geqq 2$ the transformation (13) converts the n-tuple of the linear independent solutions

$$1,\ x - x_0,\ (x - x_0)^2, \ldots, (x - x_0)^{n-1}, \quad x_0 \in I$$

of the equation $y^{(n)} = 0$ into the functions the first derivatives of which are

$$0,\ \lambda(x - x_0)^{\lambda-1},\ 2\lambda(x - x_0)^{2\lambda-1}, \ldots, (n - 1)\,\lambda(x - x_0)^{\lambda(n-1)-1}$$

for $x \geqq x_0$. At $x_0 \in I$ they all are zeros, which contradicts (**B**).

That is why for $g := h^{-1}$ every transformation (1) satisfying (**A**) and (**B**) has the form (3) if $n \geqq 2$.

Remark 3. For $n = 1$ one can show that all the transformations (1) satisfying the assumptions (**A**) and (**B**) are

$$t = g(x), \quad z = k(t) \operatorname{sign}(y) |y|^\lambda, \quad \lambda > 0,$$

where $k \in C^n(J, R)$, $k(t) \neq 0$ for every $t \in J$ and g is a C^n-diffeomorphism of I onto J (see [6] and [7]). The proof can be performed similarly as the steps (i)–(vi) in the proof of Theorem, but when using Lemma 2 some changes are necessary.

Remark 4. The statement of Theorem holds provided the transformation T in (**A**) is only a homeomorphism of $I \times \mathbf{R}$ into $\mathbf{R} \times \mathbf{R}$ and J is the smallest interval such that $T(I \times \mathbf{R}) \subset J \times \mathbf{R}$.

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