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OSCILLATION THEOREMS FOR CERTAIN NONLINEAR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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Consider the differential equation

$$x^{(n)}(t) = p(t)\, k(t, x(t), x^{(n-1)}(t))\, x^{(n-1)}(t) + q(t)\, |x[\sigma(t)]|^{\alpha} \operatorname{sgnx} [\sigma(t)] = 0 \tag{1}$$

where n is even, $\alpha \in (0, 1]$, $p, q, \sigma \colon [t_0, \infty) \to [0, \infty)$ and $k \colon [t_0, \infty) \times R^2 \to [0, \infty)$ are continuous, $q(t)$ not identically zero on any ray of the form $[t^*, \infty)$ for some $t^* \geqq t_0$ and $0 < \sigma(t) \leqq t$ and $\sigma(t) \to \infty$ as $t \to \infty$.

We assume that:

$$k(t, x, y) \leqq |y|^{\beta}, \quad -\infty < x, \ y < \infty \quad \text{and} \quad \beta \geqq 0; \tag{2}$$

$$\left(1 + \int_{t_0}^{t} p(s)\, ds\right)^{-1/\beta} \notin \mathscr{L}(t_0, \infty) \quad \text{for} \quad \beta > 0; \tag{3}$$

and

$$\int_{t_0}^{\infty} \exp\left(\int_{t_0}^{s} - p(\tau)\, d\tau\right) ds = \infty \quad \text{for} \quad \beta = 0.$$

We restrict our attention to proper solutions of (1), i.e., those solutions $x(t)$ which exist on some ray $[T_x, \infty) \subset [t_0, \infty)$ and satisfy $\sup \{|x(t)| \colon t \geqq T]\} > 0$ for any $T \geqq T_x$. A proper solution is called *oscillatory* if it has arbitrary large zeros; otherwise it is called *nonoscillatory*. Equation (1) is said to be *oscillatory* if all of its proper solutions are oscillatory. In case $n = 2$, $p(t) = 0$, $\sigma(t) = t$, Kura [6] has shown that equation (1) is oscillatory if

$$\limsup_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} \tau^{\gamma} q(\tau)\, d\tau\, ds = \infty \quad \text{for some} \quad \gamma \in [0, \alpha], \quad 0 < \alpha < 1. \tag{4}$$

Ohriska [8] considered (1) with $n = 2$, $p(t) = 0$ and $\alpha = 1$ and proved that either the condition

$$\limsup_{t \to \infty} t \int_{t}^{\infty} q(s) \frac{\sigma(s)}{s}\, ds > 1 \tag{5}$$

or

$$\limsup_{t\to\infty} t \int_{\gamma(t)}^{\infty} q(s)\,\mathrm{d}s > 1 . \tag{6}$$

is sufficient for equation (1) to be oscillatory, where

$$\gamma(t) = \sup\{s \geqq t_0 : \sigma(s) \leqq t\} \quad \text{for} \quad t \geqq t_0 .$$

The purpose of this paper is to extend and improve Kura's result to equation (1) and generalize the work of Ohriska to the more general equation (1).

We need the following three lemmas, the first is given in [3] and the second and third appeared in [7] and [4] respectively.

Lemma 1. *Let $x(t) \in C^2[T, \infty]$ and let*

$$x(t) > 0\,, \quad x^{\cdot}(t) > 0\,, \quad x^{\cdot\cdot}(t) \leqq 0 \quad \textit{for} \quad t \geqq T, \left(\cdot = \frac{\mathrm{d}}{\mathrm{d}t}\right).$$

Then for each $c \in (0, 1)$, there is a $T_c \geqq T$ such that

$$x[\sigma(t)] \geqq c\,\frac{\sigma(t)}{t}\,x(t)\,, \quad t \geqq T_c ,$$

where $\sigma(t)$ is as given above.

Lemma 2. *Let u be a positive and n-times differentiable function on $[t_0, \infty)$. If $u^{(n)}(t) \leqq 0$ for $t \geqq t_0$ and n even, and $u^{(n)}(t)$ not identically zero on any ray of the form $[t^*, \infty)$, $t^* \geqq t_0$, there exist a $t_u \geqq t_0$ and an integer l, $0 \leqq l \leqq n$ with $n + l$ odd and such that*

$$l > 0 \quad \textit{implies} \quad u^{(k)}(t) > 0 \quad \textit{for} \quad t \geqq t_u \quad (k = 0, 1, \ldots, l - 1)$$

and

$$l \leqq n - 1 \quad \textit{implies} \quad (-1)^{l+k}\,u^k(t) > 0 \quad \textit{for} \quad t \geqq t_u$$
$$(k = l,\; l + 1, \ldots, n - 1)\,.$$

Moreover, for each $\theta \in (0, 1)$, we have

$$u(t) \geqq \frac{\theta}{(n-1)!}\,t^{n-1}\,u^{(n-1)}(t) \quad \textit{for all large } t\,.$$

Lemma 3. *Let condition (2) and (3) hold. Then if $x(t)$ is a nonoscillatory solution of (1), we have*

$$x(t)\,x^{(n-1)}(t) > 0 \quad \textit{for all large} \quad t\,.$$

Our main results are as follows:

Theorem 1. *Let $n = 2$, $0 < \alpha < 1$ and conditions (2) and (3) hold. Suppose that there is a twice differentiable function $\zeta: [t_0, \infty) \to (0, \infty)$ such that*

$$\zeta^{\cdot\cdot}(t) + \left(\frac{1-\alpha}{\alpha}\right)\frac{\zeta^{\cdot 2}(t)}{\zeta(t)} \leqq 0 \quad \textit{for} \quad t \geqq t_0 \tag{7}$$

and

$$\limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{\tau}\left(\frac{\sigma(s)}{s}\right)^{\alpha}\zeta(s)\,q(s)\,\mathrm{d}s\,\mathrm{d}\tau = \infty\,, \tag{8}$$

then equation (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1), say $x(t) > 0$ for $t \geqq t_0 > 0$. There is a $t_1 \geqq t_0$ so that $x[\sigma(t)] > 0$ for $t \geqq t_1$. By Lemma 3, there exists a $t_2 \geqq t_1$, such that $x^{\cdot}(t) > 0$ for $t \geqq t_2$. Now the hypotheses of Lemma 1 are verified and hence for each $c \in (0, 1)$, there exists a $t_3 \geqq t_2$ so that

$$x[\sigma(t)] \geqq c\,\frac{\sigma(t)}{t}\,x(t) \quad \text{for all} \quad t \geqq t_3\,. \tag{9}$$

Thus, equation (1) reduces to

$$x^{\cdot\cdot}(t) + c\left(\frac{\sigma(t)}{t}\right)^{\alpha} q(t)\,x^{\alpha}(t) \leqq 0 \quad \text{for all} \quad t \geqq t_3\,. \tag{10}$$

Define

$$w(t) = \frac{1}{1-\alpha}\,\zeta(t)\,x^{1-\alpha}(t)\,.$$

Then $w(t)$ satisfies:

$$\begin{aligned}
w^{\cdot\cdot}(t) &\leqq -c^{\alpha}\,\zeta(t)\left(\frac{\sigma(t)}{t}\right)^{\alpha} q(t) + \frac{1}{1-\alpha}\,\zeta^{\cdot\cdot}(t)\,x^{1-\alpha}(t) + 2\zeta^{\cdot}(t)\,x^{-\alpha}(t)\,x^{\cdot}(t) - \\
&\quad - \alpha\,\zeta(t)\,x^{-\alpha-1}(t)\,x^{\cdot 2}(t) = \\
&= -c^{\alpha}\left(\frac{\sigma(t)}{t}\right)^{\alpha}\zeta(t)\,q(t) + \frac{1}{1-\alpha}\,\zeta^{\cdot\cdot}(t)\,x^{1-\alpha}(t) + \frac{\zeta^{\cdot 2}(t)}{\alpha\,\zeta(t)}\,x^{1-\alpha}(t) - \\
&\quad - \left((\alpha\,\zeta(t)\,x^{-\alpha-1}(t))^{1/2}\,x^{\cdot}(t) - \frac{2\,\zeta^{\cdot}(t)\,x^{-\alpha}(t)}{2(\alpha\,\zeta(t)\,x^{-\alpha-1}(t))^{1/2}}\right)^{2} \leqq \\
&\leqq -c^{\alpha}\left(\frac{\sigma(t)}{t}\right)^{\alpha}\zeta(t)\,q(t) + \frac{x^{1-\alpha}(t)}{1-\alpha}\left[\zeta^{\cdot\cdot}(t) + \frac{\zeta^{\cdot 2}(t)}{\zeta(t)}\,\frac{1-\alpha}{\alpha}\right].
\end{aligned}$$

By (7) we have

$$w^{\cdot\cdot}(t) \leqq -c^{\alpha}\left(\frac{\sigma(t)}{t}\right)^{\alpha}\zeta(t)\,q(t)\,. \tag{11}$$

Integrating (11) twice over $[t_3, t]$ we have

$$\begin{aligned}
c^{\alpha}\int_{t_3}^{t}\int_{t_3}^{s}\left(\frac{\sigma(\tau)}{\tau}\right)^{\alpha}\zeta(\tau)\,q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s &\leqq -w(t) + w(t_3) + (t - t_3)\,w^{\cdot}(t_3) \leqq \\
&\leqq w(t_3) + (t - t_3)\,w^{\cdot}(t_3)\,.
\end{aligned} \tag{12}$$

Dividing (12) by t and taking limit as $t \to \infty$ we, get

$$w^{\cdot}(t_3) \geqq \limsup_{t\to\infty}\frac{c^{\alpha}}{t}\int_{t_3}^{t}\int_{t_3}^{s}\left(\frac{\sigma(\tau)}{\tau}\right)^{\alpha}\zeta(\tau)\,q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\,.$$

This is a contradiction to (8), and the proof is complete.

Corollary 1. *Let* $n = 2, 0 < \alpha < 1$ *and conditions* (2) *and* (3) *hold. If*

$$\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} (\sigma(\tau)^{\alpha} q(\tau) \, ds = \infty \,, \tag{13}$$

then equation (1) *is oscilatory.*

Proof. The proof of the Corollary is the same as that of Theorem 1 for $\zeta(t) = t^{\alpha}$, and hence is omitted.

Remark. If $n = 2$, $p(t) \equiv 0$ and $\sigma(t) = t$, then $q(t)$ need not to be of fixed sign and hence our Theorem 1 for $\zeta(t) = t^{\gamma}$ for some $\gamma \in (0, \alpha]$ and Theorem 1 in [6] are the same.

As illustrative examples we consider the equations

$$x^{\bullet\bullet}(t) + \frac{1}{\sqrt{t}} x^{\bullet 3}(t) + t^{-4/3} x^{1/3} \left[\frac{t}{2}\right] = 0 \,, \quad t > 0 \,,$$

and

$$x^{\bullet\bullet}(t) + \frac{1}{t} x^{\bullet}(t) + t^{-7/8} x^{1/3} \left[\sqrt{t}\right] = 0 \,, \quad t > 0 \,,$$

and note that these equations are oscillatory by Corollary 1. We also note that the results in [1], [3], [5], [6] and [9] are not applicable to the above equations.

The following theorem generalize Theorem 1 of Ohriska [8].

Theorem 2. *Let conditions* (2) *and* (3) *hold and suppose that either*

$$\limsup_{t\to\infty} \sigma^{n-1}(t) \int_{t}^{\infty} q(s) \, ds > (n-1)! \quad \textit{and} \quad \sigma(t) \quad \textit{non-decreasing} \,, \tag{14}$$

or

$$\limsup_{t\to\infty} t^{n-1} \int_{\gamma(t)}^{\infty} q(s) \, ds > (n-1)! \,. \tag{15}$$

Then equation (1) *with* $\alpha = 1$ *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1). Assume $x(t) > 0$ and $x[\sigma(t)] > > 0$ for $t \geqq t_1 \geqq t_0 > 0$. By Lemma 3, there exists a $t_2 \geqq t_1$ such that $x^{(n-1)}(t) > 0$ for $t \geqq t_2$. The hypotheses of Lemma 2 are satisfied and hence there exists a $t_3 \geqq t_2$ such that

$$x^{\bullet}(t) > 0$$

and

$$x(t) \geqq \frac{c}{(n-1)!} t^{n-1} x^{(n-1)}(t) \,, \quad \text{for some} \quad c \in (0, 1) \quad \text{and} \quad t \geqq t_3 \,. \tag{16}$$

There exists a $t_4 \geqq t_3$ such that

$$x[\sigma(t)] \geqq \frac{c}{(n-1)!} \sigma^{n-1}(t) \, x^{(n-1)}(t) \,, \quad \text{for some} \quad c \in (0, 1) \quad \text{and} \quad t \geqq t_4 \,. \tag{17}$$

From (1), (16) and (17) we have

$$x(t) \geqq \frac{c}{(n-1)!}\, t^{n-1} \int_t^\infty q(s)\, x[\sigma(s)]\, ds\,, \tag{18}$$

and

$$x[\sigma(t)] \geqq \frac{c}{(n-1)!}\, \sigma^{n-1}(t) \int_t^\infty q(s)\, x[\sigma(s)]\, ds\,. \tag{19}$$

Now we shall consider two separate cases in view of conditions (14) and (15).

Case I. Assume that (14) holds. Since $x(t)$ is positive and increasing, it follows from (19) that

$$1 \geqq \frac{c}{(n-1)!}\, \sigma^{n-1}(t) \int_t^\infty q(s)\, ds\,, \quad t \geqq t_4\,. \tag{20}$$

From (20) it follows that

$$\limsup_{t\to\infty} \frac{\sigma^{n-1}(t)}{(n-1)!} \int_t^\infty q(s)\, ds < \infty\,.$$

If we put

$$\limsup_{t\to\infty} \frac{\sigma^{n-1}(t)}{(n-1)!} \int_t^\infty q(s)\, ds = A\,.$$

then, in view of (14), there exists a sequence of points $\{\tau_k\}$ such that $\tau_k \to \infty$ as $k \to \infty$ and

$$\lim_{k\to\infty} \frac{\sigma^{n-1}(\tau_k)}{(n-1)!} \int_{\tau_k}^\infty q(s)\, ds = A > 1\,.$$

So for $\varepsilon = (A-1)/2 > 0$, there exists a number K such that for every $k > K$ we have

$$\frac{A+1}{2} = A - \frac{A-1}{2} < \frac{\sigma^{n-1}(\tau_k)}{(n-1)!} \int_{\tau_k}^\infty q(s)\, ds\,. \tag{21}$$

Now, if we choose $k > K$ so that $\tau_k \geqq t_4$ and moreover, a number $c \in (0, 1)$ such that $2/(A+1) < c < 1$, then (21) implies

$$\frac{c}{(n-1)!}\, \sigma^{n-1}(\tau_k) \int_{\tau_k}^\infty q(s)\, ds > \frac{2}{A+1}\, \frac{A+1}{2} = 1$$

which contradicts (20).

Case II. Now assume (15), since $\gamma(t) \geqq t$, it follows from (18) that

$$x(t) \geqq \frac{c}{(n-1)!}\, t^{n-1} \int_{\gamma(t)}^\infty q(s)\, x[\sigma(s)]\, ds\,, \quad t \geqq t_3\,.$$

Now, using the fact that $x(t)$ is increasing and $\sigma(s) \geqq t$ for $s > \gamma(t)$, the above inequality gives

$$x(t) \geqq \frac{c}{(n-1)!}\, t^{n-1}\, x(t) \int_{\gamma(t)}^\infty q(s)\, ds\,, \quad t \geqq t_3\,.$$

or

$$1 \geqq \frac{c}{(n-1)!} t^{n-1} \int_{\gamma(t)}^{\infty} q(s) \, ds, \quad t \geqq t_3. \tag{22}$$

From (22), it follows that

$$B = \limsup_{t \to \infty} \frac{t^{n-1}}{(n-1)!} \int_{\gamma(t)}^{\infty} q(s) \, ds < \infty.$$

Suppose that (15) holds. Then similarly as above we again obtain a contradiction. This completes the proof.

Remark. Theorem 2 can be applied to equations when $q(t)$ is integrable over $[t_0, \infty)$. For $n = 2$ and $p(t) \equiv 0$, we note that our condition (15) and condition (6) due to Ohriska [8] are the same, while condition (14) is stronger than condition (5) in [8]. To see this, we consider the equations

$$x^{\cdot\cdot}(t) + \frac{c_1}{t^2} x[c_2 t] = 0, \quad t > 0 \tag{23}$$

$$x^{\cdot\cdot}(t) + \frac{\gamma_1}{t} x[t^{\gamma_2}] = 0, \quad t > 0. \tag{24}$$

where c_i, γ_i are positive constants $i = 1, 2$, $c_2 \in (0, 1]$ and $\gamma_2 \in (0, 1)$. Equation (23) is oscillatory by Theorem 1 in [8] and our Theorem 2 if $c_1 c_2 > 1$ and equation (24) is oscillatory by Theorem 1 in [8] if $\gamma_1/(1 - \gamma_2) > 1$. We note that our Theorem 2 cannot be applied to equation (24).

Next, we consider the following illustrative examples:

Example 1. Consider the equation

$$x^{(n)} + (x^{(n-1)}(t))^3 + \frac{c_1}{t^n} x[c_2 t] = 0, \quad n \text{ even}, \tag{25}$$

where c_1, c_2 are positive and $c_2 \in (0, 1]$.

By Theorem 2, all solutions of (25) are oscillatory if $c_1 c_2^{n-1} > (n-1)((n-1)!)$. One can easily verify that results in [2] and [8] fail to apply to (25) since $p(t) \not\equiv 0$.

Example 2. Consider the equation

$$x^{(n)} + \frac{1}{\sqrt{t}} |x^{(n-1)}| \, x^{(n-1)} + c_1 t^{-m} x[t^{c_2}] = 0, \quad n \text{ even}, \tag{26}$$

where c_1, c_2 and m are positive constant, $0 < c_2 \leqq 1$. All conditions of Theorem 2 are verified if $c_2(n-1) - m + 1 \geqq 0$ and $c_1/(m-1) > (n-1)!$ and hence equation (26) is oscillatory.

Remark. The results of this paper can be easily extended to more general equations of the form

$$x^{(n)}(t) + p(t) \, k(t, x(t), x^{(n-1)}(t)) \, x^{(n-1)}(t) + q(t) f(x[\sigma(t)] \, 0 = 0 \quad n \text{ even} \tag{27}$$

where p, q, and k are as above, $f: R \to R$ is continuous, $x f(x) > 0$ for $x \neq 0$, without any change in the conditions imposed in Theorem 1 and 2. We only require that $|f(x)/x^\alpha| \geqq \gamma > 0$ for $x \neq 0$, $0 < \alpha < 1$ be satisfied in Theorem 1 and we impose the condition $f(x)/x \geqq \gamma > 0$ for $x \neq 0$ in the hypotheses of Theorem 2. We note that f need not be monotonic. For illustration we consider the equations

$$(28) \qquad x^{\cdot\cdot}(t) + |x^{\cdot}|\, x^{\cdot} + t^{-4/3} x^{1/3} \left[\frac{t}{2}\right] \exp\left(\sin x \left[\frac{t}{2}\right]\right) = 0\,, \quad t_0 > 0$$

and

$$(29) \quad x^{(n)}(t) + \frac{1}{\sqrt{t}}\, |x^{(n-1)}|\, x^{(n-1)} + c t^{-n} x \left[\frac{t}{2}\right] \log\left(e + x^2 \left[\frac{t}{2}\right]\right) = 0\,, \quad t \geqq 0$$

and n is even. Equation (28) is oscillatory by Theorem 1 and equation (28) is oscillatory by Theorem 2 provided

$$\frac{c}{2^{n-1}} \frac{1}{n-1} > (n-1)!\,.$$

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