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ON GENERALIZED PERIODIC SOLUTIONS OF SOME NONLINEAR TELEGRAPH AND BEAM EQUATIONS

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1. INTRODUCTION, STATEMENT OF THE MAIN RESULTS

The purpose of this paper is to deal with *some types of nonlinear telegraph and beam equations.* We are interested in the case when the nonlinear term in the equation depends not only on the time and space variables (t and x), on the behaviour of the solution $u(t, x)$, but also when it depends on the behaviour of the solution at some fixed time $t = t_0$.

More precisely, we shall study the existence of *generalized periodic solutions* (GPS) of a nonlinear telegraph equation of the form

$$\beta u_t + u_{tt} - u_{xx} - \psi(t, x, u(t, x), u(t_0, x)) = g(t, x)\,, \tag{1.1}$$

and the existence of the GPS of a nonlinear beam equation

$$\beta u_t + u_{tt} + u_{xxxx} - \psi(t, x, u(t, x), u(t_0, x)) = g(t, x)\,, \tag{1.2}$$

(where $\beta \neq 0$).

In the sequel we shall denote by $\mathbf{I}$ the open interval $]0, 2\pi[$, $\mathbb{N}$ and $\mathbb{R}$ will denote the set of positive integers and real numbers, respectively. Let us denote by $\mathbf{H}$ the space of all measurable real valued functions $u(t, x)$ defined a.e. in $\mathbb{R}^2$ which are 2π-periodic in variables t and x, i.e. $u(t + 2\pi, x + 2\pi) = u(t + 2\pi, x) = u(t, x + 2\pi) = u(t, x)$ for almost all $(t, x) \in \mathbb{R}^2$, and which are square integrable over $\mathbf{I}^2 = \mathbf{I} \times \mathbf{I}$. The symbol $\mathbf{C}_{2\pi}^{p,q}$, where $p, q \in \mathbb{N} \cup \{0\}$, will denote the space of all continuous functions on $\mathbb{R}^2$ which are 2π-periodic in both variables and such that the partial derivatives of order p with respect to t and partial derivatives of order q with respect to x are continuous on $\mathbb{R}^2$, while $\mathbf{C}_{2\pi}$ is used for $\mathbf{C}_{2\pi}^{0,0}$. Let us denote by $\|\cdot\|$ and $(\cdot, \cdot)$ the norm and the scalar product in $\mathbf{H}$, respectively, and by $|||\cdot|||$ the usual norm in $\mathbf{C}_{2\pi}$.

Let $\mathbf{H}^{p,q}$ denote the *Sobolev space of functions from* $\mathbf{H}$, the partial derivatives of which up to order p with respect to t and up to order q with respect to x belong to $\mathbf{H}$ (see e.g. Vejvoda [8]).

Let us suppose that

$$\psi(t, x, s, z): \mathbf{I} \times \mathbf{I} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

is a *Caratheodory's function*, i.e., it is continuous in (s, z) for a.a. $(t, x) \in \mathbf{I}^2$ and measurable in (t, x) for all $(s, z) \in \mathbb{R}^2$. Suppose that there exist a function $r(t, x) \in \in \mathbf{L}^\infty(\mathbf{I}^2)$ and a real constant $c > 0$ such that

$$(1.3) \qquad |\psi(t, x, s, z)| \leqq r(t, x) + c|s|$$

for all $(t, x) \in \mathbf{I}^2$, $(s, z) \in \mathbb{R}^2$.

Let us remark that under the assumption (1.3) the *Nemytskii operator* $u \to \to \psi(t, x, u(t, x), u(t_0, x))$ is well defined from $\mathbf{C}_{2\pi}$ into $\mathbf{H}$.

Definition. GPS of (1.1) is a real valued function $u \in \mathbf{C}_{2\pi}$ such that

$$(1.4) \qquad (u, -\beta v_t + v_{tt} - v_{xx}) = (\psi(t, x, u(t, x), u(t_0, x)) + g(t, x), v)$$

holds for each $v \in C^{2,2}_{2\pi}$.

Analogously, GPS of (1.2) is a real valued function $u \in \mathbf{C}_{2\pi}$ such that

$$(1.5) \qquad (u, -\beta v_t + v_{tt} + v_{xxxx}) = (\psi(t, x, u(t, x), u(t_0, x)) + g(t, x), v)$$

holds for each $v \in \mathbf{C}^{2,4}_{2\pi}$. Our aim was to state sufficient conditions on the nonlinearity ψ for the periodic problem for the equation (1.1) or (1.2) to have at least one GPS for an arbitrary right-hand side $g \in \mathbf{H}$. Let us remark that the generalized periodic solvability of the equation (1.1) was studied by Fučík, Mawhin [3] and the generalized periodic solvability of the equation (1.2) by Krejčí [4]. In both papers it is essential that the nonlinearity ψ depends only on $u(t, x)$ and that the limits

$$\lim_{s \to +\infty} \frac{\psi(s)}{s} = \mu, \quad \lim_{s \to -\infty} \frac{\psi(s)}{s} = \nu$$

exist.

On the other hand, our assumptions allow some bounded oscillations of the nonlinear function ψ (when $|s| \to \infty$) which may be caused by the behaviour of the solution $u(t, x)$ at some fixed time level t_0.

More precisely, we shall suppose that there exist bounded measurable functions $\psi_{-\infty}(x, z)$, $\psi_{+\infty}(x, z)$ such that

$$(1.6) \qquad \lim_{s \to -\infty} \frac{\psi(t, x, s, z)}{s} = \psi_{-\infty}(x, z), \quad \lim_{s \to +\infty} \frac{\psi(t, x, s, z)}{s} = \psi_{+\infty}(x, z)$$

uniformly with respect to t, x, z, in the sense of the $\mathbf{L}^\infty$-norm.

As an example the reader may bear in mind the function

$$(1.7) \qquad \psi(t, x, s, z) = \mu(x)\, s^+(1 + \sin z) + \nu(x)\, s^-(\cos z - 1) + \sqrt{|s|} \sin t,$$

$(t, x) \in \mathbf{I}^2$, $(s, z) \in \mathbb{R}^2$, where s^+, s^- denote the positive and negative parts of s, respectively.

Let us formulate our main results. Analogously as in [3], put

$$\mathcal{M} = \{(\mu, \nu) \in \mathbb{R}^2;\ \mu < 0,\ \nu < 0\} \cup$$
$$\cup \bigcup_{k=0}^{\infty} \left\{(\mu, \nu) \in \mathbb{R}^2;\ \sqrt{\mu} > \frac{k}{2},\ \omega_k \sqrt{\mu} < \sqrt{\nu} < \omega_{k+1} \sqrt{\mu}\right\},$$

where

$$\omega_k(\tau) = \frac{k\tau}{2\tau - k}, \quad \tau \in \left]\frac{k}{2}, +\infty\right[.$$

It is easy to see that $\mathscr{M}$ is the union of components of $\mathbb{R}^2$ which have a nonempty intersection with the diagonal $\sqrt{\mu} = \sqrt{\nu}$, and which are separated by the curves

$$C_k \equiv \sqrt{\nu} = \frac{k\sqrt{\mu}}{2\sqrt{\mu} - k}, \quad \sqrt{\mu} > \frac{k}{2}, \quad k = 0, 1, 2, \ldots.$$

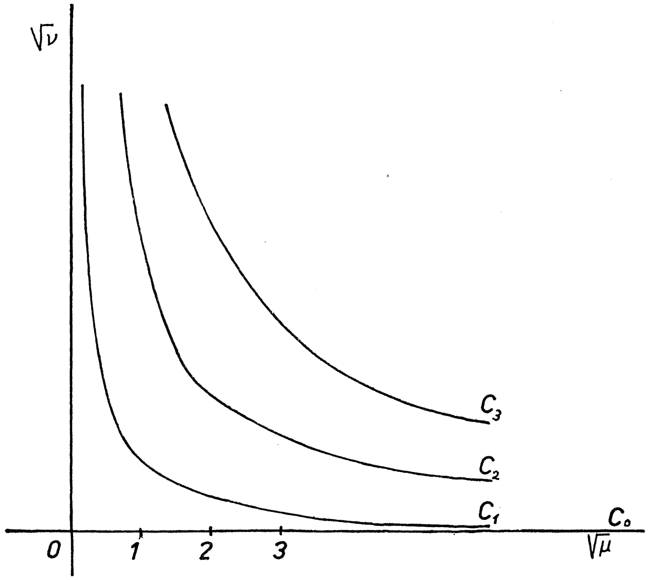

Figure 1.

Our main result concerning the solvability of the equation (1.1) is formulated in the following theorem.

Theorem 1. *Let us suppose that there exist* $k \in \mathbb{N} \cup \{0\}$, $(\mu_k, \nu_k) \in C_k$, $(\mu_{k+1}, \nu_{k+1}) \in C_{k+1}$ *and a sufficiently small* $\eta > 0$ *such that either*

(1.8) $$\mu_k + \eta \leqq \psi_{+\infty}(x, z) \leqq \mu_{k+1} - \eta, \quad \nu_k + \eta \leqq \psi_{-\infty}(x, z) \leqq \nu_{k+1} - \eta,$$

and $\mu_k + \eta$, $\nu_k + \eta$, $\mu_{k+1} - \eta$, $\nu_{k+1} - \eta$ *do not belong to any* C_i *or*

(1.8') $$\psi_{+\infty}(x, z) \leqq -\eta, \quad \psi_{-\infty}(x, z) \leqq -\eta,$$

for all $(x, z) \in \mathbf{I} \times \mathbb{R}$. *Then* (1.1) *has at least one* GPS *for an arbitrary right-hand side* $g \in \mathbf{H}$.

Let us remark that the result formulated in our Theorem 1 implies that there is a GPS for an arbitrary right-hand side $g \in \mathbf{H}$ even if the nonlinear part ψ oscillates "over" an arbitrary finite number of eigenvalues of the linear part. This oscillation, of course, is allowed only in the sense of conditions (1.8) or (1.8').

Our result concerning the solvability of the nonlinear beam equation (1.2) is formulated in the following theorem.

Theorem 2. *Let us suppose that for some* $k \in \mathbb{N} \cup \{0\}$ *either*

$$k^4 + \eta \leqq \psi_{+\infty}(x, z) \leqq (k + 1)^4 - \eta\,, \tag{1.9}$$
$$k^4 + \eta \leqq \psi_{-\infty}(x, z) \leqq (k + 1)^4 - \eta$$

or

$$\psi_{+\infty}(x, z) \leqq -\eta\,, \quad \psi_{-\infty}(x, z) \leqq -\eta \tag{1.9'}$$

holds for all $(x, z) \in \mathbf{I} \times \mathbb{R}$ *with some* $\eta > 0$. *Then* (1.2) *has at least one* GPS *for an arbitrary right-hand side* $g \in \mathbf{H}$.

The proofs of our main results are based on *the homotopy invariance property of the Leray-Schauder degree* which is combined with the "*shooting method*" *for ordinary differential equations.*

The paper is organized as follows. In Section 2 we first formulate without proofs some useful results for the linear telegraph equation giving the necessary references. Then we prove Theorem 1. Section 3 contains first some results for a linear beam equation with references concerning their proofs, and then the proof of Theorem 2. Section 4 contains some final remarks and some open problems, the solution of which, in our opinion, would lead to better understanding of periodic solvability of (1.1) and (1.2) with some other types of the nonlinearity ψ.

2. PROOF OF THEOREM 1

We first recall some *properties of the linear telegraph equation*

$$\beta u_t + u_{tt} - u_{xx} - \lambda u = h(t, x) \tag{2.1}$$

that will be useful in what follows.

Lemma 2.1. *The equation* (2.1) *with* $\beta \neq 0$ *and* λ *real constants has a* GPS *for any* $h \in \mathbf{H}$ *if and only if* $\lambda \neq m^2$; $m \in \mathbb{N}$. *In this case the solution* u *of* (2.1) *defines a linear compact operator*

$$\boldsymbol{T}_\lambda: \mathbf{H} \to \mathbf{H}\,, \quad h \mapsto u\,.$$

The operator $\boldsymbol{T}_\lambda$ *has the following regularity properties*:

$$\boldsymbol{T}_\lambda(\mathbf{H}) \subsetneq \mathbf{H}^{1,1}(\mathbf{I}^2) \cap \mathbf{C}_{2\pi}\,,$$
$$\boldsymbol{T}_\lambda(\mathbf{H}^{k,k}(\mathbf{I}^2)) \subsetneq \mathbf{H}^{k+1,k+1}(\mathbf{I}^2)\,.$$

Moreover, $\boldsymbol{T}_\lambda$ *is a completely continuous operator from* $\mathbf{H}$ *to* $\mathbf{C}_{2\pi}$.

Proof (see [5], with the exception of the complete continuity of $\boldsymbol{T}_\lambda$ from $\mathbf{H}$ into $\mathbf{C}_{2\pi}$). The complete continuity of $\boldsymbol{T}_\lambda$ from $\mathbf{H}$ into $\mathbf{C}_{2\pi}$ may be proved by using *the Fourier method* (see [8]), and one can proceed in the same way as in the proof of [2, Theorem 2.4(iii)]. ■

Using this lemma we see that the equation (1.1) is equivalent to

$$u = T_\varepsilon(g - \varepsilon u + \psi(t, x, u(t, x), u(t_0, x))) \quad \text{with} \quad 0 < \varepsilon < 1 . \tag{2.2}$$

According to Lemma 2.1 and to (1.3) the operator

$$u \mapsto \boldsymbol{T}_\varepsilon(g - \varepsilon u + \psi)$$

is completely continuous from $\mathbf{C}_{2\pi}$ into $\mathbf{C}_{2\pi}$. Let us fix a couple (μ, ν) such that

$$\mu_k + \eta < \mu < \mu_{k+1} - \eta ; \quad \nu_k + \eta < \nu < \nu_{k+1} - \eta$$

and consider the homotopy

$$\begin{aligned} \mathfrak{H}(\tau, u) = u - \boldsymbol{T}(\tau g - \varepsilon u + \tau\psi(t, x, u, u(t_0, x)) + \\ + (1 - \tau)\,\mu u^+ - (1 - \tau)\,\nu u^-) \end{aligned} \tag{2.3}$$

where $u \in \mathbf{C}_{2\pi}$, $\tau \in [0, 1]$. We claim that *the Leray-Schauder degree*

$$\deg(\mathfrak{H}(\tau, \cdot), \mathbf{B}_R(0), 0) \tag{2.4}$$

(where $\mathbf{B}_R(0) \subset \mathbf{C}_{2\pi}$ is a ball centered at the origin of $\mathbf{C}_{2\pi}$ and with radius $R > 0$) is well defined for all $\tau \in [0, 1]$, provided $R > 0$ is large enough. By contradiction, let us suppose that there exist sequences $\{\tau_n\}_{n\in N}$, $\tau_n \in [0, 1]$ and $\{u_n\}_{n\in N}$, $u_n \in \mathbf{C}_{2\pi}$ such that $|||u_n||| \to \infty$ and

$$\mathfrak{H}(\tau_n, u_n) = 0 . \tag{2.5}$$

Let us define

$$h_n(t, x) := \frac{\psi(t, x, u_n(t, x), u_n(t_0, x))}{|||u_n|||} .$$

From (2.3) and (2.5) we obtain that

$$v_n - \boldsymbol{T}_\varepsilon\left(\tau_n \frac{g}{|||u_n|||} - \varepsilon v_n + \tau_n h_n(t, x) + (1 - \tau_n)\,\mu v_n^+ - (1 - \tau_n)\,\nu v_n^-\right) = 0 \tag{2.6}$$

where

$$v_n = |||u_n|||^{-1}\, u_n . \tag{2.7}$$

We know that $h_n(t, x)$ and v_n are bounded in $\mathbf{H}$, which implies that, after passing to suitable subsequences, $\tau_n \to \tau$, $h_n(t, x) \rightharpoonup h(t, x)$ and $v_n \rightharpoonup v$ in $\mathbf{H}$.

Furthermore, using the compactness of $\boldsymbol{T}_\varepsilon$ from $\mathbf{H}$ into $\mathbf{C}_{2\pi}$ we may conclude that $v_n \to v$, $v_n^+ \to v^+$, $v_n^- \to v^-$ strongly in $\mathbf{C}_{2\pi}$. To prove our claim we first give some properties of the function $h(t, x)$.

Lemma 2.2. *Under our hypothesis we have*

$$h(t, x) = h_+(x)\, v^+(t, x) - h_-(x)\, v^-(t, x) . \tag{2.8}$$

Proof. For each $\delta > 0$ let us define

$$\begin{aligned} N_\delta &:= \{(t, x) \in \mathbf{I}^2 |\ |v(t, x)| \leqq \delta\}, \\ N_\delta^+ &:= \{(t, x) \in \mathbf{I}^2 |\ v(t, x) \geqq \delta\}, \\ N_\delta^- &:= \{(t, x) \in \mathbf{I}^2 |\ v(t, x) \leqq -\delta\}, \\ N_0 &:= N_0^+ \cap N_0^- = \{(t, x) \in \mathbf{I}^2 |\ v(t, x) = 0\}. \end{aligned}$$

Furthermore, let $\chi_\delta^+ (\chi_\delta^-)$ be the characteristic function of $N_\delta^+ (N_\delta^-)$.

First we notice that $h(t, x) = 0$ almost everywhere on N_0. In fact, on the one hand one has $h_n \to 0$ pointwise by (1.3), on the other hand, using a suitable version of *the Lebesgue dominated convergence theorem* one gets $h_n \to 0$ in $\mathbf{L}^2(N_0)$.

Let $\delta > 0$ be fixed. We want to estimate the behaviour of the limit function $h(t, x)$ on the sets $N_\delta, N_\delta^+, N_\delta^-$; to this aim we estimate, using (1.6) and (1.3), the following expression:

$$(2.9) \qquad |\phi_n(t, x)| := |h_n(t, x) - \psi_{+\infty}(x, u_n(t_0, x))\, v_n(t, x)\, \chi_0^+ - \psi_{-\infty}(x, u_n(t_0, x))\, v_n(t, x)\, \chi_0^-|\,.$$

First of all, $\exists n_0 \in \mathbb{N}$ such that $\forall (t, x) \in N_\delta$ and $\forall n > n_0$

$$(2.10) \qquad |v_n(t, x)| \leqq 2\delta$$

because $v_n \to v$ in $\mathbf{C}_{2\pi}$; then by (1.3) one has

$$(2.11) \qquad |h_n(t, x)| \leqq 2\delta c\,.$$

By $\psi_{+\infty}(x, u_n(t_0, x)), \psi_{-\infty}(x, u_n(t_0, x)) \in \mathbf{L}^\infty(\mathbf{I}^2)$ and by (2.10) we have that

$$(2.12) \qquad |\phi_n(t, x)| \leqq 2\delta\bar{c} \text{ on } N_\delta\,.$$

On the other hand, $\exists n_1 \in \mathbb{N}$ such that $\forall (t, x) \in N_\delta^+$ $\forall n > n_1$

$$(2.13) \qquad v_n(t, x) \geqq \frac{\delta}{2} > 0\,;$$

this implies

$$(2.14) \qquad \forall (t, x) \in N_\delta^+ u_n(t, x) \to +\infty \quad \text{when} \quad n \to \infty\,.$$

From (2.9) one has that $\forall n > n_1$,

$$|h_n(t, x) - \psi_{+\infty}(x, u_n(t_0, x))\, v_n(t, x)| \leqq$$
$$\leqq |v_n(t, x)| \left| \frac{\psi(t, x, u_n(t, x), u_n(t_0, x))}{u_n(t, x)} - \psi_{+\infty}(x, u_n(t_0, x)) \right| \leqq 2\delta\bar{c}$$

because v_n are bounded in $\mathbf{C}_{2\pi}$ and by (1.6).

In the same way we can prove that $\forall (t, x) \in N_\delta^-$:

$$|h_n(t, x) - \psi_{-\infty}(x, u_n(t_0, x))\, v_n(t, x)| \leqq 2\delta\bar{c} \quad \text{when} \quad n \to +\infty\,.$$

Since δ can be taken arbitrarily small we have proved that $\forall (t, x) \in \mathbf{I}^2$:

$$(2.5) \quad |h_n(t, x) - \psi_{+\infty}(x, u_n(t_0, x))\, v_n(t, x)\, \chi_0^+ - \psi_{-\infty}(x, u_n(t_0, x))\, v_n(t, x)\, \chi_0^-| \to 0$$

when $n \to \infty$, uniformly with respect to (t, x). On the other hand, $\psi_{+\infty}$ and $\psi_{-\infty}$ being bounded we have

$$(2.16) \qquad \psi_{+\infty}(x, u_n(t_0, x)) \rightharpoonup h_+(x) \quad \text{in} \quad \mathbf{L}^2(\mathbf{I}^2)\,,$$
$$\psi_{-\infty}(x, u_n(t_0, x)) \rightharpoonup h_-(x) \quad \text{in} \quad \mathbf{L}^2(\mathbf{I}^2)\,,$$

and by the previous remarks

$$(2.17) \qquad v_n(t, x)\, \chi_0^+ \to v^+(t, x) \quad \text{in} \quad \mathbf{L}^2(\mathbf{I}^2)\,,$$
$$v_n(t, x)\, \chi_0^- \to v^-(t, x) \quad \text{in} \quad \mathbf{L}^2(\mathbf{I}^2)\,.$$

Using (2.16)–(2.17) from (2.15) we obtain (2.8). ∎

Remark 2.3. From (2.16) and (1.8) one gets

$$(2.18) \qquad \mu_k + \eta \leqq h_+(x) \leqq \mu_{k+1} - \eta\,, \quad \nu_k + \eta \leqq h_-(x) \leqq \nu_{k+1} - \eta\,.$$

In fact, the order intervals $[\mu_k + \eta, \mu_{k+1} - \eta]$ and $[\nu_k + \eta, \nu_{k+1} - \eta]$ are closed convex sets in $\mathbf{L}^2(\mathbf{I}^2)$ and therefore they are weakly closed.

From the definition of GPS and from Lemma 2.3, passing to the limit in the equation (2.6) we obtain

$$(v, -\beta u_t + u_{tt} - u_{xx}) = ([\tau\, h_+(x) + (1 - \tau)\, \mu]\, v^+ - [\tau\, h_-(x) + (1 - \tau)\, \nu]\, v^-, u),$$

which is nothing else than

$$(2.20) \qquad v - \boldsymbol{T}_\varepsilon(\mu(x)\, v^+ - \nu(x)\, v^- - \varepsilon v) = 0\,,$$

where

$$\mu(x) = \tau\, h_+(x) + (1 - \tau)\, \mu\,, \quad \nu(x) = \tau\, h_-(x) + (1 - \tau)\, \nu\,.$$

Let us remark that

$$\mu_k + \eta < \mu(x) < \mu_{k+1} - \eta\,, \quad \nu_k + \eta < \nu(x) < \nu_{k+1} - \eta$$

for almost every $x \in \mathbf{I}$.

To get a contradiction and prove our claim, we have to prove that $v \equiv 0$. To this aim, first we shall show

Lemma 2.4. *The solution v of* (2.20) *does not depend on the variable t and $\tilde{v}(x) = v(t, x)$ satisfies the ordinary differential equation of the second order*

$$(2.21) \qquad -u'' = \mu(x)\, u^+ - \nu(x)\, u^-$$

almost everywhere in $\mathbf{I}$.

Finally, we shall prove

Lemma 2.5. *The only periodic solution of the equation* (2.21) *is $v \equiv 0$ in* $\mathbf{I}$.

The assertion of Lemma 2.5 contradicts the fact that $|||v||| = 1$. Hence the degree (2.4) is well defined for all $\tau \in [0, 1]$ with respect to sufficiently large $R > 0$. Using the *homotopy invariance property of the Leray-Schauder degree* we obtain

$$(2.22) \qquad \deg(u - \boldsymbol{T}_\varepsilon(g - \varepsilon u + \psi(t, x, u(t, x), u(t_0, x))), \mathbf{B}_R(0), 0) =$$

$$= \deg(\mathfrak{H}(1, u), \mathbf{B}_R(0), 0) = \deg(\mathfrak{H}(0, u), \mathbf{B}_R(0), 0) =$$
$$= \deg(u - \boldsymbol{T}_\varepsilon((\mu - \varepsilon) u^+ - (\nu - \varepsilon) u^-), \mathbf{B}_R(0), 0)$$

and due to our hypothesis on μ and ν, for ε sufficiently small $(\mu - \varepsilon, \nu - \varepsilon)$ stays in the connected component which contains (λ, λ), $\lambda \neq m^2$. Hence using again *the homotopy invariance property of the Leray-Schauder degree* it is possible to show (see for instance [2, Th. 3]) that

$$\deg(u - \boldsymbol{T}_\varepsilon((\mu - \varepsilon) u^+ - (\nu - \varepsilon) u^-), \mathbf{B}_R(0), 0) = \tag{2.23}$$
$$= \deg(u - \boldsymbol{T}_\varepsilon(\lambda u), \mathbf{B}_R(0), 0) = \pm 1$$

because $u \mapsto \boldsymbol{T}_\varepsilon(\lambda u)$ is a linear, completely continuous operator. From (2.22)–(2.23) one has

$$\deg(u - \boldsymbol{T}_\varepsilon(g - \varepsilon u + \psi(t, x, u(t, x), u(t_0, x)), \mathbf{B}_R(0), 0) \neq 0$$

and therefore, for *the basic Leray-Schauder degree property*, we have that the equation (2.2) has at least one solution; then there exists at least one GPS of (1.1). To complete the proof of Theorem 1 we just need proofs of Lemmas 2.4 and 2.5.

Proof of Lemma 2.4. To get this result we first need some estimates. Let $v \in \mathbf{C}_{2\pi}$ be a solution of the equation (2.20), where by the definition $\mu(x) \in \mathbf{L}^\infty(\mathbf{I})$ and $\nu(x) \in \mathbf{L}^\infty(\mathbf{I})$; by Lemma 2.1 one has $v \in \mathbf{H}^{1,1}(\mathbf{I}^2) \cap \mathbf{C}_{2\pi}$ and known results (Kinderleher-Stampacchia [9]) imply $v^+, v^- \in \mathbf{H}^{1,1}(\mathbf{I}^2)$. We choose two sequences $\bar{\mu}_n(x), \bar{\nu}_n(x) \in \mathbf{C}^\infty(\mathbf{I})$, equibounded and such that

$$\bar{\mu}_n(x) \to \mu(x)\,, \quad \bar{\nu}_n(x) \to \nu(x) \tag{2.24}$$

in $\mathbf{L}^2(\mathbf{I})$.

Since $v \in \mathbf{C}_{2\pi}$, we have

$$\mu_n(x)\, v \to \mu(x)\, v \tag{2.25}$$

and

$$\nu_n(x)\, v \to \nu(x)\, v \tag{2.26}$$

in the space $\mathbf{H}$. Using Lemma 2.1 we obtain that $v_n \to v$ in $\mathbf{H}^{1,1}(\mathbf{I}^2)$ and also $v_n \in \mathbf{H}^{2,2}(\mathbf{I}^2)$. We now have noughe regularity to assert that v_n is a GPS of

$$\beta v_{n_t} + v_{n_{tt}} - v_{n_{xx}} - \varepsilon v_n = \mu_n(x)\, v^+ - \nu_n(x)\, v^-\,, \tag{2.27}$$

where $\mu_n(x) = \bar{\mu}_n(x) - \varepsilon$; $\nu_n(x) = \bar{\nu}_n(x) - \varepsilon$, and (2.27) holds almost every where in $\mathbf{I}^2$. We multiply (2.27) by v_{n_t} and integrate over $\mathbf{I}^2$:

$$\beta \int_{\mathbf{I}^2} (v_{n_t})^2 \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbf{I}^2} v_{n_{tt}} v_{n_t} \,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbf{I}^2} v_{n_{xx}} v_{n_t} \,\mathrm{d}x\,\mathrm{d}t - \varepsilon \int_{\mathbf{I}^2} v_n v_{n_t} \,\mathrm{d}x\,\mathrm{d}t = \tag{2.28}$$
$$= \int_{\mathbf{I}^2} \mu_n(x)\, v^+ v_{n_t} \,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbf{I}^2} \nu_n(x)\, v^- v_{n_t} \,\mathrm{d}x\,\mathrm{d}t\,.$$

We recall that v is 2π-periodic, therefore

$$\int_{\mathbf{I}^2} \mu_n(x)\, v^+ v_t \,\mathrm{d}t\,\mathrm{d}x = \int_{\mathbf{I}^2} \nu_n(x)\, v^- v_t \,\mathrm{d}t\,\mathrm{d}x = 0\,. \tag{2.29}$$

So from (2.28) we have by integrating by parts and using (2.29)

$$\beta\int_{\mathbf{I}^2}(v_{n_t})^2\,dt\,dx+\frac{1}{2}\int_0^{2\pi}dx\int_0^{2\pi}\frac{d}{dt}(v_{n_t})^2\,dt+\int_0^{2\pi}dt\int_0^{2\pi}v_{n_x}(v_{n_x})_t\,dx-$$

$$-\frac{\varepsilon}{2}\int_0^{2\pi}dx\int_0^{2\pi}\frac{d}{dt}v_n^2dt=\int_{\mathbf{I}^2}\mu_n(x)\,v^+(v_n-v)_t\,dx\,dt-\int_{\mathbf{I}^2}\nu_n(x)\,v^-(v_n-v)_t\,dx\,dt\,,$$

that is

$$\beta\int_{\mathbf{I}^2}(v_{n_t})^2\,dt\,dx+\frac{1}{2}\int_0^{2\pi}[v_{n_t}^2(x,2\pi)-v_{n_t}^2(x,0)]\,dx+$$

$$+\frac{1}{2}\int_0^{2\pi}dx\int_0^{2\pi}\frac{d}{dx}(v_{n_x})^2\,dt-\frac{\varepsilon}{2}\int_0^{2\pi}[v_n^2(x,2\pi)-v_n^2(x,0)]\,dx=$$

$$=\int_{\mathbf{I}^2}\mu_n(x)\,v^+(v_n-v)_t\,dx\,dt-\int_{\mathbf{I}^2}\nu_n(x)\,v^-(v_n-v)_t\,dx\,dt\,.$$

Hence, recalling that v_n are 2π-periodic in x and t, we get

$$(2.30)\quad \beta\int_{\mathbf{I}^2}(v_{n_t})^2\,dx\,dt=\int_{\mathbf{I}^2}\mu_n(x)\,v^+(v_n-v)_t\,dx\,dt-\int_{\mathbf{I}^2}\nu_n(x)\,v^-(v_n-v)_t\,dx\,dt\,.$$

It is easily seen that the right-hand side of (2.30) converges to zero since it is bounded by $(\|\mu(x)\|_{\mathbf{L}^\infty}+\|\nu(x)\|_{\mathbf{L}^\infty})\,\|v\|_{\mathbf{L}^2}\,\|v_n-v\|_{\mathbf{H}^{1,1}}$; therefore $v_{n_t}\to 0$. But we have already seen that $v_n\to v$ in $\mathbf{H}^{1,1}(\mathbf{I}^2)$, that is $v_t=0$, and so $v(t,x)=\tilde{v}(x)$.

We know that v solves

$$(v,\,-\beta\mu_t+u_{tt}-u_{xx})=(\mu(x)\,v^+-\nu(x)\,v^-,\,u)\quad\forall u\in\mathbf{C}_{2\pi}^{2,2}\,,$$

that is, $\tilde{v}$ is a weak solution of

$$(2.31)\quad \int_0^{2\pi}u'(x)\,\tilde{v}'(x)\,dx=\int_0^{2\pi}\mu(x)\,\tilde{v}^+u\,dx-\int_0^{2\pi}\nu(x)\,\tilde{v}^-u\,dx\,,\quad\forall u\in\mathbf{C}_{2\pi}^2(\mathbf{I})\,;$$

by standard regularity results for ordinary differential equations there exists $\tilde{v}''(x)$ almost everywhere in $\mathbf{I}$, so we can integrate (2.31) by parts and, recalling that $\tilde{v}$ is 2π-periodic, we conclude that

$$(2.32)\quad -\int_0^{2\pi}u(x)\,\tilde{v}''(x)\,dx=\int_0^{2\pi}\mu(x)\,\tilde{v}^+(x)\,u(x)\,dx-\int_0^{2\pi}\nu(x)\,\tilde{v}^-(x)\,u(x)\,dx\,,$$

$$\forall u\in\mathbf{C}_{2\pi}^2(\mathbf{I})\,,$$

from which one has that $\tilde{v}$ is a 2π-periodic solution of

$$(2.33)\qquad -\tilde{v}''(x)=\mu(x)\,\tilde{v}^+(x)-\nu(x)\,\tilde{v}^-(x)$$

almost everywhere in $\mathbf{I}$. ■

Before proving Lemma 2.5 we point out some properties of 2π-solutions of the equation

$$(2.34)\qquad -w''(x) = \mu\, w^+(x) - \nu\, w^-(x)\,, \quad w(0) = w(2\pi)\,, \quad w'(0) = w'(2\pi)\,,$$

where μ, ν do not depend on x.

Remark 2.6. Solutions of (2.34) are invariant under translations.

Remark 2.7. Let $]a, b[$ be an interval where the solution u of (2.34) is a positive solution of the Dirichlet problem

$$(2.35)\qquad -w''(x) = \mu\, w(x)\,, \quad w(a) = w(b) = 0\,.$$

Let $\bar{x} > b (\bar{x} < b)$ be fixed. We can modify the equation (2.35) in such a way that u is a positive solution of

$$(2.36)\qquad -w''(y) = \tilde{\mu}\, w(y)\,, \quad w(a) = w(\bar{x}) = 0$$

with

$$\tilde{\mu} = \left(\frac{b-a}{\bar{x}-a}\right)^2 \mu < \mu \quad \left(\tilde{\mu} = \left(\frac{b-a}{\bar{x}-a}\right)^2 \mu > \mu\right).$$

Remark 2.8. The solution of (2.34) is such that, when denoting by $x_1, \ldots, x_{2n}$ its zeroes in $[0, 2\pi[$, we have

$$d(x_i, x_{i+1}) = \begin{cases} C_1 = C_1(\mu) \text{ constant } \forall i \text{ such that } w(x) > 0 \text{ in }]x_i, x_{i+1}[\\ C_2 = C_2(\nu) \text{ constant } \forall i \text{ such that } w(x) < 0 \text{ in }]x_i, x_{i+1}[\end{cases},$$

with, possibly, $C_1 = C_2$ if $\mu = \nu$ (see [11]).

Remark 2.9. The number of zeros (in $[0, 2\pi[$) of solutions of (2.34) is constant when $(\mu, \nu) \in C_k$, and this number increases by two when going from C_k to C_{k+1} (see [11]).

Proof of Lemma 2.5. Consider now a solution v of (2.21), where the following inequalities hold:

$$\mu_k < \mu(x) < \mu_{k+1}\,, \quad \nu_k < \nu(x) < \nu_{k+1}\,, \quad k \geqq 1$$

with $(\mu_k, \nu_k) \in C_k$, $(\mu_{k+1}, \nu_{k+1}) \in C_{k+1}$. The solutions of (2.34) with coefficients on C_k have $2k$ zeros in $[0, 2\pi[$ and those with coefficients on C_{k+1} have $(2k + 2)$ zeros in $[0, 2\pi[$ (see [11]).

Let $]x_i, x_{i+1}[$ be an interval where v is a positive solution of

$$(2.37)\qquad -v''(x) = \mu(x)\, v(x)\,, \quad v(x_i) = v(x_{i+1}) = 0\,.$$

Using well-known results about *eigenvalue problems with weight* (see Manes-Micheletti [10]) one gets

$$(2.38)\qquad 1 = \lambda_1(\mu(x)) = \inf_{|||v|||=1} \frac{\int_{x_i}^{x_{i+1}} (v'(x))^2\, dx}{\int_{x_i}^{x_{i+1}} \mu(x)\, v^2(x)\, dx}\,.$$

On the other hand, by Remark 2.6 we know that it is possible to translate the equation (2.34) in such a way that x_i becomes a zero of the solution w of (2.34) from which the solution starts to be positive.

Let $]x_i, b_k[$ be an interval in which $w(x) > 0$ is a solution of (2.35) with $x_i = a$, $b = b_k$ and $\mu = \mu_k$. We prove that $b_k > x_{i+1}$. Suppose, by contradiction, that $b_k \leqq x_{i+1}$; by Remark 2.7 w is a positive solution of

$$-w''(x) = \mu\, w(x)\,, \quad w(x_i) = w(x_{i+1}) = 0$$

with $\tilde{\mu} \leqq \mu_k < \mu(x) \quad \forall x \in \mathbf{I}$.

By (2.37) we get

$$(2.39) \qquad 1 = \inf_{|||v|||=1} \frac{\displaystyle\int_{x_i}^{x_{i+1}} (v')^2\, dx}{\displaystyle\int_{x_i}^{x_{i+1}} \tilde{\mu} v^2\, dx} > \lambda_1(\mu(x)) = 1\,,$$

which is a contradiction. Therefore necessarily $b_k > x_{i+1}$. In a similar way we can prove that $x_{i+1} > b_{k+1}$, where as before, $]x_i, b_{k+1}[$ is an interval in which $w(x) > 0$ is a solution of (2.35) with $x_i = a$, $b = b_{k+1}$ and $\mu = \mu_{k+1}$. In fact in this case we can use reverse inequalities, that is (2.39) with $\tilde{\mu} \geqq \mu_{k+1} > \mu(x)$ (see Remark 2.7).

Similar arguments hold in those intervals where v is negative. From these remarks it turns out that v cannot be 2π-periodic. In fact, as the distance between its consecutive zeros is always larger than the distance between the zeros of solutions of (2.34) with $\mu = \mu_k$, $\nu = \nu_k$, v cannot have $2k$ zeros; in the same way it cannot have $(2k + 2)$ zeros because the distance of its nodes is always less than that between the zeros of the solution of (2.34) when $\mu = \mu_{k+1}$, $\nu = \nu_{k+1}$. Obviously it cannot have $(2k + 1)$ zeros and be 2π-periodic.

To complete the proof we only have to consider the cases $\mu(x) \leqq -\eta$, $\nu(x) \leqq -\eta$ and $k = 0$.

Let us first consider the case $k = 0$. As before we prove that the distance between the zeros of v must be less than the distance between the zeros of (2.34) with $\mu = \mu_1$, $\nu = \nu_1$ which is a contradiction with v being 2π-periodic. Let us now suppose $\mu(x) \leqq$ $\leqq -\eta$, $\nu(x) \leqq -\eta$. Then there are no nontrivial 2π-periodic solutions of (2.21) because of *the maximum principle.* ∎

3. PROOF OF THEOREM 2

First we shall formulate some useful properties of the linear beam equation

$$(3.1) \qquad \beta u_t + u_{tt} + u_{xxxx} - \lambda u = h(t, x)\,.$$

Lemma 3.1. *Let $\beta > 0$ and λ be real constants. Then (3.1) has for an arbitrary $h \in \mathbf{H}$ a unique* GPS *if and only if $\lambda \neq m^4$, $m \in \mathbb{N} \cup \{0\}$. The mapping*

$$\boldsymbol{T}_\lambda \colon \mathbf{H} \to \mathbf{H}\,, \quad h \mapsto u\,,$$

where u is the unique GPS *of (3.1) with the right-hand side $h \in \mathbf{H}$, is linear and compact. Moreover,* $\operatorname{Im} \boldsymbol{T}_\lambda \subset \mathbf{C}_{2\pi}$ *and the mappings*

$$\boldsymbol{T}_\lambda \colon \mathbf{H} \to \mathbf{C}_{2\pi}\,, \quad \boldsymbol{T}_\lambda|_{C_{2\pi}^{0,0}} \colon \mathbf{C}_{2\pi} \to \mathbf{C}_{2\pi}$$

are compact. If $p, r \in \mathbb{N} \cup \{0\}$ *then*

$$T_\lambda(\mathbf{H}^{p,r}) \subset \mathbf{H}^{p+1,r+2}\,.$$

Remark 3.1. For the proof of Lemma 3.1 see [2]. ∎

The idea of the proof of Theorem 2 is the same as that of the proof of Theorem 1. Let us choose again a sufficiently small $0 < \varepsilon < 1$. Then $u \in \mathbf{C}_{2\pi}$ is a GPS of (1.2) if and only if

$$u = T_\varepsilon(g + \psi(t, x, u(t, x), u(t_0, x)) - \varepsilon u)\,. \tag{3.2}$$

According to Lemma 3.1 and the growth restrictions (1.3) the operator

$$u \mapsto T_\varepsilon(g + \psi(t, x, u(t, x), u(t_0, x)) - \varepsilon u)$$

is a completely continuous mapping from $\mathbf{C}_{2\pi}$ into $\mathbf{C}_{2\pi}$. Choose $(\mu, \nu) \in \mathbb{R}^2$ such that (1.9) or (1.9'), holds with $\psi_{+\infty}, \psi_{-\infty}$, replaced by μ, ν, respectively. Let us define the homotopy $\mathfrak{H}$ in the same way as in (2.3). To show that *the Leray-Schauder degree* (2.4) is well defined we proceed again via contradiction and obtain the existence of such $v \in \mathbf{C}_{2\pi}$ and $\tau \in]0, 1]$ that $|||v||| = 1$ and

$$v - T_\varepsilon((\mu(x) - \varepsilon)\, v^+ - (\nu(x) - \varepsilon)\, v^-) = 0\,, \tag{3.3}$$

where $\mu(x), \nu(x)$ are bounded measurable functions such that either

$$k^4 + \eta \leqq \mu(x)\,, \quad \nu(x) \leqq (k+1)^4 - \eta \tag{$\overline{1.9}$}$$

with some $k \in \mathbb{N} \cup \{0\}$, or

$$\mu(x), \nu(x) \leqq -\eta \tag{$\overline{1.9'}$}$$

for $x \in \mathbf{I}$. We want to prove that $|||v||| = 1$ and (3.3) do not hold simultaneously. First we prove the following lemma.

Lemma 3.2. *The solution* $v \in \mathbf{C}_{2\pi}$ *of* (3.3) *is independent of* t *and the function* $\tilde{v}(x) = v(t, x)$ *is the solution in the sense of Carathéodory of the periodic problem for the equation*

$$u^{(\mathrm{IV})} = \mu(x)\, u^+ - \nu(x)\, u^- \tag{3.4}$$

on the interval $\mathbf{I}$.

Proof. The idea is quite similar to that of the proof of Lemma 2.2. Let us take $\{\mu_n(x)\}_{n=1}^\infty$, $\{\nu_n(x)\}_{n=1}^\infty$, sequences of infinitely differentiable 2π-periodic functions such that $\mu_n \to \mu$, $\nu_n \to \nu$ in $\mathbf{L}^2(\mathbf{I})$, and define $\{v_n\}_{n=1}^\infty$ by the relation

$$v_n = T_\varepsilon((\mu_n(x) - \varepsilon)\, v^+ - (\nu_n(x) - \varepsilon)\, v^-)\,. \tag{3.5}$$

Then obviously (with respect to Lemma 3.1) $v_n \to v$ in $\mathbf{C}_{2\pi}$, $v_n \in \mathbf{H}^{1,1}$ and hence $[(\mu_n(x) - \varepsilon)\, v^+ - (\nu_n(x) - \varepsilon)\, v^-] \in \mathbf{H}^{1,1}$ for all $n \in \mathbb{N}$. Applying once more Lemma 3.1 we obtain that $v_n \in \mathbf{H}^{2,3}$ for all $n \in \mathbb{N}$. Writting (3.5), using the definition of GPS and integrating by parts we obtain

$$(3.6) \qquad (\beta(v_n)_t + (v_n)_{tt}, w) - ((v_n)_{xxx}, w_x) - \varepsilon(v_n, w) =$$
$$= ((\mu_n(x) - \varepsilon)\, v^+, w) - ((v_n(x) - \varepsilon)\, v^-, w)\,, \quad w \in \mathbf{C}^{2,4}_{2\pi}\,.$$

Since the set $\mathbf{C}^{2,4}_{2\pi}$ is dense in $\mathbf{H}$, the relation (3.6) remains valid also for $w = v_{n_t}$. We have

$$\beta\|v_{n_t}\|^2_{\mathbf{H}} = \int_{\mathbf{I}^2} (\mu_n(x) - \varepsilon)\, v^+ v_{n_t}\, dt\, dx - \int_{\mathbf{I}^2} (v_n(x) - \varepsilon)\, v^- v_{n_t}\, dt\, dx\,.$$

From this equality, exactly as in the proof of Lemma 2.4, we first derive that $\|v_{n_t}\|_{\mathbf{H}} \to \to 0$ for $u \to \infty$, and then $\|v_t\|_{\mathbf{H}} = 0$. If we write (3.3) using the definition of GPS, we obtain that the function $\tilde{v}(x) = v(t, x)$ satisfies

$$\int_{\mathbf{I}} \tilde{v}_{xx} \cdot w_{xx}\, dx = \int_{\mathbf{I}} (\mu(x)\, \tilde{v}^+ - v(x)\, \tilde{v}^-)\, w\, dx$$

for all $w \in \mathbf{C}^4_2[0, 2\pi]$. Using the standard regularity argument for ordinary differential equations we obtain that $\tilde{v}$ is the solution in *the sense of Carathéodory* of (3.4). This completes the proof of Lemma 3.2. ■

Let us define linear operator $\boldsymbol{L}\colon \mathbf{W}^{2,2}_{2\pi}(\mathbf{I}) \to \mathbf{W}^{2,2}_{2\pi}(\mathbf{I})$, $\boldsymbol{S}\colon \mathbf{W}^{2,2}_{2\pi}(\mathbf{I}) \to \mathbf{W}^{2,2}_{2\pi}(\mathbf{I})$ by the relations

$$(\boldsymbol{L}u, v)_{\mathbf{W}^{2,2}_{2\pi}} = \int_0^{2\pi} u''v''\, dx\,, \quad (\boldsymbol{S}(u), v)_{\mathbf{W}^{2,2}_{2\pi}} = \int_0^{2\pi} uv\, dx\,,$$

for $u, v \in \mathbf{W}^{2,2}_{2\pi}(\mathbf{I})$, where $\mathbf{W}^{2,2}_{2\pi}(\mathbf{I})$ is *the Sobolev space of periodic functions* the first derivatives of which are completely continuous and the second ones are square integrable over $(\mathbf{I})$ *). It is easy to see that the operator $\boldsymbol{S}$ is completely continuous and that the eigenvalues of the eigenvalue problem

$$\boldsymbol{L}u - \lambda \boldsymbol{S}(u) = 0$$

are $\lambda_n = n^4$, $n \in \mathbb{N} \cup \{0\}$. From the theory of linear completely continuous operators (see e.g. [7]) we obtain that for an arbitrary $\lambda \in \mathbb{R}$,

$$(3.7) \qquad \|\boldsymbol{L}u - \lambda \boldsymbol{S}(u)\|_{\mathbf{W}^{2,2}_{2\pi}} \geqq \min_{n \in N \cup \{0\}} \operatorname{dist}(\lambda, n^4)\, \|u\|_{\mathbf{W}^{2,2}_{2\pi}}\,.$$

Let us take $\bar{\lambda} = ((k + 1)^4 - k^4)/2$ if $\mu(x)$, $v(x)$ satisfy $\overline{(1.9)}$. Then according to Lemma 3.2,

$$\tilde{v}^{(\mathrm{IV})} - \bar{\lambda}\tilde{v} = (\mu(x) - \bar{\lambda})\, \tilde{v}^+ - (v(x) - \bar{\lambda})\, \tilde{v}^-$$

in *the sense of Carathéodory*. Hence

$$(3.8) \qquad \int_0^{2\pi} \tilde{v}''u''\, dt - \bar{\lambda} \int_0^{2\pi} \tilde{v}u\, dt = \int_0^{2\pi} (\mu(x) - \bar{\lambda})\, \tilde{v}^+ u\, dt - \int_0^{2\pi} (v(x) - \bar{\lambda})\, \tilde{v}^- u\, dt$$

*) Equipped with the norm

$$\|u\|_{2,2} = (u, u)^{\frac{1}{2}}_{\mathbf{W}^{2,2}_{2\pi}} = \int_0^{2\pi} u''u''\, dt + \int_0^{2\pi} uu\, dt\,.$$

for all $u \in \mathbf{W}^{2,2}_{2\pi}(0, 2\pi)$. But according to (3.7) we have

$$(3.9) \qquad \sup_{\|u\|_{2,2} \leqq 1} \left[\int_0^{2\pi} \tilde{v}'' u''\, dt - \bar{\lambda} \int_0^{2\pi} \tilde{v} u\, dt \right] \geqq \frac{(k+1)^4 - k^4}{2} \|\tilde{v}\|_{2,2}\,.$$

On the other hand, since $\mu(x)$, $\nu(x)$ satisfy $\overline{(1.9)}$ we have

$$(3.10) \qquad \sup_{\|u\|_{2,2} \leqq 1} \left[\int_0^{2\pi} (\mu(x) - \bar{\lambda})\, \tilde{v}^+ u\, dt - \int_0^{2\pi} (\nu(x) - \bar{\lambda})\, \tilde{v}^- u\, dt \right] \leqq$$

$$\leqq \max \{ \sup_{x \in [0, 2\pi]} |\mu(x) - \bar{\lambda}|,\ \sup_{x \in [0, 2\pi]} |\nu(x) - \bar{\lambda}| \} \|\tilde{v}\|_{2,2} \leqq$$

$$\leqq \left[\frac{(k+1)^4 - k^4}{2} - \eta \right] \|\tilde{v}\|_{2,2}\,,$$

where $\eta > 0$. Then (3.9) and (3.10) contradict (3.8) if $\|\tilde{v}\|_{2,2} \neq 0$. This means that $|||v||| = 1$ and (3.3) do not hold simultaneously. The same argument leads to a contradiction if $\mu(x)$, $\nu(x)$ satisfy $\overline{(1.9')}$. Hence the homotopy $\mathfrak{H}$ is well defined and we have

$$(3.11) \qquad \deg(u - \boldsymbol{T}_\varepsilon(g - \varepsilon u + \psi(t, x, u(t, x), u(t_0, x)); \mathbf{B}_R(0), 0) =$$

$$= \deg(\mathfrak{H}(1, u); \mathbf{B}_R(0), 0) = \deg(\mathfrak{H}(0, u); \mathbf{B}_R(0), 0) =$$

$$= \deg(u - \boldsymbol{T}_\varepsilon((\mu - \varepsilon)\, u^+ - (\nu - \varepsilon)\, u^-); \mathbf{B}_R(0), 0)\,.$$

But according to the choice of μ, ν and because ε was chosen sufficiently small we have

$$k^4 < \mu - \varepsilon < (k+1)^4\,, \quad k^4 < \nu - \varepsilon < (k+1)^4\,.$$

Using *the homotopy invariance property* once again (see [2]) we obtain

$$\deg(u - \boldsymbol{T}_\varepsilon((\mu - \varepsilon)\, u^+ - (\nu - \varepsilon)\, u^-); \mathbf{B}_R(0), 0) =$$

$$= \deg(u - \boldsymbol{T}_\varepsilon(\lambda u); \mathbf{B}_R(0), 0) = \pm 1 \quad \text{with} \quad k^4 < \lambda < (k+1)^4\,.$$

Then the existence of at least one GPS of (1.2) follows from *the basic property of the Leray-Schauder degree.* ■

4. FINAL REMARKS

Remark 4.1. In the proofs of our main results it was essential that the limit equation (2.20) or (3.3) had a solution which was indendent of t in order to justify the application of *the shooting method* which is *typical for ordinary differential equations.* That is why we were allowed to treat the class of nonlinearities which may oscillate only with respect to x. It would be interesting to check if the following assertions are true.

Open problem 4.1. Let us suppose that $\psi(t, x, s)$: $\mathbf{I}^2 \times \mathbb{R} \to \mathbb{R}$ is a *Carathéodory function*, satisfying the growth restriction (1.3) and such that

$$m^2 < C_1 < \limsup_{s \to \pm\infty} \frac{\psi(t, x, s)}{s} < C_2 < (m + 1)^2\,,$$

$$m^2 < C_1 < \liminf_{s \to \pm\infty} \frac{\psi(t, x, s)}{s} < C_2 < (m + 1)^2$$

for a.a. $(t, x) \in \mathbf{I}^2$ with some $m \in \mathbb{N} \cup \{0\}$. Then (1.1) with $\psi(t, x, u(t, x), u(t_0, x))$ replaced by $\psi(t, x, u(t, x))$ has at least one GPS for each right-hand side $g \in \mathbf{H}$.

Open problem 4.2. Let us suppose that a *Carathéodory function* $\psi(t, x, s)$: $\mathbf{I}^2 \times \times \mathbb{R} \to \mathbb{R}$ satisfies (1.3) and

$$m^4 < C_1 < \limsup_{s \to \pm\infty} \frac{\psi(t, x, s)}{s} < C_2 < (m + 1)^4,$$

$$m^4 < C_1 < \liminf_{s \to \pm\infty} \frac{\psi(t, x, s)}{s} < C_2 < (m + 1)^4$$

for a.a. $(t, x) \in \mathbf{I}^2$, with some $m \in \mathbb{N} \cup \{0\}$. Then (1.2) with $\psi(t, x, u(t, x), u(t_0, x))$ replaced by $\psi(t, x, u(t, x))$ has at least one GPS for an arbitrary right-hand side $g \in \mathbf{H}$.

Remark 4.2. According to the authors' knowledge, *neither the affirmative nor the negative answer* to the above problems has yet been published.

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