Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 36 (1986), No. 4, 617–622

Persistent URL: http://dml.cz/dmlcz/102120

Czechoslovak Mathematical Journal, 36 (111) 1986, Praha

TOLERANCE DISTRIBUTIVE AND TOLERANCE BOOLEAN VARIETIES OF SEMIGROUPS

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(Received July 23, 1985)

The aim of this paper consists in a characterization of varieties of semigroups whose tolerance lattices are distributive or boolean. The present result generalizes some results from [1] to arbitrary semigroups.

Recall that a tolerance on a semigroup S is a reflexive and symmetric subsemigroup of the direct product $S \times S$. By $LT(S)$ we denote the lattice of all tolerances on S with respect to set inclusion (see [2] and [3]). A semigroup S is said to be *tolerance distributive* (*boolean*) if the lattice $LT(S)$ is *distributive* (*boolean*). A variety V of semigroups is called *tolerance distributive* (*boolean*) if each S from V is tolerance distributive (boolean) (see [4]).

Terminology and notation not defined here may be found in [5] and [6].

Denote by $\vee$ or $\wedge$ the join or meet in $LT(S)$, respectively. The meet evidently coincides with set intersection. For $M \subseteq S \times S$, we denote by $T(M)$ the least tolerance on S containing M. It is easy to show the following:

(1) $(x, y) \in T(M)$ if and only if $x = x_1 x_2 \ldots x_m$ and $y = y_1 y_2 \ldots y_m$, where either $(x_i, y_i) \in M$ or $(y_i, x_i) \in M$ or $x_i = y_i \in S$ for $i = 1, 2, \ldots, m$;

(2) $$A \vee B = T(A \cup B) \quad \text{for any} \quad A, B \in LT(S).$$

By $W(i_1 = i_2)$ we denote the variety of all semigroups satisfying the identity $i_1 = i_2$.

Theorem 1. *A variety V of semigroups is tolerance distributive if and only if V is a subvariety of $W(xyz = xz)$.*

First, we shall prove the following lemmas.

Lemma 1. *The variety $W(xyz = xz)$ is tolerance distributive.*

Proof. Suppose that S is a semigroup from $W(xyz = xz)$, which is not tolerance distributive. Then there exist $A, B, C \in LT(S)$ such that $(A \wedge C) \vee (B \wedge C) \neq (A \vee B) \wedge C$. Since $(A \wedge C) \vee (B \wedge C) \subseteq (A \vee B) \wedge C$, there exists $(u, v) \in (A \vee B) \wedge C$ such that $(u, v) \notin (A \wedge C) \vee (B \wedge C)$. By (1) and (2) we have $u = u_1 u_2 \ldots u_m$ and $v = v_1 v_2 \ldots v_m$, where $(u_i, v_i) \in A \cup B$ and $m \geqq 2$. We have

$S \in W(xyz = xz)$ and so we can suppose (without loss of generality) that $u = u_1 u_m$ and $v = v_1 v_m$, where $(u_1, v_1) \in A$ and $(u_m, v_m) \in B$. Then we obtain $(u_1 u_m, v_1 u_m) = (u, v)(u_m, u_m) \in A \wedge C$ and $(u_1 u_m, u_1 v_m) = (u_1, u_1)(u, v) \in B \wedge C$. It follows from (2) that $(u, v) = (u_1 u_m, v_1 u_m)(u_1 u_m, u_1 v_m) \in (A \wedge C) \vee (B \wedge C)$, which is a contradiction.

Lemma 2. *Let $\{a, b, c, p, q, 0\}$ be a semigroup with the multiplication table*

	a	b	c
a	0	p	0
b	0	0	q
c	0	0	0

and $xy = 0 = yx$ for $x \in P$ and $y \in \{p, q, 0\}$. Then the lattice $LT(P)$ is not modular.

Proof. Clearly we have $(xy)\,z = x(yz)$ for all $x, y\ z \in P$. It is easy to show that $D = \{(p, 0), (0, p), (q, 0), (0, q)\} \cup \mathrm{id}_P \in LT(P)$, $A = \{(a, b), (b, a)\} \cup D \in LT(P)$, $B = \{(b, c), (c, b)\} \cup D \in LT(P)$, and $C = \{(p, q), (q, p)\} \cup A \in LT(P)$. It follows from (2) that $(p, q) = (a, b)(b, c) \in (A \vee B) \wedge C$. We have $(p, q) \notin A = A \vee D = A \vee (B \wedge C)$ and so the lattice $LT(P)$ is not modular.

Lemma 3. *Let $Q = \{a, e, f, g\}$ be a semigroup with the multiplication table*

	a	e	f	g
a	e	e	f	g
e	e	e	f	g
f	g	e	f	g
g	e	e	f	g

Then the lattice $LT(Q \times Q \times Q)$ is not modular.

Proof. It is easy to show that

$$(3) \qquad Q \in W(xyz = yz)\,.$$

Put $A = T((e, f, e), (f, e, e))$, $B = T((a, a, e), (a, a, f))$, and $C = T((e, g, e), (g, e, f)) \vee A$. It follows from (1) and (2) that $(u, v) = ((e, g, e), (g, e, f)) = ((e, f, e), (f, e, e))((a, a, e), (a, a, f)) \in (A \vee B) \wedge C$.

First we shall prove the following implication. Let $s, t \in Q \times Q \times Q$.

$$(4) \qquad \text{If} \quad u \in (Q \times Q \times Q)\,s \quad \text{and} \quad v \in (Q \times Q \times Q)\,t\,,$$

then $s \neq t$.

Suppose that $u, v \in (Q \times Q \times Q)\,s$. It is clear that $s \in Q \times Q \times \{w\}$ for some $w \in Q$. If $w = a$, then $f \in Qa$, which is impossible. If $w \neq a$, then $e = w = f$, a contradiction.

Now, we shall show that $(u, v) \notin A \vee (B \wedge C)$. On the contrary, suppose that $(u, v) \in A \vee (B \wedge C)$. Then by (1), (2) and (3) we have $u = u^2 = u_1 u_2$ and $v = v^2 = v_1 v_2$, where $(u_i, v_i) \in A \cup (B \wedge C)$ for $i = 1, 2$. We have $(u_2, v_2) \in C$. It follows from (4) that $u_2 \neq v_2$. Then $u_2 = u_2^2$, $v_2 = v_2^2$ and according to (3), we have $(u, v) = (u_2, v_2) \in A \cup B$.

Case 1. $(u, v) \in A$. It follows from (1), (3) and (4) that $(e, g, e) = u \in \{(e, f, e), (f, e, e)\}$, a contradiction.

Case 2. $(u, v) \in B$. According to (1), (3) and (4) we have $(u, v) = (s, t)((a, a, e), (a, a, f))$, where either $(s, t) = ((a, a, e), (a, a, f))$ or $(s, t) = ((a, a, f), (a, a, e))$ or $s = t \in Q \times Q \times Q$, a contradiction.

Therefore the lattice $LT(Q \times Q \times Q)$ is not modular.

Proof of Theorem 1. Let V be a tolerance distributive variety of semigroups. By Lemma 1 it suffices to show that V is a subvariety of $W(xyz = xz)$.

It follows from Theorem 2 of [1] that

(5) every commutative semigroup from V is zero.

Let S be an arbitrary semigroup from V. Let $u \in S$. By X we denote the subsemigroup of S generated by u. Clearly X is commutative and belongs to V. It follows from (5) that $u^2 = u^3$. We have

(6) $$V \subseteqq W(x^2 = x^3).$$

Let $u, v \in S$. By Y we denote the subsemigroup of S generated by u^2, u^2vu^2. It follows from (6) that Y is commutative. Since $Y \in V$, according to (5), Y is zero. Therefore $u^2vu^2 = (u^2vu^2)\,u^2 = u^2u^2 = u^2$. We have

(7) $$V \subseteqq W(x^2yx^2 = x^2).$$

This implies $(x^2y)^2 = x^2yx^2y = x^2y$ and so

(8) $$V \subseteqq W((x^2y)^2 = x^2y).$$

Dually we can get

(9) $$V \subseteqq W((yx^2)^2 = yx^2).$$

For any semigroup S we denote by $E(S)$ the set of all idempotents of S. According to (8) and (9), we have the following:

(10) *Let S be a semigroup from V. Then $E(S)$ is an ideal of S.*

Now, we shall show that

(11) $$V \subseteqq W((xy)^2 = xy).$$

Suppose that there exist elements u, v belonging to a semigroup S from V such that $uv \neq (uv)^2$. By U we denote the subsemigroup of S generated by u, v. Clearly $U \in V$. If $uv = vu$, then U is commutative and so, by (5), U is zero, which is a contradiction. Thus we have $uv \neq vu$. Put $I = \{vu, uvu, vuv\} \cup E(U)$. According to (6) and (10), I is an ideal of U. By (6) and (10) it is easy to show that $u \neq v \neq uv \neq u$ and $u, v, uv \notin I$. This implies that the Rees quotient $R = U/I$ has exactly four elements u, v, uv and 0.

We shall define a mapping $\varphi: P \to R \times R$, where P is the semigroup from Lemma 2. Let us put $\varphi(a) = (u, 0)$, $\varphi(b) = (v, u)$, $\varphi(c) = (0, v)$, $\varphi(p) = (uv, 0)$, $\varphi(q) = (0, uv)$, and $\varphi(0) = (0, 0)$. It is easy to show that φ is an isomorphism. Since $R \times R \in V$, we have $P \in V$, which contradicts Lemma 2. Therefore (11) is true.

Further, we shall prove that

$$(12) \qquad V \subseteq W(x^2yx = x^2).$$

Suppose that there exist elements u, v belonging to a semigroup S from V such that $u^2vu \neq u^2$. Define a mapping $\varphi: Q \to S$, where Q is the semigroup from Lemma 3. Put $\varphi(a) = u$, $\varphi(e) = u^2$, $\varphi(f) = u^2v$ and $\varphi(g) = u^2vu$. Using (6) and (7) it is easy to show that φ is a homomorphism. We shall prove that φ is injective. If $u = u^2$, then by (7) we have $u^2 = u^2vu^2 = u^2vu$, a contradiction. If $u = u^2v$, then $u^2 = u^2vu$, a contradiction. If $u = u^2vu$, then by (6) we have $u^2 = u^3vu = u^2vu$, a contradiction. If $u^2 = u^2v$, then by (6) we have $u^2 = u^3 = u^2vu$, a contradiction. If $u^2v = u^2vu$, then by (7) we have $u^2vu = u^2vu^2 = u^2$, a contradiction. Since φ is an isomorphism, we obtain that $Q \in V$, which is a contradiction (see Lemma 3). Therefore (12) is satisfied.

Dually we can get

$$(13) \qquad V \subseteq W(xyx^2 = x^2).$$

Finally, we shall show that $V \subseteq W(xyz = xz)$. Using (7), (11), (12) and (13) we obtain $xyz = (xyz)^2 = (xy)(zx)(yz) = (xy)^2(zx)(yz)^2 = (xy)^2(x^2z^2)(yz)^2(xy)^2 \,.\, (zx)(yz)^2 = (xy)^2(x^2z^2)(yz)^2 = (xy)(x^2z^2)(yz) = (xyx^2)(z^2yz) = x^2z^2 = (xzx^2)(z^2xz) = (xz)(x^2z^2)(xz) = (xz)^2(x^2z^2)(xz)^2 = (xz)^2 = xz$.

Theorem 2. *A variety V of semigroups is tolerance boolean if and only if either $V = W(x_1y_1 = x_ky_2)$ or V is a variety of rectangular bands.*

Note that the variety of all rectangular bands is $RB = W(x^2 = x) \cap W(xyx = x)$. It is well known (see [7]) that the only non-trivial and proper subvariety of RB is either $W(xy = x)$ or $W(xy = y)$.

Before the proof we formulate three lemmas.

Lemma 4. *The variety RB is tolerance boolean.*

Proof. First we shall show

$$(14) \qquad RB \subseteq W(xyz = xz).$$

Indeed, we have $xyz = xy(zxz) = x(yz)xz = xz$.

Let S be a rectangular band. It follows from (14) and Lemma 1 that the lattice $LT(S)$ is distributive. We shall prove that it is boolean. Let $A \in LT(S)$. Choose $e \in S$ and put $B = T((Se \times Se \cup eS \times eS) \setminus A)$.

Let $u, v \in S$. According to (14), we have $(u, v) = (ue, ve)(eu, ev)$. Clearly (ue, ve), $(eu, ev) \in A \cup B$. It follows from (1) and (2) that $(u, v) \in A \vee B$. Therefore $A \vee B = S \times S$.

Suppose that $A \wedge B \neq \mathrm{id}_S$. Then there exist $u, v \in S$ such that $(u, v) \in A \cap B$

and $u \neq v$. According to (1), (2) and (14), we have $(u, v) = (u_1, v_1)(u_2, v_2)$, where either $u_1 = v_1$ or $(u_1, v_1) \in (Se \times Se \cup eS \times eS) \setminus A$ and either $u_2 = v_2$ or $(u_2, v_2) \in \in (Se \times Se \cup eS \times eS) \setminus A$. If $(u_1, v_1) \in Se \times Se \setminus A$, then by (14) we obtain $(u_1, v_1) = (u_1 e, v_1 e) = (u_1, v_1)(u_2, v_2)(e, e) = (u, v)(e, e) \in A$, which is a contradiction. Thus we have $(u_1, v_1) \notin Se \times Se \setminus A$. Dually we obtain that $(u_2, v_2) \notin eS \times \times eS \setminus A$. Consequently we have the following possibilities:

Case 1. $u_1 = v_1$. Then $(u_2, v_2) \in Se \times Se \setminus A$ and so by (14) we have $u = u_1 e = = v_1 e = v$, a contradiction.

Case 2. $u_2 = v_2$. Then dually we obtain a contradiction.

Case 3. $(u_1, v_1) \in eS \times eS$ and $(u_2, v_2) \in Se \times Se$. According to (14) we have $u = u_1 u_2 = e = v_1 v_2 = v$, a contradiction.

Therefore $A \wedge B = \mathrm{id}_S$. Consequently, the lattice $LT(S)$ is boolean.

Lemma 5. *The variety $ZS = W(x_1 y_1 = x_2 y_2)$ is tolerance boolean.*

Proof. It follows from Theorem 1 that ZS is a tolerance distributive variety of semigroups. Let $S \in ZS$. Clearly S is a zero-semigroup. Let $A \in LT(S)$. Put $B = = T(S \times S \setminus A) = (S \times S \setminus A) \cup \mathrm{id}_S$. We have $A \wedge B = \mathrm{id}_S$ and $A \vee B = = S \times S$. Therefore ZS is tolerance boolean.

Lemma 6. *Let $P = \{a, e, f\}$ be a semigroup with the multiplication table*

	a	e	f
a	e	e	e
e	e	e	e
f	f	f	f

Then the lattice $LT(P)$ is not boolean.

Proof. It is easy to show that P is a semigroup and $A = \{(a, e), (e, a), (e, f), (f, e)\} \cup \mathrm{id}_P \in LT(P)$. Suppose that there exists $B \in LT(P)$ such that

$$(15) \qquad A \wedge B = \mathrm{id}_P \quad \text{and} \quad A \vee B = P \times P\,.$$

If $(a, f) \in B$, then $(e, f) = (a, f)^2 \in A \wedge B$, which is a contradiction. It follows from (15) that $B = \mathrm{id}_P$ and so $A = P \times P$, a contradiction. Therefore the lattice $LT(P)$ is not bollean.

Proof of Theorem 2. Let V be a tolerance boolean variety of semigroups. By Lemma 4 and Lemma 5, it suffices to show that either $V = ZS$ or V is a subvariety of RB. According to Theorem 1, we have

$$(16) \qquad V \subseteqq W(xyz = xz)$$

and so $V \subseteqq W(xy)^2 = xy)$. Hence every semigroup from V has idempotents.

Case 1. Every semigroup from V has exactly one idempotent. Then $V \subseteqq \subseteqq W(x_1 y_1 = x_2 y_2) = ZS$. It is well known that the variety ZS is minimal and so either $V = ZS$ or V is trivial (which means that $V \subseteqq RB$).

Case 2. There is a semigroup S from V having at least two idempotents (say j and k). We shall show that

$$V \subseteqq W(x^2 = x). \tag{17}$$

Subcase 2a. $j = jk$. Then by (16) $kj = kjk = k$ and so $J = \{j, k\}$ is a subsemigroup of S. Hence we have $J \in V$. Suppose that there exists a semigroup X from V having an element $u \neq u^2$. It follows from (16) that $u^2 = u^3$ and so $U = \{u, u^2\}$ is a subsemigroup of X. Therefore we have $U \in V$ and so $J \times U \in V$. We shall define a mapping $\varphi: P \to J \times U$, where P is the semigroup of Lemma 6. Let us put $\varphi(a) = (j, u)$, $\varphi(e) = (j, u^2)$ and $\varphi(f) = (k, u^2)$. It is easy to show that φ is an isomorphism. This implies that $P \in V$, which contradicts Lemma 6. Therefore we have (17).

Subcase 2b. $j \neq jk$. Put $h = jk$. It follows from (16) that $h^2 = j(kj)\,k = jk = h$ and $j = jkj = hj$. This implies that our subcase ($j = hj$) is dual to Subcase 2a. Consequently, we obtain (17).

According to (16) and (17), we have $V \subseteqq RB$.

Corollary. *For a non-trivial variety V of commutative semigroups the following conditions are equivalent:*

1. *V is tolerance distributive,*
2. *V is tolerance boolean,*
3. *V is the variety of all zero-semigroups.*

This follows from Theorem 2 and Theorem 2 of [1].

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