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PRIME IDEALS IN AUTOMETRIZED ALGEBRAS

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A system $(A, +, \leqq, *)$ is called an *autometrized algebra* if
(1) $(A, +)$ is a commutative semigroup with zero element 0;
(2) $(A, \leqq)$ is an ordered set and
$\forall a, b, c \in A;\ a \leqq b \Rightarrow a + c \leqq b + c;$
(3) $*\colon A \times A \to A$ is a mapping such that
$\forall a, b \in A;\ a * b \geqq 0$ and $a * b = 0 \Leftrightarrow a = b,$
$\forall a, b \in A;\ a * b = b * a,$
$\forall a, b, c \in A;\ a * c \leqq (a * b) + (b * c).$

An autometrized algebra $(A, +, \leqq, *)$ is called
a) an *l-algebra* if $(A, \leqq)$ is a lattice and
$\forall a, b, c \in A;\ a + (b \vee c) = (a + b) \vee (a + c),$
$a + (b \wedge c) = (a + b) \wedge (a + c);$
b) *semiregular* if
$\forall a \in A;\ a \geqq 0 \Rightarrow a * 0 = a;$
c) *normal* if
$\forall a \in A;\ a \leqq a * 0,$
$\forall a, b, c, d \in A;\ (a + c) * (b + d) \leqq (a * b) + (c * d),$
$\forall a, b, c, d \in A;\ (a * c) * (b * d) \leqq (a * b) + (c * d),$
$\forall a, b \in A;\ (a \leqq b \Rightarrow \exists x \geqq 0;\ a + x = b).$

Note. Every commutative *DRl*-semigroup (for instance, every commutative *l*-group and every Brouwerian algebra) is a semiregular normal autometrized *l*-algebra.

If $A = (A, +, \leqq, *)$ is an autometrized algebra, then $\emptyset \neq I \subseteq A$ is called an *ideal* in A if and only if
$\forall a, b \in I;\ a + b \in I;$
$\forall a \in I,\ x \in A;\ x * 0 \leqq a * 0 \Rightarrow x \in I.$

The principal ideal in A generated by $a \in A$ is denoted by $I(a)$ and we have $I(a) = \{x \in A;\ x * 0 \leqq m(a * 0)$ for some $m \geqq 0\}$. Let us denote the set of all ideals in an algebra A by $\mathscr{I}(A)$. K. L. N. Swamy and N. P. Rao [2] studied the properties of ideals in normal autometrized algebras. They showed that in those algebras the ideals are exactly the kernels of homomorphisms and that each epimorphic image is isomorphic to the factor-algebra over its kernel. Moreover, in [2] it is proved that the set of all ideals in a normal autometrized algebra, ordered by set inclusion, is a complete algebraic lattice.

In the paper prime ideals in autometrized algebras are introduced and studied.

Let $A = (A, +, \leqq)$ be an ordered semigroup with zero element 0. Then A is called an *interpolation semigroup* if

$\forall a, b, c \in A;\ [0 \leqq a, b, c,\ a \leqq b + c \Rightarrow (\exists 0 \leqq b_1 \leqq b,\ 0 \leqq c_1 \leqq c;$
$a = b_1 + c_1)]$.

Note. It is clear that, for example, commutative l-groups and Brouwerian algebras are interpolation semigroups.

Lemma 1. *If* $0 \leqq a, b, c$ *are elements of an interpolation l-semigroup* A, *then* $a \wedge (b + c) \leqq (a \wedge b) + (a \wedge c)$.

Proof. Let $0 \leqq a, b, c \in A$. Then $0 \leqq a \wedge (b + c) \leqq b + c$, hence $a \wedge (b + c) = u + v$ for some $0 \leqq u \leqq b$, $0 \leqq v \leqq c$. Moreover, $u \leqq u + v \leqq a$, thus $u \leqq a \wedge b$, and similarly $v \leqq a \wedge c$, which means $a \wedge (b + c) \leqq (a \wedge b) + (a \wedge c)$.

Proposition 2. *The intersection of any two principal ideals in an interpolation semiregular autometrized l-algebra* A *is a principal ideal in* A. *If* $a, b \in A$, *then* $I(a) \cap I(b) = I((a * 0) \wedge (b * 0))$; *in particular*, $I(a) \cap I(b) = I(a \wedge b)$ *for* $0 \leqq a, b \in A$.

Proof. Since A is semiregular, $I(a) = I(a * 0)$ for each $a \in A$.

Let $a, b \in A$. Then $0 \leqq (a * 0) \wedge (b * 0) \leqq a * 0,\ b * 0$, hence $(a * 0) \wedge (b * 0) \in I(a) \cap I(b)$, therefore $I((a * 0) \wedge (b * 0)) \subseteq I(a) \cap I(b)$.

Conversely, if $x \in I(a) \cap I(b)$, then there exist $m, n \geqq 0$ such that $x * 0 \leqq m(a * 0)$, $x * 0 \leqq n(b * 0)$. By Lemma 1 we have $m(a * 0) \wedge n(b * 0) \leqq mn[(a * 0) \wedge (b * 0)]$, thus $x \in I((a * 0) \wedge (b * 0))$, i.e. $I(a) \cap I(b) \subseteq I((a * 0) \wedge (b * 0))$.

Proposition 3. *If* A *is a semiregular normal autometrized l-algebra*, $a, b \in A$, *then* $I(a) \vee I(b) = I((a * 0) \vee (b * 0)) = I((a * 0) + (b * 0))$; *in particular*, $I(a) \vee I(b) = I(a \vee b) = I(a + b)$ *for* $0 \leqq a, b \in A$.

Proof. Let $a, b \in A$. Then $0 \leqq a * 0,\ b * 0 \leqq (a * 0) \vee (b * 0)$, hence $a, b \in I((a * 0) \vee (b * 0))$, therefore $I(a) \vee I(b) \subseteq I((a * 0) \vee (b * 0))$.

Let $I \in \mathscr{I}(A)$, $I(a), I(b) \subseteq I$, $x \in I((a * 0) \vee (b * 0))$. Then there exists $m \geqq 0$ such that $x * 0 \leqq m[(a * 0) \vee (b * 0)]$. Since $a, b \in I$, $(a * 0) + (b * 0) \in I$. Moreover, $0 \leqq (a * 0) \vee (b * 0) \leqq (a * 0) + (b * 0)$, hence $(a * 0) \vee (b * 0) \in I$, thus also $m[(a * 0) \vee (b * 0)] \in I$. Therefore $x \in I$, i.e. $I(a) \vee I(b) = I((a * 0) \vee (b * 0))$.

The equality $I(a) \vee I(b) = I((a * 0) + (b * 0))$ is satisfied by [2, Lemma 3].

If A is an autometrized algebra, then we say that an ideal I in A is a *prime ideal* if

$$\forall J, K \in \mathscr{I}(A);\quad J \cap K = I \Rightarrow J = I \quad \text{or} \quad K = I\,.$$

Theorem 4. *If A is a semiregular normal autometrized l-algebra, then for $I \in \mathscr{I}(A)$ the following conditions are equivalent:*

1. *I is a prime ideal.*
2. *$\forall J, K \in \mathscr{I}(A); J \cap K \subseteq I \Rightarrow J \subseteq I$ or $K \subseteq I$.*
3. *$\forall a, b \in A$; $0 \leqq a \wedge b \in I \Rightarrow a \in I$ or $b \in I$.*

Proof. $1 \Rightarrow 2$: Let $J \cap K \subseteq I$. Then $I = I \vee (J \cap K)$, and since $\mathscr{I}(A)$ is (by [2, Lemma 6]) distributive, $I = (I \vee J) \cap (I \vee K)$. Hence $I = I \vee J$ or $I = I \vee K$, that is $J \subseteq I$ or $K \subseteq I$.

$2 \Rightarrow 3$: Let $0 \leqq a \wedge b \in I$. By Proposition 2, we have $I(a) \cap I(b) = I(a \wedge b) \subseteq I$, thus $I(a) \subseteq I$ or $I(b) \subseteq I$, and so $a \in I$ or $b \in I$.

$3 \Rightarrow 1$: Let $J, K \in \mathscr{I}(A)$, $J \cap K = I$. Let us suppose that $a \in J \setminus I$, $b \in K \setminus I$. A is semiregular, hence we can suppose $0 < a, b$. Then $0 \leqq a \wedge b \leqq a, b$, thus $a \wedge b \in$ $\in J \cap K = I$, therefore $a \in I$ or $b \in I$, a contradiction. That means $J = I$ or $K = I$.

Corollary 5. *If I is a prime ideal, then*

$$\forall a, b \in A ; \quad 0 = a \wedge b \Rightarrow a \in I \quad \text{or} \quad b \in I .$$

Let us recall the notion of a dually residuated lattice ordered semigroup (*DRl*-semigroup) that has been introduced by Swamy in [1].

A system $A = (A, +, \leqq, -)$ is called a *DRl-semigroup* if

(1) $(A, +, \leqq)$ is a commutative lattice ordered semigroup with zero element 0;

(2) for each $a, b \in A$ there exists the least element $x \in A$ such that $b + x \geqq a$ (such x is denoted by $a - b$);

(3) $\forall a, b \in A$; $(a - b) \vee 0 + b \leqq a \vee b$;

(4) $\forall a \in A$; $a - a \geqq 0$.

Let us denote $a * b = (a - b) \vee (b - a)$ for $a, b \in A$. Then $(A, +, \leqq, *)$ is an autometrized algebra (see [1, Theorem 9]) which by [2] is normal and semiregular.

A *DRl*-semigroup A is called *representable* (see [3]) if $(a - b) \wedge (b - a) \leqq 0$ for each $a, b \in A$. (For example, commutative l-groups and Boolean algebras are representable *DRl*-semigroups.)

Lemma 6. *If A is a representable DRl-semigroup, $a, b \in A$, then*

$$a = (a \wedge b) + [a - (a \wedge b)] , \quad b = (a \wedge b) + [b - (a \wedge b)] ,$$
$$[a - (a \wedge b)] \wedge [b - (a \wedge b)] = 0 .$$

Proof. Let $a, b \in A$. Since $a \geqq a \wedge b$, by [1, Lemma 8] we have $[a - (a \wedge b)] +$ $+ (a \wedge b) = a$. Similarly $[b - (a \wedge b)] + (a \wedge b) = b$. Moreover, by [1, Lemma 5 and Theorem 2],

$$[a - (a \wedge b)] \wedge [b - (a \wedge b)] = [(a - a) \vee (a - b)] \wedge$$
$$\wedge [(b - a) \vee (b - b)] = (0 \wedge 0) \vee [(a - b) \wedge (b - a)] \vee$$
$$\vee [0 \wedge (b - a)] \vee [(a - b) \wedge 0] .$$

By the assumption, A is representable, hence

$$[a - (a \wedge b)] \wedge [b - (a \wedge b)] = 0 .$$

Theorem 7. *If A is a representable DRl-semigroup and I an ideal in A such that*

$$\forall a, b \in I ; \quad 0 = a \wedge b \Rightarrow a \in I \quad or \quad b \in I ,$$

then the set of all classes of the congruence corresponding to I is linearly ordered.

Proof. Let $\bar{a}, \bar{b} \in A/I$. We know that $a = (a \wedge b) + x$, $b = (a \wedge b) + y$, where $x \wedge y = 0$. Hence $x \in I$ or $y \in I$. If $x \in I$, then $\bar{a} = \overline{a \wedge b}$. We always have $\overline{a \wedge b} \leqq \bar{b}$, thus in this case $\bar{a} \leqq \bar{b}$. If $y \in I$, then similarly $\bar{b} \leqq \bar{a}$.

Theorem 8. *If $(P_i; i \in \Gamma)$ is a linearly ordered system of prime ideals in a semiregular normal autometrized algebra A, then $P = \bigcap_{i \in \Gamma} P_i$ is a prime ideal in A.*

Proof. Let $a, b \in A$, $0 \leqq a \wedge b \in P$, $a \notin P$, $b \notin P$. Then there exist $j, k \in \Gamma$ such that $a \notin P_j$, $b \notin P_k$. Let $j \leqq k$. Then $a \notin P_k$, $b \notin P_k$, a contradiction. Therefore, by Theorem 4, P is a prime ideal in A.

Corollary 9. *Every prime ideal contains a minimal prime ideal.*

Let us denote by $\mathscr{P}(A)$ the set of all prime ideals in a normal autometrized algebra A.

Theorem 10. *Let A be a semiregular interpolation normal autometrized l-algebra, $C \in \mathscr{I}(A)$. Then the mapping $\varphi: P \mapsto P \cap C$ is a bijection of the set of all prime ideals in A that do not contain C onto the set of all proper prime ideals in C. For any $K \in \mathscr{P}(C)$ we have $\varphi^{-1}(K) = \{x \in A; (x * 0) \wedge (c * 0) \in K$ for each $c \in C\}$.*

Proof. Clearly $\varphi(P) \in \mathscr{P}(C)$ for $P \in \mathscr{P}(A)$. Let $K \in \mathscr{P}(C)$. Let us denote $L = \{x \in A; (x * 0) \wedge (c * 0) \in K$ for each $c \in C\}$. Let $x, y \in L$, $c \in C$. Then the normality of the algebra A and Lemma 1 yield

$$0 \leqq [(x + y) * 0] \wedge (c * 0) \leqq [(x * 0) + (y * 0)] \wedge (c * 0) \leqq$$
$$\leqq [(x * 0) \wedge (c * 0)] + [(y * 0) \wedge (c * 0)] \in K .$$

Hence $[(x + y) * 0] \wedge (c * 0) \in K$, and thus $x + y \in L$.

Let now $x \in L$, $z \in A$, $z * 0 \leqq x * 0$, $c \in C$. Then $0 \leqq (z * 0) \wedge (c * 0) \leqq (x * 0) \wedge (c * 0) \in K$, hence $(z * 0) \wedge (c * 0) \in K$, i.e. $z \in L$. Therefore $L \in \mathscr{I}(A)$.

Let $x, y \in A$, $x \notin L$, $y \notin L$, $0 \leqq x \wedge y$. Then there exist $c_1, c_2 \in C$ such that

$$x \wedge (c_1 * 0) \notin K , \quad y \wedge (c_2 * 0) \notin K .$$

Since $K \in \mathscr{P}(C)$, we have

$$[x \wedge (c_1 * 0)] \wedge [y \wedge (c_2 * 0)] \notin K ,$$

hence $(x \wedge y) \wedge [(c_1 * 0) \wedge (c_2 * 0)] \notin K$, but this means $x \wedge y \notin L$. Thus $L \in \mathscr{P}(A)$.

Let $x \in L \cap C$. Then $(x * 0) \wedge (x * 0) \in K$, thus $x \in K$, i.e. $L \cap C \subseteq K$. The converse inclusion is evident.

Let us suppose that $P \in \mathscr{P}(A)$, P does not contain C, $P' = \{x \in A;\ (x * 0) \wedge \wedge (c * 0) \in P \cap C$ for each $c \in C\}$, $y \in P'$. Let $c_1 \in C \setminus P$. Then also $c_1 * 0 \in C \setminus P$, and since $(y * 0) \wedge (c_1 * 0) \in P$, we have $y * 0 \in P$, therefore also $y \in P$, i.e. $P' \subseteq P$. The converse inclusion is again evident.

References

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