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SOME CONVERSE THEOREMS IN THE ASYMPTOTIC THEORY OF ORDINARY DIFFERENTIAL EQUATIONS

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1. ASYMPTOTIC THEOREMS OF TRENCH

Trench recently gave sufficient conditions for a scalar differential equation

$$x^{(n)} + p_1(t)\, x^{(n-1)} + \ldots + p_n(t)\, x = 0 \tag{1}$$

to have a solution which behaves for $t \to \infty$ like a given polynomial of degree $< n$ (see [5]), and for an equation

$$x^{(n)} + [a_1 + p_1(t)]\, x^{(n-1)} + \ldots + [a_n + p_n(t)]\, x = 0 \tag{2}$$

to have a solution like $\exp(\lambda_0 t)$ asymptotically, where λ_0 is a root of the polynomial equation

$$\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n = 0 \tag{3}$$

with constant coefficients a_k (see [6]). Trench's integrability conditions on p_k are stated largely in terms of ordinary integral convergence. This presents a significant weakening of the classical conditions that require the absolute convergence ([1, Chapter X]). The aim of the present paper is to show that Trench's sufficient conditions are close to necessary.

Throughout the paper, all functions considered are complex- or real-valued and continuous on $[T, \infty)$, for some real T. In all hypotheses (conclusions), the improper integrals are assumed (concluded) to converge. The symbols "o" and "O" refer to the behavior for $t \to \infty$.

The above mentioned results of Trench imply the following two assertions on the existence of fundamental systems of solutions of (1) and (2), with prescribed asymptotic behavior.

Theorem A. *Assume that φ is positive and nonincreasing on $[T, \infty)$, ϱ is a nonnegative constant and*

$$\textstyle\int_t^\infty \varphi^2(s)\,\mathrm{d}s = O(\varphi(t)) \quad \textit{if} \quad \varrho = 0\,. \tag{4}$$

Further, assume that (3) has n distinct roots λ_j such that $\operatorname{Re}\lambda_1 \geqq \operatorname{Re}\lambda_2 \geqq \ldots$

$\ldots \geqq \operatorname{Re} \lambda_n$, *and the functions* p_k *satisfy*

$$(5) \qquad \int_t^\infty p_k(s)\, e^{\varrho s}\, ds = o(\varphi(t)) \quad (1 \leqq k \leqq n)$$

and

$$(6) \qquad \int_t^\infty |p_1(s)|\, \varphi(s)\, ds = o(\varphi(t))\,.$$

If $\operatorname{Re}(\lambda_1 - \lambda_n) > \varrho$, *assume also that* $e^{\alpha t}\, \varphi(t)$ *is nondecreasing on* $[T, \infty)$, *for some* α *smaller than any positive value from the set*

$$\{\operatorname{Re}(\lambda_j - \lambda_m) - \varrho \mid 1 \leqq j < m \leqq n\}\,.$$

Finally, assume that

$$(7) \qquad \int_t^\infty \Big(\sum_{k=1}^{n} \lambda_j^{n-k}\, p_k(s)\Big)\, e^{(\lambda_j - \lambda_m)\varrho}\, ds = o(\varphi(t))\,,$$

whenever $\operatorname{Re}(\lambda_j - \lambda_m) = \varrho$. *Then* (2) *has n solutions* x_j $(1 \leqq j \leqq n)$ *satisfying*

$$(8) \qquad x_j^{(k)}(t) = (\lambda_j^k + o(e^{-\varrho t}\, \varphi(t)))\, e^{\lambda_j t} \quad (0 \leqq k \leqq n-1)\,.$$

Theorem B. *Assume that* ψ *is positive and nonincreasing on* $[T, \infty)$, v *is a nonnegative integer and*

$$(9) \qquad \int_t^\infty \frac{\psi^2(s)}{s}\, ds = O(\psi(t)) \quad \text{if} \quad v = 0\,.$$

If $v < n - 1$, *assume also that* $t^\alpha\, \psi(t)$ *is nondecreasing on* $[T, \infty)$ *for some constant* $\alpha < 1$. *Finally, assume that the functions* p_k *satisfy*

$$(10) \qquad \int^\infty |p_1(t)|\, dt < \infty\,,$$

$$(11) \qquad \int_t^\infty p_k(s)\, s^{k-1}\, ds = o(t^{-v}\, \psi(t)) \quad (1 \leqq k \leqq n)$$

and, if $v < n$,

$$(12) \qquad \int_t^\infty g_j(s)\, s^{n-j+v-1}\, ds = o(\psi(t)) \quad (v \leqq j \leqq n-1)\,,$$

where the functions g_j *are given by*

$$(13) \qquad g_j(t) = \sum_{k=1}^{n} p_k(t)\, (t^j)^{(n-k)} \quad (v \leqq j \leqq n-1)\,.$$

Then (1) *has n solutions* x_j $(0 \leqq j \leqq n-1)$ *satisfying*

$$(14) \qquad x_j^{(k)}(t) = (t^j)^{(k)} + o(t^{j-k-v}\, \psi(t)) \quad (0 \leqq k \leqq n-1)\,.$$

Remark 1. Theorem A with $\varphi(t) = t^{-q}$ (q = const. $\geqq 0$) was essentially proved in [2]. Then (4) means that $q \geqq 1$ if $\varrho = 0$. As shown in [3], Theorem A becomes false without this restriction on ϱ and q. The case $\varrho = 0$ and $q < 1$ was discussed in [4].

Remark 2. In an unpublished work the author observed that Theorem A holds with (5) replaced by the weaker assumption

$$(15) \qquad \int_t^\infty p_k(s)\, ds = o(e^{-\varrho t}\, \varphi(t)) \quad (1 \leqq k \leqq n)\,.$$

(Integration by parts shows that (5) implies (15); the converse implication is false.)

2. THE FIRST CONVERSE THEOREM

Theorem 1. *Let φ and ϱ be as in the first sentence of Theorem* A, *including* (4). *Assume that* (3) *has n distinct roots λ_j,* (2) *has n solutions x_j satisfying* (8), *and* (6) *holds. Then the functions p_k satisfy* (15). *Moreover,* (7) *holds whenever* $\operatorname{Re}(\lambda_j - \lambda_m) = \varrho$.

It is convenient to state two preparatory lemmas separately from the proof of Theorem 1.

Lemma 1. *Let x be a function in $C^{(n)}[T, \infty)$ satisfying*

$$x^{(k)}(t) = (\lambda^k + o(e^{-\varrho t}\varphi(t))\, e^{\lambda t} \quad (0 \leqq k \leqq n-1), \tag{16}$$

where ϱ and φ are as in the first sentence of Theorem A *and λ is a constant. Then the functions*

$$h_k(t) = \frac{x^{(k)}(t)}{x(t)} - \lambda^k \quad (1 \leqq k \leqq n) \tag{17}$$

satisfy

$$h_k(t) = o(e^{-\varrho t}\varphi(t)) \quad (1 \leqq k \leqq n-1), \tag{18}$$

$$h_k'(t) = o(e^{-\varrho t}\varphi(t)) \quad (1 \leqq k \leqq n-2) \tag{19}$$

and

$$\int_t^\infty h_k(s)\, ds = o(e^{-\varrho t}\varphi(t)) \quad (1 \leqq k \leqq n). \tag{20}$$

Lemma 2. *Suppose that the equation*

$$u^{(n)} + a_1 u^{(n-1)} + \ldots + a_n u = f(t) \tag{21}$$

has a solution $u = u(t)$ satisfying

$$u^{(k)}(t) = o(e^{\beta t}\varphi(t)) \quad (0 \leqq k \leqq n-1), \tag{22}$$

where β is a real constant and φ is positive and nonincreasing on $[T, \infty)$. If λ_m is a root of (3) *with* $\operatorname{Re}\lambda_m = \beta$, *then*

$$\int_t^\infty f(s)\, e^{-\lambda_m s}\, ds = o(\varphi(t)). \tag{23}$$

We leave the proofs of Lemmas 1 and 2 for the appendix.

Proof of Theorem 1. We proceed by induction with respect to n, the order of (2). In the case $n = 1$, any solution x of (2) satisfies

$$x(t) = C \exp\left[-a_1 t - \int_T^t p_1(s)\, ds\right], \tag{24}$$

where C is a constant and $t \geqq T$. If x_1 is a solution of (2) as in (8), with $\lambda_1 = -a_1$, then (15) follows from (8) and (24) with $x = x_1$. Obviously, if $n = 1$, then $\operatorname{Re}(\lambda_j - \lambda_m) = \varrho$ holds only if $j = m = 1$ and $\varrho = 0$, which reduces (7) to (15).

Assume now that (2) satisfies the hypotheses of Theorem 1 with $n > 1$. We use reduction of order. Given n solutions x_j of (2) as in (8), we introduce constants b_k

and functions h_k, q_k and z_j by

(25) $$b_k = \sum_{j=0}^{k} \binom{n-j}{k-j} a_j \lambda_1^{k-j}, \quad h_k(t) = \frac{x_1^{(k)}(t)}{x_1(t)} - \lambda_1^k, \quad (1 \leqq k \leqq n,\ a_0 = 1),$$

(26) $$q_k(t) = p_k(t) + \sum_{j=0}^{k-1} \binom{n-j}{k-j} a_j\, h_{k-j}(t) + \\ + \sum_{j=1}^{k-1} \binom{n-j}{k-j} p_j(t)\, h_{k-j}(t) + \sum_{j=1}^{k-1} \binom{n-j}{k-j} p_j(t)\, \lambda_1^{k-j}, \quad (1 \leqq k \leqq n)$$

and

(27) $$z_j(t) = (\lambda_j - \lambda_1)^{-1} \left[x_j(t)/x_1(t)\right]' \quad (2 \leqq j \leqq n).$$

Since x_1 is as in (8), the functions in (25)–(27) are defined on $[T_1, \infty)$ for some real $T_1 \geqq T$. Moreover, Lemma 1 with $x = x_1$ and $\lambda = \lambda_1$ implies that (18)–(20) hold for our functions h_k in (25). The constants b_k are chosen in (25) so that the polynomial $\lambda^n + b_1\lambda^{n-1} + \ldots + b_{n-1}\lambda + b_n$ has n distinct zeros $\lambda_j - \lambda_1$, $1 \leqq j \leqq n$.

The following assertion makes the meaning of the definitions (26) and (27) clear: the equation

(28) $$z^{(n-1)} + [b_1 + q_1(t)]\, z^{(n-2)} + \ldots + [b_{n-1} + q_{n-1}(t)]\, z = 0$$

has $(n-1)$ solutions (27) that satisfy

(29) $$z_j^{(k)}(t) = [(\lambda_j - \lambda_1)^k + o(e^{-\varrho t} \varphi(t))]\, e^{(\lambda_j - \lambda_1)t}, \quad (0 \leqq k \leqq n-2).$$

To see this, we first put $x = x_1(t)\, y$. A routine computation shows that (2) is transformed into

(30) $$y^{(n)} + [b_1 + q_1(t)]\, y^{(n-1)} + \ldots + [b_n + q_n(t)]\, y = 0,$$

with b_k and q_k as in (25) and (26). Since x_1 is a solution of (2) and λ_1 is a root of (3), we have

(31) $$b_n = 0 \quad \text{and} \quad q_n(t) = 0 \quad (t \geqq T).$$

Consequently, we may put $z = y' = (x/x_1(t))'$ to obtain the equation (28) with $(n-1)$ solutions (27). To prove (29), we need to show that the functions y_j in

(32) $$x_j(t) = x_1(t)\, y_j(t) \quad (2 \leqq j \leqq n)$$

satisfy

(33) $$y_j^{(k)}(t) = [(\lambda_j - \lambda_1)^k + o(e^{-\varrho t} \varphi(t))]\, e^{(\lambda_j - \lambda_1)t}$$

for $k = 1, 2, \ldots, n-1$. First we note that (33) with $k = 0$ follows from (8) with $k = 0$. Further, assume that (33) holds with any $k \leqq m - 1$ for some m, $1 \leqq \leqq m \leqq n - 1$. If we differentiate (32) m times, we obtain

$$x_j^{(m)}(t) = x_1(t)\, y_j^{(m)}(t) + \sum_{k=1}^{m} \binom{m}{k} x_1^{(k)}(t)\, y_j^{(m-k)}(t)$$

and, therefore,

$$y_j^{(m)}(t) = x_j^{(m)}(t)/x_1(t) - \sum_{k=1}^{m} \binom{m}{k} (\lambda_1^k + h_k(t))\, y_j^{(m-k)}(t) \tag{34}$$

(see the definition of h_k in (25)). Now (8), (18), (33) with $k \leqq m - 1$ and (34) imply that

$$e^{(\lambda_1 - \lambda_j)t}\, y_j^{(m)}(t) = \frac{\lambda_j^m + o}{1 + o} - \sum_{k=1}^{m} \binom{m}{k} (\lambda_1^k + o)\,[(\lambda_j - \lambda_1)^{m-k} + o] =$$

$$= \lambda_j^m - \sum_{k=1}^{m} \binom{m}{k} \lambda_1^k (\lambda_j - \lambda_1)^{m-k} + o = (\lambda_j - \lambda_1)^m + o\,,$$

where, for brevity, "o" stands for "$o(e^{-\varrho t}\, \varphi(t))$". Thus (33) with $k = m$ holds, which proves (29).

Assuming now that Theorem 1 holds if (2) is of order $n - 1$, we conclude from (29) that

$$\int_t^\infty q_k(s)\, ds = o(e^{-\varrho t}\, \varphi(t)) \quad (1 \leqq k \leqq n - 1)\,, \tag{35}$$

because, as we now verify,

$$\int_t^\infty |q_1(s)|\, \varphi(s)\, ds = o(\varphi(t))\,. \tag{36}$$

Indeed, we see from (18) and (26) that

$$q_1(t) - p_1(t) = n\, h_1(t) = o(e^{-\varrho t}\, \varphi(t))\,, \tag{37}$$

hence (36) follows from (6), (37) and the fact that

$$\int_t^\infty o(e^{-\varrho s}\, \varphi^2(s))\, ds = o(\varphi(t))\,. \tag{38}$$

The last relation follows either from (4), or from

$$\int_t^\infty e^{-\varrho s}\, \varphi^2(s)\, ds \leqq \varphi^2(t) \int_t^\infty e^{-\varrho s}\, ds = o(\varphi(t)) \quad \text{if} \quad \varrho > 0\,.$$

The next step of our proof is to show that

$$\int_t^\infty p_k(s)\, ds = o(e^{-\varrho t}\, \varphi(t)) \tag{39}$$

holds for $k = 1, 2, \ldots, n$. If $k = 1$, then (39) follows from (18), (35) and (37). Assuming now that (39) holds with $k \leqq m - 1$ for some m, $1 < m \leqq n$, we obtain from (20), (26), (35) and (39) with $k \leqq m - 1$ that

$$\int_t^\infty p_m(s)\, ds = o(e^{-\varrho t}\, \varphi(t)) - \sum_{j=1}^{m-1} \binom{n-j}{m-j} \int_t^\infty p_j(s)\, h_{m-j}(s)\, ds\,,$$

provided the integrals on the right hand side converge. (Note that (35) holds also with $k = n$ because of (31).) Consequently, (39) with $k = m$ holds if

$$\int_t^\infty p_j(s)\, h_{m-j}(s)\, ds = o(e^{-\varrho t}\, \varphi(t)) \tag{40}$$

is valid for $j = 1, 2, \ldots, m - 1$. If $j = 1$, (40) follows from (6) and (18), because

$$\int_t^\infty |p_1(s)\, h_{m-1}(s)|\, ds = \int_t^\infty |p_1(s)|\, \varphi(s)\, o(e^{-\varrho s})\, ds = o(e^{-\varrho t}\, \varphi(t))\,.$$

If $1 < j \leqq m - 1$, then integration by parts yields

$$(41) \qquad \int_t^{t_1} p_j(s)\, h_{m-j}(s)\, \mathrm{d}s = -P_j(s)\, h_{m-j}(s)\big|_t^{t_1} + \int_t^{t_1} P_j(s)\, h'_{m-j}(s)\, \mathrm{d}s\,,$$

where $P_j(t) = o(\mathrm{e}^{-\varrho t}\, \varphi(t))$ is the integral (39) with $k = j$. Since both $P_j(s)\, h_{m-j}(s)$ and $P_j(s)\, h'_{m-j}(s)$ are $o(\mathrm{e}^{-2\varrho s}\, \varphi^2(s))$ (see (18) and (19)), we can let $t_1 \to \infty$ in (41) and use (38) to obtain (40). Thus (15) is proved.

To complete the proof of Theorem 1, we need to show that (7) holds if $\mathrm{Re}\,(\lambda_j - \lambda_m) = \varrho$. We put $u_j(t) = x_j(t) - \exp(\lambda_j t)$, where x_j is the solution of (2) that satisfies (8). Then

$$(42) \qquad u_j^{(k)}(t) = o(\mathrm{e}^{(\mathrm{Re}\,\lambda_j - \varrho)t}\, \varphi(t)) \quad (0 \leqq k \leqq n - 1)\,.$$

Moreover, u_j is a solution of (21) with f given by

$$(43) \qquad f(t) = -\sum_{k=1}^{n} p_k(t)\, \lambda_j^{n-k} \mathrm{e}^{\lambda_j t} - \sum_{k=1}^{n} p_k(t)\, u_j^{(n-k)}(t)\,.$$

Since $\mathrm{Re}\,(\lambda_j - \lambda_m) = \varrho$, Lemma 2 with $\beta = \mathrm{Re}\,\lambda_j - \varrho$ and (42) imply (23) with f as in (43). Now (23) and (43) imply (7) provided

$$(44) \qquad \int_t^{\infty} p_k(s)\, u_j^{(n-k)}(s)\, \mathrm{e}^{-\lambda_m s}\, \mathrm{d}s = o(\varphi(t))$$

for $k = 1, 2, \ldots, n$. If $k = 1$, then (44) follows from (6) and (42):

$$\int_t^{\infty} |p_1(s)\, u_j^{(n-1)}(s)\, \mathrm{e}^{-\lambda_m s}|\, \mathrm{d}s = \int_t^{\infty} |p_1(s)|\, o(\varphi(s)) = o(\varphi(t))\,.$$

If $1 < k \leqq n$, then integration by parts yields

$$(45) \qquad \int_t^{t_1} p_k(s)\, u_j^{(n-k)}(s)\, \mathrm{e}^{-\lambda_m s}\, \mathrm{d}s = -P_k(s)\, u_j^{(n-k)}(s)\, \mathrm{e}^{-\lambda_m s}\big|_t^{t_1} + \int_t^{t_1} P_k(s)\, [u_j^{(n-k+1)}(s) - \lambda_m u_j^{(n-k)}(s)]\, \mathrm{e}^{-\lambda_m s}\, \mathrm{d}s\,,$$

where $P_k(t) = o(\mathrm{e}^{-\varrho t}\, \varphi(t))$ is the integral (39). By virtue of (39) and (42), the integrand and the outintegral function on the right hand side of (45) are $o(\mathrm{e}^{-\varrho t}\, \varphi^2(t))$. In view of (38), we can let $t_1 \to \infty$ in (45) to obtain (44). This completes the proof of Theorem 1.

3. THE SECOND CONVERSE THEOREM

Theorem 2. *Let ψ and ν be as in the first sentence of Theorem* B, *including* (9). *If* (1) *has n solutions x_j $(0 \leqq j \leqq n - 1)$ satisfying* (14) *and* (10) *holds, then the functions p_k satisfy* (11) *and, if $\nu < n$, also* (12).

Proof of Theorem 2. We will show that Theorem 2 is a consequence of Theorem 1. We introduce new variables y and τ by

$$(46) \qquad \tau = \log t\,, \quad y(\tau) = x(t)\,.$$

Then

$$(47) \quad x^{(k)}(t) = \mathrm{e}^{-k\tau} Q_k(D)\, y(\tau) \quad \left(D = \frac{\mathrm{d}}{\mathrm{d}\tau}\,, \quad Q_k(\lambda) = \prod_{j=0}^{k-1} (\lambda - j)\,, \quad 1 \leqq k \leqq n\right)$$

and therefore, (1) is transformed into

$$Q_n(D)\, y + \sum_{k=1}^{n} q_k(\tau)\, D^{n-k} y = 0\,, \tag{48}$$

where

$$q_k(\tau) = \frac{1}{(n-k)!} \sum_{m=1}^{k} Q_{n-m}^{(n-k)}(0)\, p_m(e^\tau)\, e^{m\tau}\,, \quad 1 \leqq k \leqq n\,. \tag{49}$$

Now we verify that (48) satisfies the conditions of Theorem 1 with t replaced by τ, $\varrho = \nu$, $\lambda_j = j - 1$ $(1 \leqq j \leqq n)$ and $\varphi(\tau) = \psi(e^\tau)$. Namely, we show that (48) has n solutions y_j $(0 \leqq j \leqq n - 1)$ satisfying

$$D^k\, y_j(\tau) = [j^k + o(e^{-\nu\tau}\, \varphi(\tau))]\, e^{j\tau} \quad (0 \leqq k \leqq n-1)\,, \tag{50}$$

$$\textstyle\int_\tau^\infty |q_1(r)|\, \varphi(r)\, dr = o(\varphi(\tau))\,, \tag{51}$$

and that φ obeys

$$\textstyle\int_\tau^\infty \varphi^2(r)\, dr = O(\varphi(\tau)) \quad \text{if} \quad \nu = 0\,. \tag{52}$$

Indeed, if we put $x = x_j$ in (46), where x_j are solutions of (1) as in (14), we obtain n solutions y_j of (48) satisfying

$$F(D)\, y_j(\tau) = [F(j) + o(e^{-\nu\tau}\, \psi(e^\tau))]\, e^{j\tau} \tag{53}$$

for any polynomial F of degree $<n$. The last relation holds, because (14), (46) and (47) imply (53) with $F = 1, Q_1, \ldots, Q_{n-1}$ (note that $(t^j)^{(k)} = Q_k(j)\, t^{j-k}$). Thus (50) is proved. To verify (51) and (52), we substitute $s = e^r$ in the integrals on their left hand sides. Since $q_1(\tau) = e^\tau p_1(e^\tau)$ (see (49)), we obtain

$$\textstyle\int_\tau^\infty |q_1(r)|\, \varphi(r)\, dr \leqq \varphi(\tau) \int_t^\infty |p_1(s)|\, ds$$

and

$$\int_\tau^\infty \varphi^2(r)\, dr = \int_t^\infty \frac{\psi^2(s)}{s}\, ds\,,$$

where $t = e^\tau$ (see (46)). Consequently, (51) and (52) follow from (10) and (9), respectively.

Applying Theorem 1 to (48), we conclude that

$$\textstyle\int_\tau^\infty q_k(r)\, dr = o(e^{-\nu\tau}\varphi(\tau)) \quad (1 \leqq k \leqq n) \tag{54}$$

and, if $\nu < n$,

$$\int_\tau^\infty \sum_{k=1}^{n} j^{n-k}\, q_k(r)\, e^{\nu r}\, dr = o(\varphi(\tau)) \quad (\nu \leqq j \leqq n-1)\,. \tag{55}$$

Using (49) and substituting $s = e^r$, we find that

$$\int_\tau^\infty q_k(r)\, dr = \frac{1}{(n-k)!} \int_t^\infty \sum_{m=1}^{k} Q_{n-m}^{(n-k)}(0)\, p_m(s)\, s^{m-1}\, ds \quad (1 \leqq k \leqq n) \tag{56}$$

and

$$\int_\tau^\infty \sum_{k=1}^{n} j^{n-k}\, q_k(r)\, e^{\nu r}\, dr = \int_t^\infty \sum_{m=1}^{n} Q_{n-m}(j)\, p_m(s)\, s^{m+\nu-1}\, ds\,. \tag{57}$$

Since $Q_{n-k}^{(n-k)}(0) = (n-k)! \neq 0$, (11) follows from (54), (56) by induction. Finally, (55) and (57) imply (12), because

$$\sum_{m=1}^{n} Q_{n-m}(j)\, p_m(s)\, s^{m+\nu-1} = g_j(s)\, s^{n-j+\nu-1}$$

(see (13)). This completes the proof of Theorem 2.

4. APPENDIX

Proof of Lemma 1. First we note that (18) follows immediately from (16) and (17). Routine manipulations with (17) show that

$$h_k' = h_{k+1} - \lambda h_k - h_k h_1 - \lambda^k h_1 \quad (1 \leqq k \leqq n-1)\,. \tag{58}$$

Now (18) and (58) imply (19). Further, (16) and (17) imply

$$\int_t^\infty h_1(s)\, \mathrm{d}s = -\log\left[\mathrm{e}^{-\lambda t}\, x(t)\right].$$

Since $\exp(-\lambda t)\, x(t) = 1 + o(\mathrm{e}^{-\varrho t}\varphi(t))$ (see (16)), the first relation in (20) holds. Integrating (58) we obtain

$$\int_t^{t_1} h_{k+1}(s)\, \mathrm{d}s = h_k(s)\big|_t^{t_1} + \int_t^{t_1} (\lambda\, h_k(s) + \lambda^k\, h_1(s) + h_k(s)\, h_1(s))\, \mathrm{d}s\,,$$

for $k = 1, 2, \ldots, n-1$. Consequently, (20) is proved by induction, because, as we now verify,

$$\int_t^\infty |h_k(s)\, h_1(s)|\, \mathrm{d}s = o(\mathrm{e}^{-\varrho t}\, \varphi(t))\,. \tag{59}$$

Indeed, if $\varrho = 0$, then (59) follows from (4) and (18). If $\varrho > 0$, then (59) follows from (18) and the inequality

$$\int_t^\infty \mathrm{e}^{-2\varrho s}\varphi^2(s)\, \mathrm{d}s \leqq \varphi^2(t) \int_t^\infty \mathrm{e}^{-2\varrho s}\, \mathrm{d}s = (2\varrho)^{-1}\, \varphi^2(t)\, \mathrm{e}^{-2\varrho t}\,.$$

This completes the proof of Lemma 1.

Proof of Lemma 2. We put $u = \exp(\lambda_m t)\, v$ to transform (21) into

$$v^{(n)} + b_1 v^{(n-1)} + \ldots + b_{n-1} v' + b_n v = f(t)\, \mathrm{e}^{-\lambda_m t} \tag{60}$$

with constant coefficients b_k. Let $u(t)$ be a solution of (21) as in (22). Then the solution $v(t) = \exp(-\lambda_m t)\, u(t)$ of (60) obeys

$$v^{(k)}(t) = o(\varphi(t)) \quad (0 \leqq k \leqq n-1) \tag{61}$$

because $\operatorname{Re} \lambda_m = \beta$. Since λ_m is a root of (3), we have $b_n = 0$ in (60). Consequently, integrating (60) with $v = v(t)$, we obtain

$$(v^{(n-1)}(s) + b_1 v^{(n-2)}(s) + \ldots + b_{n-1}\, v(s))\big|_t^{t_1} = \int_t^{t_1} f(s)\, \mathrm{e}^{-\lambda_m s}\, \mathrm{d}s\,.$$

This together with $t_1 \to \infty$ and (61) implies (23), which completes the proof of Lemma 2.

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