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Gary D. Jones

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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A FOURTH ORDER LINEAR DIFFERENTIAL EQUATION

GARY D. JONES, Murray

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1. Introduction. The asymptotic behavior of solutions of the second order differential equation

(1) $$y'' + p(t)\, y = 0$$

assuming either

(2) $$p \in C^1[a, \infty)\,, \quad p'(t) \geqq 0 \quad \text{and} \quad \lim_{t \to \infty} p(t) = \infty$$

or

(3) $$p \in C^1[a, \infty)\,, \quad p'(t) \leqq 0 \quad \text{and} \quad \lim_{t \to \infty} p(t) = 0$$

has been widely studied. (Reference [2] gives a history of the study of (1) with condition (2)). It is known, for example, that assuming condition (2), (1) has at least one non-trivial solution which tends to zero as t tends to infinity [3]. It need not be the case, however, that all solutions of (1) tend to zero [3]. Similarly, assuming condition (3), (1) has at least one non-trivial solution y such that $\limsup |y(t)| = \infty$. Again, however, it need not be the case that $\limsup |y(t)| = \infty$ for every non-trivial solution of (1).

We call a non-trivial solution $y(t)$ of the fourth order differential equation

(4_i) $$y^{\mathrm{IV}} + (-1)^i\, p(t)\, y = 0 \quad i = 1, 2$$

oscillatory if the set of zeros of $y(t)$ is not bounded above.

Assuming condition (2), Hastings and Lazer [1] show that unlike (1), every oscillatory solution of (4_1) tends to zero.

The purpose of this note is to study asymptotic behavior of solutions of (4_1) under condition (3) and (4_2) under conditions (2) and (3). In each case we show that stronger conclusions can be made for (4_i) than for (1).

2. Preliminary results. In this section we will give some simple results for (4_i) $i = 1, 2$ that will be used to prove our main theorems.

We define, for $y \in C^3[a, \infty)$ and $i = 1, 2$

(5_i) $$G_i[y(t)] \equiv (y'''(t))^2/(-1)^{i+1}\, p(t) - 2\, y(t)\, y''(t) + y'^2(t)\,,$$

(6$_i$) $$H_i[y(t)] \equiv (-1)^{i+1}\, p(t)\, y^2(t) - 2\, y'(t)\, y'''(t) + y''^2(t)\,,$$
and
(7) $$F[y(t)] \equiv y'(t)\, y''(t) - y(t)\, y'''(t)\,.$$

Lemma. *Let $y_i(t)$ be a solution of (4_i) for $i = 1, 2$.*

a) *Assuming* (3)
   1. *$G_1[y_1(t)]$ is increasing and $H_1[y_1(t)]$ is decreasing, while*
   2. *$G_2[y_2(t)]$ is decreasing and $H_2[y_2(t)]$ is increasing.*

b) *Assuming* (2), *$G_2[y_2(t)]$ is increasing, $H_2[y_2(t)]$ is decreasing and $F[y_2(t)]$ is increasing.*

Proof. The proof of each statement of the Lemma follows from the facts that for $i = 1, 2$
(8$_i$) $$G_i'[y_i(t)] = (-1)^i\, p'(t)\,(y_i'''(t)/p(t))^2\,,$$
(9$_i$) $$H_i'[y_i(t)] = (-1)^{i+1}\, p'(t)\, y_i^2(t)$$
and
(10) $$F'[y_2(t)] = (y_2''(t))^2 + p(t)\, y_2^2(t)\,.$$

3. In this section we consider (4_1) assuming (3).

Theorem 1. *If $p \in C^1[a, \infty)$, $p'(t) \leqq 0$ and $\lim_{t\to\infty} p(t) = 0$, then every oscillatory solution of $y^{\mathrm{IV}} - p(t)\, y = 0$ is unbounded.*

Proof. Suppose $y(t)$ is an oscillatory solution of (4_1) that is bounded. Let $\{b_n\}$ be the divergent sequence along which y'' assumes its relative maximum or relative minimum values. Then by the Lemma and (6_1)
(11) $$(y''(b_n))^2 \leqq p(b_n)\, y^2(b_n) + (y''(b_n))^2 = H[y(b_n)] \leqq H[y(a)]\,.$$
Hence y'' is bounded. Let $\{c_n\}$ be the divergent sequence along which y''' assumes its relative maximum or a relative minimum values. If n is such that $b_n > c_1$, then by (5_1) and the Lemma
(12) $$0 \leqq G_1[y(c_1)] < G_1[y(b_n)] = -2\, y(b_n)\, y''(b_n) + (y'(b_n))^2 \leqq$$
$$\leqq |2\, y(b_n)\, y''(b_n)| + (y'(b_n))^2\,.$$
Assuming that y' is bounded, then from the monotone property of $G[y(t)]$ we conclude that $G[y(t)]$ is also bounded. Now
$$(y'''(c_n))^2/p(c_n) \leqq (y'''(c_n))^2/p(c_n) + (y'(c_n))^2 = G_1[y(c_n)]\,.$$
Since $G[y(t)]$ is bounded and
(13) $$\lim_{t\to\infty} p(t) = 0$$
we conclude that
(14) $$\lim_{t\to\infty} y'''(t) = 0\,.$$
Let $\{d_n\}$ be the divergent sequence along which y' assumes its relative maximum or

relative minimum values. Then

$$H_1[y(d_n)] = p(d_n)\, y^2(d_n) - 2\, y'(d_n)\, y'''(d_n)\,.$$

From the monotone property of $H_1[y(t)]$, the assumptions that y and y' are bounded, (13) and (14), we conclude

$$\lim_{t\to\infty} H[y(t)] = 0\,. \tag{15}$$

It now follows from (11) and (15) that

$$\lim_{t\to\infty} y''(t) = 0\,. \tag{16}$$

If n is large enough so that $b_n > c_2$ by the Lemma we have

$$0 \leqq G_1[y(c_1)] < G_1[y(c_2)] < G_1[y(b_n)] = -2\, y(b_n)\, y''(b_n) + y'^2(b_n)\,. \tag{17}$$

Hence from (17) and (16) we have

$$\limsup |y'(t)| = A \neq 0\,. \tag{18}$$

Suppose, without loss of generality, that $\limsup y'(t) = A$. Let $\{x_n\}$ be a divergent sequence such that $y'(x_n) = A/2$ and $\{t_n\}$ be a divergent sequence such that on $[t_n, x_n]$, $y'(x) \geqq A/4$ with $y'(t_n) = A/4$. Then by the Mean Value Theorem there is an $s_n \in [t_n, x_n]$ so that

$$(A/4)/(x_n - t_n) = [y'(x_n) - y'(t_n)]/(x_n - t_n) = y''(s_n)\,. \tag{19}$$

Because of (16), it follows from (19) that

$$\lim_{n\to\infty} (x_n - t_n) = \infty\,. \tag{20}$$

Hence, since

$$y(x_n) - y(t_n) = \int_{t_n}^{x_n} y'(t)\, dt \geqq (A/4)(x_n - t_n)$$

either y is not bounded or y' is not bounded. Assume y' is not bounded and without loss of generality that $\limsup y' = \infty$. Let $\{s_n\}$ be a divergent sequence on which y' assumes a relative maximum and where

$$\lim_{n\to\infty} y'(s_n) = \infty \tag{21}$$

and $y'(s_n) > 1$ for all n. Let $\{t_n\}$ be a divergent sequence so that $y'(t_n) = 1$, $t_n < s_n$ and $y'(t) \geqq 1$ for $t \in [t_n, s_n]$. By (11) y'' is bounded. Let $B > 0$ be such that $|y''(t)| < B$. Then

$$|y'(s_n) - 1| = |y'(s_n) - y'(t_n)| = \left|\int_{t_n}^{s_n} y''(t)\, dt\right| \leqq \int_{t_n}^{s_n} |y''(t)|\, dt \leqq B(s_n - t_n)\,.$$

Thus by (21)

$$\lim_{n\to\infty} (s_n - t_n) = \infty\,. \tag{22}$$

Hence

$$|y(s_n) - y(t_n)| = \left|\int_{t_n}^{s_n} y'(t)\, dt\right| \geqq \int_{t_n}^{s_n} dt = s_n - t_n\,.$$

As a consequence of (22), y is not bounded.

4. It is known that either all or none of the solutions of (4_2) oscillate. Assuming (2) or (3) we get information about pairs of oscillatory solutions.

Theorem 2. *If* $p \in C^1[a, \infty)$, $p'(t) \leqq 0$, $\lim_{t \to \infty} p(t) = 0$ *and* $y^{IV} + py = 0$ *is oscillatory, then there is a pair of linearly independent solutions that are unbounded.*

Proof. Let y be a solution of (4_2) such that $y(a) = y'(a) = 0$. From the Lemma, (5_2) and (6_2) $H_2[y(t)]$ is positive and $G_2[y(t)]$ is negative. If y is bounded then

$$\lim_{t \to \infty} p(t)\, y^2(t) = 0 \,. \tag{23}$$

Integrating (9_2) shows $H_2[y(t)]$ to be bounded. As in Theorem 1, we let $\{b_n\}$ be the divergent sequence along which y'' assumes its relative maximum or relative minimum values. Then

$$H_2[y(b_n)] = -p(b_n)\, y^2(b_n) + y''^2(b_n) \,. \tag{24}$$

Since $H_2[y(t)]$ is bounded and (23),

$$0 < \lim\sup |y''| = A < \infty \,. \tag{25}$$

Letting $\{d_n\}$ be the divergent sequence along which y' assumes its relative maximum or relative minimum values, then

$$H_2[y(d_n)] = -p(d_n)\, y^2(d_n) - 2\, y'(d_n)\, y'''(d_n) \,. \tag{26}$$

Again using the boundedness of $H_2[y(t)]$ and (23),

$$0 < \lim\sup |y' y'''| = B < \infty \,. \tag{27}$$

Suppose y''' does not go to zero. If

$$\lim_{t \to \infty} G_2[y(t)] = -\infty \tag{28}$$

then $G_2[y(b_n)] = -2\, y(b_n)\, y''(b_n) + y'^2(b_n)$ yields

$$\lim\sup |y\, y''| = \infty \,. \tag{29}$$

Since we are assuming y is bounded (29) implies y'' is not bounded contrary to (25). Let $\{c_n\}$ be the divergent sequence along which y''' assumes its relative maximum or relative minimum values. Then

$$G_2[y(c_n)] = y'''^2(c_n)/(-p(c_n)) + y'^2(c_n) \,. \tag{30}$$

Since we are assuming y''' does not go zero, $\lim\sup y'''^2(c_n)/p(c_n) = \infty$. Thus if $G_2[y(t)]$ is bounded (30) implies $\lim\sup\, y'^2(c_n) = \infty$. Hence

$$\lim\sup |y'(c_n)\, y'''(c_n)| = \infty \,. \tag{31}$$

Now

$$H_2[y(c_n)] = -2\, y'(c_n)\, y'''(c_n) + y''^2(c_n) \,. \tag{32}$$

Thus by (31) and (25), $H_2[y(c_n)]$ is unbounded which is a contradiction. Hence either

$$\lim_{t\to\infty} y'''(t) = 0 \tag{33}$$

or y is unbounded. Assuming (33), since $H_2[y(t)]$ is increasing, it follows from (26) and (23) that $-2\,y'(d_n)\,y'''(d_n)$ is bounded away from zero. Hence from (33)

$$\limsup |y'(t)| = +\infty . \tag{34}$$

Let $\{a_n\}$ be a divergent sequence such that

$$\lim_{n\to\infty} y'(a_n) = +\infty . \tag{35}$$

By (25) y'' is bounded. Assume $|y''(t)| < C$. Then for $x \in [a_n, a_n + 1]$

$$y'(x) - y'(a_n) = \int_{a_n}^{x} y''(t)\,dt > -C$$

or

$$y'(x) > y'(a_n) - C . \tag{36}$$

By The Mean Value Theorem

$$y(a_n + 1) - y(a_n) = y'(\varepsilon_n) \quad \text{for } a_n < \varepsilon_n < a_n + 1 . \tag{37}$$

From (37), (36) and (35) it follows that y is not bounded. Since y_1 and y_2 satisfying $y_1(a) = y_1'(a) = y_1''(a) = 0$, $y'''(a) = 1$ and $y_2(a) = y_2'(a) = y_2'''(a) = 0$ and $y_2''(a) = 1$ are independent, the theorem follows.

Theorem 3. *If* $p \in C'[a, \infty)$, $p'(t) \geqq 0$ *and* $\lim_{t\to\infty} p(t) = \infty$, *then* $y^{\mathrm{IV}} + py = 0$ *has a pair of oscillatory solutions that go to zero.*

Proof. We first show that the conclusion of the Theorem holds for any solution y of (4_2) for which

$$G_2[y(t)] < 0 \quad t \in [a, \infty), \tag{38}$$

$$H_2[y(t)] > 0 \quad t \in [a, \infty) \text{ and} \tag{39}$$

$$F[y(t)] < 0 \quad t \in [a, \infty). \tag{40}$$

Later we will show the existence of two such solutions which are linearly independent.

Suppose y is a solution of (4_2) that satisfies (38), (39) and (40). Suppose

$$\lim_{t\to\infty} y(t) \neq 0 . \tag{41}$$

Let $\{t_n\}$ be the divergent sequence along which y assumes its relative maximum or relative minimum values. Then from (41) follows

$$\limsup p(t_n)\, y^2(t_n) = \infty . \tag{42}$$

From (6_2)

$$H_2[y(t_n)] = -p(t_n)\, y^2(t_n) + y''^2(t_n) .$$

Hence since $H_2[y(t)]$ is positive and decreasing, (42) implies

$$\limsup y''^2(t_n) = +\infty . \tag{43}$$

Hence

$$(44)\qquad \limsup |y(t_n)\, y''(t_n)| = +\infty\,.$$

But

$$(45)\qquad G_2[y(t_n)] = -y'''^2(t_n)/p(t_n) - 2\, y(t_n)\, y''(t_n)\,.$$

Since $G_2[y(t_n)]$ is negative and increasing, (44) implies

$$(46)\qquad \limsup y'''^2(t_n)/p(t_n) = \infty\,,$$

which in turn implies

$$(47)\qquad \limsup |y'''(t_n)| = \infty\,.$$

But

$$(48)\qquad F[y(t_n)] = -y(t_n)\, y'''(t_n)\,.$$

Since $F[y(t)]$ is negative and increasing, $F[y(t_n)]$ is bounded, while (47) implies

$$(49)\qquad \limsup |y(t_n)\, y'''(t_n)| = \infty\,.$$

Hence $\lim_{t\to\infty} y(t) = 0$.

To show the existence of two linearly independent solutions of (4_2) that satisfy (38), (39) and (40), we use standard compactness arguments in the following way. Let Z_i for $i = 0, 1, 2, 3$ be solutions of (4_2) defined by the initial conditions

$$Z_i^{(j)}(a) = \delta_{ij} = 0\,, \quad i \neq j\,,$$
$$\qquad\qquad = 1\,, \quad i = j\,.$$

For each integer $n > a$, let $b_{0n}, b_{2n}, b_{3n}, c_{1n}, c_{2n}, c_{3n}$ be numbers such that

$$(50)\qquad b_{0n}^2 + b_{2n}^2 + b_{3n}^2 = 1\,, \quad c_{1n}^2 + c_{2n}^2 + c_{3n}^2 = 1$$

and

$$(51)\qquad b_{0n}\, Z_0^{(i)}(n) + b_{2n}\, Z_2^{(i)}(n) + b_{3n}\, Z_3^{(i)}(n) = 0 \quad \text{for} \quad i = 0, 1\,,$$
$$c_{1n}\, Z_1^{(i)}(n) + c_{2n}\, Z_2^{(i)}(n) + c_{3n}\, Z_3^{(i)}(n) = 0 \quad \text{for} \quad i = 0, 1\,.$$

Let

$$U_n(t) = b_{0n}\, Z_0(t) + b_{2n}\, Z_2(t) + b_{3n}\, Z_3(t)\,,$$
$$V_n(t) = c_{1n}\, Z_1(t) + c_{2n}\, Z_2(t) + c_{3n}\, Z_3(t)\,.$$

By (50), there exists a sequence of integers $\{n_j\}$ such that the sequences $\{b_{in_j}\}$ and $\{c_{in_j}\}$ converge to numbers b_i and c_i respectively. Let u and v be solutions of (4_2) defined by

$$(52)\qquad u(t) = b_0\, Z_0(t) + b_2\, Z_2(t) + b_3\, Z_3(t)$$
$$v(t) = c_1\, Z_1(t) + c_2\, Z_2(t) + c_3\, Z_3(t)\,.$$

From (50) it follows that neither $u(t)$ nor $v(t)$ are identically zero. Clearly the sequences $\{U_{n_j}(t)\}$ and $\{V_{n_j}(t)\}$ converge uniformly on compact intervals to $u(t)$ and $v(t)$ respectively. From (50) and the monotone properties of G_2, H_2 and F it follows that

$$(53)\qquad G_2[y_n(x)] \leqq 0\,, \quad H_2[y_n(x)] \geqq 0 \quad \text{and} \quad F[y_n(x)] \leqq 0 \quad \text{on} \quad [a, n]$$
$$\text{for} \quad y_n = u_n \quad \text{or} \quad v_n\,.$$

Thus

$$G_2[y(x)] \leqq 0\,, \quad H_2[y(x)] \geqq 0 \quad \text{and} \quad F[y(x)] \leqq 0 \quad \text{on} \quad [a, \infty) \tag{54}$$

$$\text{for} \quad y = u \quad \text{or} \quad v\,.$$

If u and v are linearly dependent then by (52)

$$u(t) = k\,v(t) = a_2\,z_2(t) + a_3\,z_3(t)\,.$$

In that case by the Lemma and (7) $F[u(t)] > 0$ for $t > a$ contrary to (54). Hence u and v are linearly independent.

Added in proof. M. Švec, Sur le comportement asymptotique des intégrales de l'équation différentielle $y^{(4)} + Q(x)\,y = 0$, Czech. Math. J. *8(83)* (1958), pp. 230–245, gets the conclusion of Theorem 3 assuming only that $p(x) \geqq m > 0$. The author has been able to prove a theorem with the same conclusion as Theorem 2 assuming only that $0 < p(x) < M$.

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Author's address: Murray State University, College of Science, Department of Mathematics, Murray, Ky. 42071-3306, U.S.A.