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ON THE OSCILLATION OF A LINEAR DIFFERENTIAL EQUATION OF SECOND ORDER

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This paper is a continuation of the paper [1]. Here we shall supplement the results of the paper [1] with results concerning the linear differential equation

$$(r(t)\, y'(t))' + p(t)\, y(t) = 0\,. \tag{1}$$

The aim of this paper is to present oscillation and nonoscillation theorems for the equation (1) as well as to introduce necessary and sufficient conditions for (1) to be oscillatory. The technique used in the paper is established on the notion of the v-derivative of a function.

1. THE v-DERIVATIVE AND THE v-TRANSFORMATION

Suppose throughout this section that f, g, v, φ are real-valued functions of one real variable. Let the interval $(-\infty, \infty)$ be denoted by $\mathbb{R}$. We introduce the following definitions, remarks and theorems from [1].

Definition 1.1. Let functions f and v be defined on some neighborhood $O(t)$ of a point $t \in \mathbb{R}$ and let the conditions $x \in O(t)$, $x \neq t$ imply $v(x) \neq v(t)$. If the limit

$$\lim_{x \to t} \frac{f(x) - f(t)}{v(x) - v(t)}$$

is finite, then it is called the *v-derivative* of the function f at the point t and denoted by $f'_v(t)$ or $\mathrm{d}f(t)/\mathrm{d}v$.

Remark 1.1. It follows from Definition 1.1 that $f'_f(t) = 1$ for every t such that f is defined on $O(t)$ and the conditions $x \in O(t)$, $x \neq t$ imply $f(x) \neq f(t)$.

Theorem 1.1. *Let the following conditions be satisfied:*

(i) *a function v is continuous at a point t,*

(ii) *a function g has the v-derivative at the point t,*

(iii) *a function f has the ordinary derivative at the point $g(t)$.*

Then the composite function $f(g)$ has the v-derivative at the point t and

$$(f(g))'_v\,(t) = f'(g(t))\, g'_v(t)\,.$$

Theorem 1.2. *Let there exist $v'(t) \neq 0$ on an interval I. Then for $t \in I$ the v-derivative $f'_v(t)$ exists if and only if the derivative $f'(t)$ exists. At the same time,*

$$f'_v(t) = \frac{f'(t)}{v'(t)}.$$

In this paper we shall need the following simple form of Definition 1.2 from [1].

Definition 1.2. Let functions f and v be as in Definition 1.1. Let the function f'_v be defined on some neighborhood $O(t)$ of a point $t \in \mathbb{R}$. If the limit

$$\lim_{x \to t} \frac{f'_v(x) - f'_v(t)}{v(x) - v(t)}$$

is finite, then it is called the *second v-derivative* of the function f at the point t and denoted by

$$f''_{v^2}(t) \quad \text{or} \quad \frac{\mathrm{d}^2 f(t)}{\mathrm{d}v^2}.$$

For the purposes of this paper we also simplify the notion of a v-transformation of a differential equation, presented in [1]. Thus suppose that the following conditions are satisfied:

a) I and I_1 are intervals in $\mathbb{R}$,

b) $v \in C(I_1)$, v is a strictly monotone function, $v: I_1 \to I$,

c) φ is the inverse function to v,

d) $p: I \to \mathbb{R}$.

Consider the differential equation

$$y''(t) + p(t)\, y(t) = 0\,, \quad t \in I\,. \tag{2}$$

If the independent variable t is replaced by the function $v(t)$ in the coefficient $p(t)$ of the equation (2) and $y''(t)$ is replaced by $y''_{v^2}(t)$ in the sense that $v(t)$ replaces even the independent variable as the argument of the function with respect to which the derivatives of the unknown function are calculated ($y'(t) = y'_w(t)$, where $w(t) \equiv t$), then the equation (2) is transformed into the equation

$$y''_{v^2}(t) + p(v(t))\, y(t) = 0\,, \quad t \in I_1\,, \tag{3}$$

In the sequel we shall call the above mentioned process of obtaining (3) from (2) the *v-transformation of the differential equation* (2).

It is useful to note that a φ-transformation of (3) leads again to (2).

Now we can introduce the following result which is a special case of Theorem 2.1 proved in [1].

Theorem 1.3. *Let the conditions* a)–d) *be satisfied. A function $u(t)$ is a solution of the equation* (2) *on I if and only if the function $u(v(t))$ is a solution of the equation* (3) *on I_1.*

Note that by Theorem 1.1 and Remark 1.1 we can verify Theorem 1.3 directly.

2. ON THE OSCILLATION OF THE EQUATION (1)

Consider the equation (1) on an interval $[t_0, T)$, where $T \leqq \infty$, and suppose throughout this section that the following conditions and notation hold:

(i_1) $p \in C([t_0, T))$,

(ii_1) $r \in C([t_0, T))$, $r(t) > 0$,

(iii_1) $R(t) = \int_{t_0}^{t} (\mathrm{d}s/r(s))$ for $t \in [t_0, T)$.

Let Φ be the inverse function to R.

In the sequel we shall restrict our attention to nontrivial solutions of the equations considered. In the case $T = \infty$ such a solution is called *oscillatory* if it has arbitrarily large zeros, and *nonoscillatory* otherwise. An equation is said to be *oscillatory* if all its solutions are oscillatory; otherwise it is said to be *nonoscillatory.*

Theorem 2.1. *Let* $(\mathrm{i}_1,)$, (ii_1), (iii_1) *be satisfied. Let there exist a positive number* $a \in \mathbb{R}$ *such that*

$$r(t)\, p(t) \geqq a \quad for \quad t \in [t_0, T)\,.$$

Then the number of zero points of each solution of (1) *on the interval* $[t_C, T)$ *equals at least*

$$\left[\frac{\sqrt{a}}{\pi} \int_{t_0}^{T} \frac{\mathrm{d}t}{r(t)}\right].$$

Proof. From Theorem 1.2 we know that the equation

$$(r(t)\, y'(t))' + \frac{a}{r(t)}\, y(t) = 0 \tag{4}$$

can be written in the form

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}R^2} + a\, y(t) = 0\,. \tag{4_1}$$

From Theorem 1.3 we see that the functions

$$y_1(t) = \cos\left((\sqrt{a})\, R(t)\right), \quad y_2(t) = \sin\left((\sqrt{a})\, R(t)\right)$$

are linearly independent solutions of (4_1) and also of (4). Since $p(t) \geqq a/r(t)$ for $t \in [t_0, T)$ so the well known Sturm comparison theorem implies our assertion and the proof is complete.

The following two results immediately follow from Theorem 2.1.

Corollary 2.1. *Let* $T = \infty$ *and* $R(t) \to \infty$ *as* $t \to \infty$. *Let the assumptions of Theorem* 2.1 *be satisfied. Then the equation* (1) *is oscillatory.*

Corollary 2.2. *Let* $T = \infty$. *Let* (i_1), (ii_1), (iii_1) *be satisfied,* $R(t) \to \infty$ *as* $t \to \infty$ *and*

$$\liminf_{t \to \infty} r(t)\, p(t) > 0\,.$$

Then the equation (1) *is oscillatory.*

Note that Corollary 2.2 extends part a) of Theorem 3.1 in [1] for $n = 2$.

Theorem 2.2. *Let* $(\mathrm{i}_1), (\mathrm{ii}_1), (\mathrm{iii}_1)$ *be satisfied. Let there exist a number* $b \in \mathbb{R}$ *such that*

$$r(t)\, p(t) \leqq b \quad \text{for} \quad t \in [t_0, T)\,.$$

Then the number of zero points of each solution of (1) *on the interval* $[t_0, T)$ *equals at most*

$$\left[\frac{\sqrt{|b|}}{\pi} \int_{t_0}^{T} \frac{\mathrm{d}t}{r(t)}\right] + 1\,.$$

Proof. The equation

$$(5) \qquad (r(t)\, y'(t))' + \frac{b}{r(t)}\, y(t) = 0$$

can be written in the form

$$(5_1) \qquad \frac{\mathrm{d}^2 y(t)}{\mathrm{d}R^2} + b\, y(t) = 0\,.$$

Then by Theorem 1.3 we know that the functions

$$y_1(t) = \cos\left((\sqrt{b})\, R(t)\right), \quad y_2(t) = \sin\left((\sqrt{b})\, R(t)\right) \quad \text{for} \quad b > 0\,,$$
$$y_1(t) = 1\,, \quad y_2(t) = R(t) \quad \text{for} \quad b = 0\,,$$
$$y_1(t) = \exp\left((\sqrt{-b})\, R(t)\right), \quad y_2(t) = \exp\left((-\sqrt{-b})\, R(t)\right) \quad \text{for} \quad b < 0$$

are linearly independent solutions of (5_1) and also of (5).

Since $p(t) \leqq b/r(t)$ for $t \in [t_0, T)$ so the Sturm comparison theorem implies our assertion and the proof is complete.

Now we see that the following results hold true.

Corollary 2.3. *Let* $T = \infty$ *and* $\lim_{t\to\infty} R(t) < \infty$. *Let the assumptions of Theorem* 2.2 *be satisfied. Then the equation* (1) *is nonoscillatory.*

Corollary 2.4. *Let* $T = \infty$. *Let* $(\mathrm{i}_1), (\mathrm{ii}_1), (\mathrm{iii}_1)$, *be satisfied,* $\lim_{t\to\infty} R(t) < \infty$ *and*

$$\limsup_{t\to\infty} r(t)\, p(t) < \infty\,.$$

Then the equation (1) *is nonoscillatory.*

From the proof of Theorem 2.2 it is easy to see that the following extension of part b) of Theorem 3.1 in [1] for $n = 2$ holds true.

Corollary 2.5. *Let* $(\mathrm{i}_1), (\mathrm{ii}_1), (\mathrm{iii}_1)$ *be satisfied for* $T = \infty$. *Let* $R(t) \to \infty$ *as* $t \to \infty$ *and*

$$\limsup_{t\to\infty} r(t)\, p(t) < 0\,.$$

Then the equation (1) *is nonoscillatory.*

Corollary 2.2 and Corollary 2.4 yield

Corollary 2.6. *Let* $(\mathrm{i}_1), (\mathrm{ii}_1), (\mathrm{iii}_1)$ *be satisfied for* $T = \infty$. *Let*

$$\liminf_{t\to\infty} r(t)\,p(t) > 0$$

and

$$\limsup_{t\to\infty} r(t)\,p(t) < \infty\,.$$

Then the equation (1) *is oscillatory if and only if*

$$\int^{\infty} \frac{\mathrm{d}t}{r(t)} = \infty\,.$$

Now we will use the Sturm comparison theorem and the Euler equation to state other conditions for the equation (1) to be either oscillatory or nonoscillatory.

Theorem 2.3. *Let* $T = \infty$. *Let* $(\mathrm{i}_1), (\mathrm{ii}_1), (\mathrm{iii}_1)$ *be satisfied and* $R(t) \to \infty$ *as* $t \to \infty$. *Then the equation* (1) *is oscillatory if*

$$\liminf_{t\to\infty} R^2(t)\,r(t)\,p(t) > 1/4$$

and the equation (1) *is nonoscillatory if*

$$\limsup_{t\to\infty} R^2(t)\,r(t)\,p(t) < 1/4\,.$$

Proof. Put

$$a = \liminf_{t\to\infty} R^2(t)\,r(t)\,p(t)\,.$$

Then for every a_1 with the property $1/4 < a_1 < a$ there exists t_1 $(\geqq t_0)$ such that

$$R^2(t)\,r(t)\,p(t) \geqq a_1 \quad \text{if} \quad t \geqq t_1\,.$$

The equation

$$(6) \qquad (r(t)\,y'(t))' + \frac{a_1}{r(t)\,R^2(t)}\,y(t) = 0$$

can be written in the form

$$(6_1) \qquad \frac{\mathrm{d}^2 y(t)}{\mathrm{d}R^2} + \frac{a_1}{R^2(t)}\,y(t) = 0\,.$$

By the v-transformation of (6_1) with $v = \Phi$ we obtain the Euler equation

$$y''(t) + \frac{a_1}{t^2}\,y(t) = 0$$

which is oscillatory because $a_1 > 1/4$. Hence, according to Theorem 1.3, the equation (6) is oscillatory. However,

$$p(t) \geqq \frac{a_1}{r(t)\,R^2(t)} \quad \text{if} \quad t \geqq T_1$$

and thus according to the Sturm comparison theorem the equation (1) is oscillatory.

Analogously, if we put

$$b = \limsup_{t \to \infty} R^2(t)\, r(t)\, p(t)\,,$$

then for every b_1 with the property $b < b_1 < 1/4$ there exists t_2 $(\geqq t_0)$ such that

$$R^2(t)\, r(t)\, p(t) \leqq b_1 \quad \text{if} \quad t \geqq t_2\,.$$

The equation

$$(r(t)\, y'(t))' + \frac{b_1}{r(t)\, R^2(t)}\, y(t) = 0 \tag{7}$$

can be written in the form

$$\frac{d^2 y(t)}{dR^2} + \frac{b_1}{R^2(t)}\, y(t) = 0\,. \tag{7_1}$$

The v-transformation of (7_1) with $v = \Phi$ gives the Euler equation

$$y''(t) + \frac{b_1}{t^2}\, y(t) = 0$$

which is nonoscillatory because $b_1 < 1/4$. Hence, according to Theorem 1.3, the equation (7) is nonoscillatory. Since now

$$p(t) \leqq \frac{b_1}{r(t)\, R^2(t)} \quad \text{if} \quad t \geqq t_2\,,$$

the Sturm comparison theorem yields that the equation (1) is nonoscillatory. The proof is complete.

Note that Theorem 2.3 extends a result of E. Hille (see [2], p. 194) concerning the differential equation $y''(t) + p(t)\, y(t) = 0$.

Using Theorem 2.3 we can state the following improvement of Corollary 2.6.

Corollary 2.7. *Let* $(i_1), (ii_1), (iii_1)$ *be satisfied for* $T = \infty$. *Let either*

$$\liminf_{t \to \infty} r(t)\, p(t) > 0$$

or

$$\liminf_{t \to \infty} R^2(t)\, r(t)\, p(t) > 1/4$$

and let

$$\limsup_{t \to \infty} r(t)\, p(t) < \infty\,.$$

Then the equation (1) *is oscillatory if and only if*

$$\int^{\infty} \frac{dt}{r(t)} = \infty\,.$$

In the literature we can find several sufficient conditions for (1) to be oscillatory in the case $\int^{\infty} dt/r(t) = \infty$ (see e.g. [2]). Other conditions of this kind may be obtained by applying the technique from Section 3 of [1] to such results for the equation $y''(t) + p(t)\,y(t) = 0$. On the other hand, we have only a few sufficient conditions for (1) to be oscillatory in the case $\int^\infty \mathrm{d}t/r(t) < \infty$. One of such results may be found in [2], p. 196. Now we present an other result of this kind.

Theorem 2.4. *Let* $(\mathrm{i}_1), (\mathrm{ii}_1), (\mathrm{iii}_1)$ *be satisfied for* $T = \infty$. *Let*

$$\int_{t_0}^{\infty} \frac{\mathrm{d}t}{r(t)} = K < \infty$$

and

$$\lim_{t\to\infty} r(t)\,p(t) = \infty\,. \tag{8}$$

In addition, for every sufficiently large $L > 0$ *let there exist* $M \geqq t_0$ *such that*

$$R(M) \leqq K - \frac{2\pi}{\sqrt{L}}$$

and $r(t)\,p(t) \geqq L$ *for* $t \geqq M$.

Then the equation (1) *is oscillatory.*

Proof. We know that

$$R\colon [t_0, \infty) \to [0, K) \quad \text{and} \quad \Phi\colon [0, K) \to [t_0, \infty)\,.$$

If we rewrite the equation (1) to the form

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}R^2} + r(t)\,p(t)\,y(t) = 0\,, \quad t \in [t_0, \infty)\,, \tag{9}$$

then by the v-transformation of (9) with $v = \Phi$ we obtain

$$y''(t) + r(\Phi(t))\,p(\Phi(t))\,y(t) = 0\,, \quad t \in [0, K)\,. \tag{10}$$

Since (8) holds true thus

$$\lim_{t\to K-} r(\Phi(t))\,p(\Phi(t)) = \infty$$

and, in addition, for every sufficiently large $L > 0$ there exists $s = R(M)$ such that

$$r(\Phi(t))\,p(\Phi(t)) \geqq L \quad \text{if} \quad t \in [s, K)\,.$$

Now consider the equation

$$y''(t) + L\,y(t) = 0\,, \quad t \in [s, K)\,,$$

the linearly independent solutions of which are

$$y_1(t) = \cos(t\sqrt{L})\,, \quad y_2(t) = \sin(t\sqrt{L})\,.$$

The distance between neighbouring zeroes of these functions is $\pi/\sqrt{L}$ and thus we see that they have at least two zeros in the interval $[s, K)$. By the Sturm comparison theorem we see that any solution of (10) has at least one zero in the interval $[s, K)$. It means that the solutions of the equation (9) and also of (1) have arbitrarily large zeros, i.e. the equation (1) is oscillatory and the proof is complete.

References

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