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# OPERATOR-VALUED ANALYTIC FUNCTIONS OF CONSTANT NORM

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Let $X$ be a complex Banach space with norm $\|\cdot\|$. Following Globevnik [2], for any element $a$ of $X$ we define $E(a)$ to be the set of elements $b$ of $X$ such that $\|a + \lambda b\| = \|a\|$ for all complex numbers $\lambda$ in some nonempty open disk about the origin. The set $E(a)$ is a (not necessarily closed) linear manifold in $X$. It has interesting properties, which include a key role in an extension of the strong maximum modulus principle [3, 5].

**Theorem 1** (Globevnik [2]). *Let $f(z)$ be an $X$-valued analytic function on an open connected set $\Omega$ in the complex plane.*

(i) *If $\|f(z)\|$ is constant for $z$ in $\Omega$, then $M = E(f(z))$ is independent of $z$ in $\Omega$, and $f(u) - f(v) \in M$ for all $u$ and $v$ in $\Omega$.*

(ii) *If the closed manifold $N = (E(f(z)))^-$ is independent of $z$ in $\Omega$ and $f(u) - f(v) \in N$ for all $u$ and $v$ in $\Omega$, then $\|f(z)\|$ is constant for $z$ in $\Omega$.*

In this paper we compute $E(A)$ for any element $A$ of $\mathscr{B}(\mathscr{H}, \mathscr{K})$, the space of bounded linear operators on a Hilbert space $\mathscr{H}$ to a Hilbert space $\mathscr{K}$ in the operator norm. The result has features in common with the theorem on completing two-by-two operator matrix contractions, a recent account of which is given in Pták and Vrbová [4]. Our derivation of the result is independent of the latter theorem. It is sufficient to treat the case $\|A\| = 1$.

**Theorem 2.** *Let $A$ be an element of $\mathscr{B}(\mathscr{H}, \mathscr{K})$ with $\|A\| = 1$. Then $E(A)$ is the set of operators in $\mathscr{B}(\mathscr{H}, \mathscr{K})$ of the form*

$$B = (1 - AA^*)^{1/2} C (1 - A^*A)^{1/2}, \tag{1}$$

*where $C$ belongs to $\mathscr{B}(\mathscr{H}, \mathscr{K})$.*

Here and below, underlying spaces are assume to be Hilbert spaces. The identity operator on any space is written 1. We use triangular brackets $\langle \cdot, \cdot \rangle$ for inner products and double bars $\|\cdot\|$ for norms, with subscripts to indicate the underlying spaces.

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**Lemma 1.** *Assume $A, B \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ and $\|A\| = 1$. Then $B \in E(A)$ if and only if there is a $\delta > 0$ such that*

$$\|Bf\|_{\mathscr{K}}^2 \leqq \delta\langle(1 - A^*A)f, f\rangle_{\mathscr{H}} \tag{2}$$

*and*

$$|\langle Af, Bg\rangle_{\mathscr{K}}|^2 \leqq \delta\langle(1 - A^*A)f, f\rangle_{\mathscr{H}} \langle(1 - A^*A)g, g\rangle_{\mathscr{H}} \tag{3}$$

*for all $f$ and $g$ in $\mathscr{H}$.*

Proof. Assume that $B \in E(A)$. Then there is an $R > 0$ such that $\|(A + \lambda B)f\|_{\mathscr{K}}^2 \leqq \|f\|_{\mathscr{H}}^2$ for all $f$ in $\mathscr{H}$ and $|\lambda| \leqq R$. Hence for any $f$ in $\mathscr{H}$ and $|\lambda| \leqq R$,

$$2 \operatorname{Re} \bar{\lambda}\langle Af, Bf\rangle_{\mathscr{K}} + |\lambda|^2 \|Bf\|_{\mathscr{K}}^2 \leqq \|f\|_{\mathscr{H}}^2 - \|Af\|_{\mathscr{K}}^2 .$$

It follows that

$$2R|\langle Af, Bf\rangle_{\mathscr{K}}| + R^2\|Bf\|_{\mathscr{K}}^2 \leqq \langle(1 - A^*A)f, f\rangle_{\mathscr{H}} . \tag{4}$$

Therefore (2) holds with $\delta = 1/R^2$, and

$$|\langle Af, Bf\rangle_{\mathscr{K}}| \leqq (2R)^{-1} \langle(1 - A^*A)f, f\rangle_{\mathscr{H}} . \tag{5}$$

We show that (3) also holds with $\delta = 1/R^2$. Consider first any $f$ and $g$ in $\mathscr{H}$ such that

$$\langle(1 - A^*A)f, f\rangle_{\mathscr{H}} = \langle(1 - A^*A)g, g\rangle_{\mathscr{H}} = 1 .$$

Applying (5) with $f$ replaced by $f \pm g$ and $f \pm ig$, we obtain

$$\begin{aligned} |\langle Af, Bg\rangle_{\mathscr{K}}| &= \tfrac{1}{4}|\langle A(f + g), B(f + g)\rangle_{\mathscr{K}} - \langle A(f - g), B(f - g)\rangle_{\mathscr{K}} + \\ &\quad + i\langle A(f + ig), B(f + ig)\rangle_{\mathscr{K}} - i\langle A(f - ig), B(f - ig)\rangle_{\mathscr{K}}| \leqq \\ &\leqq (8R)^{-1} [\langle(1 - A^*A)(f + g), f + g\rangle_{\mathscr{H}} + \langle(1 - A^*A)(f - g), f - g\rangle_{\mathscr{H}} + \\ &\quad + \langle(1 - A^*A)(f + ig), f + ig\rangle_{\mathscr{H}} + \langle(1 - A^*A)(f - ig), f - ig\rangle_{\mathscr{H}}] = \\ &= (2R)^{-1} [\langle(1 - A^*A)f, f\rangle_{\mathscr{H}} + \langle(1 - A^*A)g, g\rangle_{\mathscr{H}}] = R^{-1} . \end{aligned}$$

Assuming only that $\langle(1 - A^*A)f, f\rangle_{\mathscr{H}} \neq 0$ and $\langle(1 - A^*A)g, g\rangle_{\mathscr{H}} \neq 0$ and replacing $f$ and $g$ in the preceding calculation by

$$f/\langle(1 - A^*A)f, f\rangle_{\mathscr{H}}^{1/2} \quad \text{and} \quad g/\langle(1 - A^*A)g, g\rangle_{\mathscr{H}}^{1/2} ,$$

we obtain (3) with $\delta = 1/R^2$.

It remains to show that (3) holds with $\delta = 1/R^2$ if either $\langle(1 - A^*A)f, f\rangle_{\mathscr{H}}$ or $\langle(1 - A^*A)g, g\rangle_{\mathscr{H}}$ is zero. For definiteness, suppose $\langle(1 - A^*A)f, f\rangle_{\mathscr{H}} = 0$. Repeating the estimate of the preceding paragraph up to the next to last stage, we obtain

$$|\langle Af, Bg\rangle_{\mathscr{K}}| \leqq (2R)^{-1} \langle(1 - A^*A)g, g\rangle_{\mathscr{H}} .$$

Replace $g$ by $\varepsilon g$ and let $\varepsilon$ tend to zero to see that $\langle Af, Bg\rangle_{\mathscr{K}} = 0$. We have shown that (2) and (3) hold in all cases with $\delta = 1/R^2$.

Conversely, suppose that (2) and (3) hold for some $\delta > 0$ and all $f$ and $g$ in $\mathscr{H}$. Then we may choose $R > 0$ such that (4) holds for all $f$ in $\mathscr{H}$. It follows from (4) that $\|(A + \lambda B)f\|_{\mathscr{K}}^2 \leqq \|f\|_{\mathscr{H}}^2$ for all $f$ in $\mathscr{H}$ and $|\lambda| \leqq R$. Since $\|A\| = 1$, $\|A + \lambda B\| = \|A\|$ for $|\lambda| < R$, and hence $B$ belongs to $E(A)$. ∎

**Lemma 2.** *Given any operators* $U \in \mathscr{B}(\mathscr{H}_1, \mathscr{K})$ *and* $V \in \mathscr{B}(\mathscr{H}_2, \mathscr{K})$, *the following assertions are equivalent*:

(i) $U = VW$ *for some* $W \in \mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$;

(ii) $U\mathscr{H}_1 \subseteq V\mathscr{H}_2$;

(iii) $UU^* \leqq \lambda VV^*$ *for some positive real number* $\lambda$.

Proof. See Douglas [1]. ■

Proof of Theorem 2. Suppose that $B$ has the form (1) for some $C$ in $\mathscr{B}(\mathscr{H}, \mathscr{K})$. For any $f$ in $\mathscr{H}$,

$$\|Bf\|_{\mathscr{K}}^2 = \|(1 - AA^*)^{1/2} C(1 - A^*A)^{1/2} f\|_{\mathscr{K}}^2 \leqq \delta_1 \|(1 - A^*A)^{1/2} f\|_{\mathscr{H}}^2 = \\ = \delta_1 \langle (1 - A^*A) f, f \rangle_{\mathscr{H}},$$

where $\delta_1 = \|(1 - AA^*)^{1/2} C\|^2$. For any $f$ and $g$ in $\mathscr{H}$,

$$|\langle Af, Bg \rangle_{\mathscr{K}}|^2 = |\langle f, A^*(1 - AA^*)^{1/2} C(1 - A^*A)^{1/2} g \rangle_{\mathscr{H}}|^2 = \\ = |\langle f, (1 - A^*A)^{1/2} A^*C(1 - A^*A)^{1/2} g \rangle_{\mathscr{H}}|^2 \leqq \\ \leqq \delta_2 \langle (1 - A^*A) f, f \rangle_{\mathscr{H}} \langle (1 - A^*A) g, g \rangle_{\mathscr{H}},$$

where $\delta_2 = \|A^*C\|^2$. By Lemma 1, $B$ belongs to $E(A)$.

Conversely suppose that $B$ belongs to $E(A)$. Then $B^*$ belongs to $E(A^*)$. Choose $\delta$ for $A$, $B$ and $A^*$, $B^*$ as in Lemma 1. By (2),

$$B^*B \leqq \delta(1 - A^*A) \quad \text{and} \quad BB^* \leqq \delta(1 - AA^*) .$$

By Lemma 2 we can write

$$B = T(1 - A^*A)^{1/2} \quad \text{and} \quad B^* = R(1 - AA^*)^{1/2}$$

for some $T \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ and $R \in \mathscr{B}(\mathscr{K}, \mathscr{H})$. In particular,

$$T(1 - A^*A)^{1/2} \mathscr{H} = B\mathscr{H} = (1 - AA^*)^{1/2} R^*\mathscr{H} \subseteq (1 - AA^*)^{1/2} \mathscr{K} = \\ = (1 - AA^*)^{1/2} \mathscr{D}(A^*) ,$$

where $\mathscr{D}(A^*) = ((1 - AA^*)^{1/2} \mathscr{K})^-$.

Let $\mathscr{H}_A$ be the range of $(1 - A^*A)^{1/2}$, viewed as a Hilbert space in the inner product which makes $(1 - A^*A)^{1/2}$ a partial isometry from $\mathscr{H}$ onto $\mathscr{H}_A$; the isometric set of the partial isometry is $\mathscr{D}(A) = ((1 - A^*A)^{1/2} \mathscr{H})^-$. Since the inclusion of $\mathscr{H}_A$ in $\mathscr{H}$ is continuous, there is an operator $T_A \in \mathscr{B}(\mathscr{H}_A, \mathscr{K})$ such that

$$T_A g = Tg , \quad g \in \mathscr{H}_A .$$

By what was shown above, $T_A \mathscr{H}_A \subseteq (1 - AA^*)^{1/2} \mathscr{D}(A^*)$. Hence by Lemma 2, there is an operator $C_A \in \mathscr{B}(\mathscr{H}_A, \mathscr{D}(A^*))$ such that

$$T_A = (1 - AA^*)^{1/2} C_A .$$

We show that $C_A$ is bounded relative to the norms of $\mathscr{H}$ and $\mathscr{K}$. Consider vectors $u = (1 - A^*A)^{1/2} f$ and $v = (1 - A^*A)^{1/2} g$ in $\mathscr{H}$, where $f, g \in \mathscr{H}$. For the positive number $\delta$ chosen above, we have

$$(6) \qquad \|(1 - AA^*)^{1/2} C_A u\|^2_{\mathscr{K}} = \|Bf\|^2_{\mathscr{K}} \leqq \delta\langle(1 - A^*A) f, f\rangle_{\mathscr{H}} = \delta\|u\|^2_{\mathscr{H}},$$

and by (3),

$$(7) \qquad |\langle v, A^*C_A u\rangle_{\mathscr{H}}|^2 = |\langle g, (1 - A^*A)^{1/2} A^*C_A(1 - A^*A)^{1/2} f\rangle_{\mathscr{H}}|^2 = = |\langle g, A^*(1 - AA^*)^{1/2} C_A(1 - A^*A)^{1/2} f\rangle_{\mathscr{H}}|^2 = |\langle g, A^*Bf\rangle_{\mathscr{H}}|^2 \leqq \leqq \delta\langle(1 - A^*A) g, g\rangle_{\mathscr{H}} \langle(1 - A^*A) f, f\rangle_{\mathscr{H}} = \delta\|u\|^2_{\mathscr{H}} \|v\|^2_{\mathscr{H}}.$$

By (7), since $A^*C_A u \in A^* \mathscr{D}(A^*) \subseteq \mathscr{D}(A)$,

$$(8) \qquad \|A^*C_A u\|^2_{\mathscr{H}} \leqq \delta\|u\|^2_{\mathscr{H}}.$$

Combining (6) and (8), we obtain

$$\|C_A u\|^2_{\mathscr{H}} = \langle(1 - AA^*) C_A u, C_A u\rangle_{\mathscr{K}} + \langle AA^*C_A u, C_A u\rangle_{\mathscr{K}} \leqq 2\delta\|u\|^2_{\mathscr{H}}.$$

This shows that $C_A$ is bounded relative to the norms of $\mathscr{H}$ and $\mathscr{K}$, and so there is an operator $C \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ such that $C_A f = Cf$ for all $f$ in $\mathscr{H}_A$. By construction, for any $f$ in $\mathscr{H}$,

$$Bf = T(1 - A^*A)^{1/2} f = T_A(1 - A^*A)^{1/2} f = (1 - AA^*)^{1/2} C_A(1 - A^*A)^{1/2} f = = (1 - AA^*)^{1/2} C(1 - A^*A)^{1/2} f.$$

Therefore $B$ has the form (1). ∎

It is natural to ask if a similar result holds for any $C^*$ algebra. John Erdos has shown that the answer is negative, but there may be algebras other than $\mathscr{B}(\mathscr{H})$ for which the result holds. The author thanks John Erdos and Vlastimil Pták for discussions of the ideas in this paper.

### *References*

[1] *R. G. Douglas:* On majorization, factorization, and range inclusion of operators on Hilbert space, Proc. Amer. Math. Soc. *17* (1966), 413–415.

[2] *J. Globevnik:* On vector-valued analytic functions with constant norm, Studia Math. *53* (1975), 29–37.

[3] *L. A. Harris:* Schwarz's lemma in normed linear spaces, Proc. Nat. Acad. Sci. *62* (1969), 1014–1017.

[4] *V. Pták* and *P. Vrbová:* Lifting intertwining relations, Integral Equations and Operator Theory, *11* (1988), 128–147.

[5] *E. Thorp* and *R. Whitley:* The strong maximum modulus theorem for analytic functions into a Banach space, Proc. Amer. Math. Soc. *18* (1967), 640–646.

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