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ON A CLASS OF LINEAR n-TH ORDER DIFFERENTIAL EQUATIONS

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Dedicated to Academician Michal Greguš on the occasion of his sixtieth birthday

1. INTRODUCTION

Consider the n-th order $(n \geqq 2)$ linear differential equation

$$(\mathrm{E}) \qquad (L[y] \equiv)\, y^{(n)} + \sum_{k=2}^{n} p_k(t)\, y^{(n-k)} = 0\,,$$

where the coefficients $p_k(t)$, $k = 2, \ldots, n$, are real-valued continuous functions on the interval $I = \langle a, \infty)$, $-\infty < a < \infty$. Sometimes the following assumptions will be required:

$$(\mathrm{A}) \qquad \sum_{k=2}^{n} p_k(t) \frac{x^{k-2}}{(k-2)!} \leqq 0 \quad \text{for all} \quad t \in I\,, \quad x \in R\,;$$

(B) the hypothesis (A) is satisfied, $n = 2m$, $p_k(t) \leqq 0$ for all $t \in I$, $k = 2, 3, \ldots, n$, and $p_n(t)$ is not identically zero in any subinterval of I;

(C) the hypothesis (B) is satisfied and $p_3(t) \equiv 0$, $p_5(t) \equiv 0, \ldots, p_{n-1}(t) \equiv 0$ for all $t \in I$.

For the orders $n = 2$, $n = 3$ and $n = 4$, the condition (A) is satisfied by the equations

$$y'' + p_2(t)\, y = 0 \quad \text{with} \quad p_2(t) \leqq 0 \quad \text{in} \quad I\,,$$

$$y''' + p_2(t)\, y' = 0 \quad \text{with} \quad p_2(t) \leqq 0 \quad \text{in} \quad I\,,$$

and

$$y^{(4)} + p_2(t)\, y'' + p_3(t)\, y' + p_4(t)\, y = 0\,, \quad p_2(t) \leqq 0\,,$$

$$p_3^2(t) \leqq 2\, p_2(t)\, p_4(t) \quad \text{in} \quad I\,,$$

respectively. The last equation has been studied by J. Regenda in several papers, e.g. [8], [9], [10], [11].

It is clear that if the equation (E) satisfies the assumption (A), then $p_2(t) \leqq 0$ in I and for $n = 2m + 1$ we have $p_n(t) \equiv 0$ in I, while for $n = 2m$ we have $p_n(t) \leqq 0$ in this interval. Conversely, if $n = 2m$, $p_n(t) < 0$ in I, then for any $p_3(t), \ldots, p_{n-1}(t)$ there exists (a sufficiently great in absolute value) $p_2(t) \leqq 0$ such that the equation (E) satisfies (A).

Although the equations (E) of the second and third orders satisfying the condition (A) are disconjugate, the equation $y^{(4)} - y = 0$ (having the property (A)) possesses a fundamental system of solutions $y_1(t) = e^t$, $y_2(t) = e^{-t}$, $y_3(t) = \cos t$, $y_4(t) = \sin t$, and, thus, has oscillatory solutions.

A nontrivial solution of the differential equation (E) is called *oscillatory* if its set of zeros is not bounded from above. Otherwise, it is called *nonoscillatory*. The equation (E) will be called *nonoscillatory* when all its solutions are nonoscillatory; *oscillatory* when at least one of its solutions is oscillatory. It is said to be *disconjugate in an interval* $J \subset I$ iff each of its non trivial solutions has at most $n - 1$ zeros in J, counting each zero so many times as its multiplicity indicates. It is eventually disconjugate (on I) if it is disconjugate on an interval of the type (b, ∞), where $b \in I$.

In the paper fundamental properties of the equation (E) are derived under some of the assumptions (A), (B), (C), such as the existence of solutions without zeros, a comparison theorem, the existence of a bundle of solutions and the properties of nonoscillatory solutions of the equation.

2. PRELIMINARIES

We begin by formulating and proving the results which are needed later on.

Lemma 1. *Suppose that* $t_0 \in I$, y_0^i, $i = 0, 1, \ldots, n - 1$, *are arbitrary numbers. Then the initial value problem*

$$(1) \qquad L[y] = 0 \,, \quad y^{(i)}(t_0) = y_0^i \,, \quad i = 0, 1, \ldots, n - 1 \,,$$

is equivalent to the following Volterra's integral equation

$$(2) \qquad y^{(n-1)}(t) = g(t) + \int_{t_0}^t A(t, s)\, y^{(n-1)}(s)\, ds \,, \quad t \in I \,,$$

where

$$(3) \qquad g(t) = y_0^{n-1} - \sum_{j=0}^{n-2} y_0^j \sum_{k=n-j}^{n} \int_{t_0}^t p_k(s) \frac{(s - t_0)^{j-n+k}}{(j - n + k)!}\, ds \,,$$

$$(4) \qquad A(t, s) = - \sum_{k=2}^{n} \int_s^t p_k(u) \frac{(u - s)^{k-2}}{(k - 2)!}\, du \,, \quad t, s \in I \,.$$

Proof. Integrating the equation (E) from t_0 to t and taking the initial conditions in (1) into consideration, we get

$$(5) \qquad y^{(n-1)}(t) = y_0^{n-1} - \sum_{k=2}^{n} \int_{t_0}^t p_k(s) \left(\sum_{l=0}^{k-2} \frac{y_0^{n-k+l}}{l!} (s - t_0)^l + \int_{t_0}^s \frac{(s - u)^{k-2}}{(k - 2)!}\, y^{(n-1)}(u)\, du \right) ds \,.$$

We put (5) into the form (2). First we denote

$$g(t) = y_0^{n-1} - \sum_{k=2}^{n} \int_{t_0}^t p_k(s) \left(\sum_{l=0}^{k-2} \frac{y_0^{n-k+l}}{l!} (s - t_0)^l \right) ds \,.$$

Then

$$g(t) = y_0^{n-1} - \sum_{k=2}^{n} \int_{t_0}^{t} p_k(s) \sum_{j=n-k}^{n-2} \frac{y_0^j}{(j-n+k)!} (s-t_0)^{j-n+k} \, ds ,$$

which implies (3).

Similarly we consider the function

$$-\sum_{k=2}^{n} \int_{t_0}^{t} p_k(u) \left(\int_{t_0}^{u} \frac{(u-s)^{k-2}}{(k-2)!} y^{(n-1)}(s) \, ds \right) du =$$

$$= - \int_{t_0}^{t} \left[\sum_{k=2}^{n} \int_{s}^{t} p_k(u) \frac{(u-s)^{k-2}}{(k-2)!} \, du \right] y^{(n-1)}(s) \, ds .$$

Then, on the basis of (3) and (4), we get (2).

By virtue of the assumption (A), the function $A(t, s)$ given by (4) is continuous and nonnegative for $t_0 \leqq s \leqq t$ as well as nonpositive for $a \leqq t \leqq s \leqq t_0$. The following lemma deals with the equation (2) in this case.

Lemma 2 ([8], p. 331). *Let $A(t, s)$ be a nonnegative and continuous function for $t_0 \leqq s \leqq t$ (a nonpositive and continuous function for $a \leqq t \leqq s$). If $g(t)$, $\varphi(t)$ ($\psi(t)$) are continuous functions in the interval $\langle t_0, \infty)$ ($\langle a, t_0 \rangle$) and*

$$\varphi(t) \leqq g(t) + \int_{t_0}^{t} A(t, s) \varphi(s) \, ds \quad \textit{for} \quad t \in \langle t_0, \infty)$$

$$(\psi(t) \geqq g(t) + \int_{t_0}^{t} A(t, s) \psi(s) \, ds \quad \textit{for} \quad t \in \langle a, t_0 \rangle) ,$$

then every solution $y(t)$ of the integral equation

$$y(t) = g(t) + \int_{t_0}^{t} A(t, s) y(s) \, ds \tag{6}$$

satisfies the inequality

$$y(t) \geqq \varphi(t) \quad in \quad \langle t_0, \infty) \quad (y(t) \leqq \psi(t) \quad in \quad \langle a, t_0 \rangle) .$$

If we suppose in addition that $g(t) \geqq 0$ for $t \in \langle t_0, \infty)$ ($g(t) \leqq 0$ for $t \in \langle a, t_0 \rangle$), then the solution $y(t)$ of (6) satisfies the inequality

$$y(t) \geqq g(t) \geqq 0 \quad for \quad t \in \langle t_0, \infty) \quad (y(t) \leqq g(t) \leqq 0 \; for \; t \in \langle a, t_0 \rangle) .$$

We shall show that under the assumption (A) neither the solution $y(t)$ of the equation (E) satisfying the conditions

$$y^{(i)}(t_0) = 0 , \quad i = 0, 1, \ldots, n-2 , \quad y^{(n-1)}(t_0) \neq 0 \tag{7}$$

nor any of its derivatives $y^{(j)}(t)$, $j = 1, \ldots, n-1$, has a zero at $t \in I$, $t \neq t_0$.

Lemma 3. *Suppose that* (A) *holds and let $y(t)$ be the solution of* (E) *satisfying the initial conditions* (7) *with $y^{(n-1)}(t_0) > 0$. Then:*

(8) (i) $y^{(i)}(t) > 0$ *for all* $t > t_0$, $i = 0, 1, \ldots, n-1$.

(ii) If $a < t_0$, *then*

$$(-1)^{i+1} y^{(n-i)}(t) > 0 \quad for \; all \quad t \in \langle a, t_0), \; i = 1, 2, \ldots, n. \tag{9}$$

Proof. If the solution y of (E) satisfies (7) and $y_0^{n-1} = y^{(n-1)}(t_0) > 0$, then the function $g(t)$ determined by (2) is $g(t) = y_0^{n-1} > 0$ and, by Lemma 2, $y^{(n-1)}(t) \geqq y_0^{n-1}$ for all $t \geqq t_0$, which in view of (7) leads to the inequalities (8).

If $a < t_0$ and the solution y of (E) satisfies (7) with $y_0^{n-1} = y^{(n-1)}(t_0) < 0$, then $g(t) = y_0^{n-1} < 0$ and, by Lemma 2, $y^{(n-1)}(t) \leqq y_0^{n-1} < 0$ for all $t \in \langle a, t_0 \rangle$. This, with respect to (7), implies $(-1)^i y^{(n-i)}(t) > 0$ in $\langle a, t_0)$, $i = 1, 2, \ldots, n$. Hence the inequalities (9) for the solution y with $y_0^{n-1} > 0$ are true.

Under the condition (B) or the condition (C) stronger results can be proved.

Lemma 3'. *Suppose that* (B) *holds and let $y(t)$ be a nontrivial solution of* (E) *satisfying at $t_0 \in I$ the initial conditions*

$$y^{(i)}(t_0) = y_0^i \geqq 0\,, \quad i = 0, 1, \ldots, n-1\,.$$

Then

$$y^{(i)}(t) > 0 \quad \textit{for all} \quad t > t_0\,, \quad i = 0, 1, \ldots, n-1\,.$$

Lemma 3''. *Suppose that* (C) *holds and $t_0 > a$. Let $y(t)$ be a nontrivial solution of* (E) *satisfying the initial conditions*

$$(-1)^i\, y^{(i)}(t_0) = (-1)^i\, y_0^i \geqq 0\,, \quad i = 0, 1, \ldots, n-1\,.$$

Then

$$(-1)^i\, y^{(i)}(t) > 0 \quad \text{for all } t, \quad a \leqq t < t_0\,, \quad i = 0, 1, \ldots, n-1\,.$$

The proofs are similar to that of Lemma 3 and will be omitted.

Using the well-known Kiguradze lemmas ([5], pp. 289–290, [12], p. 94) we get the following lemma.

Lemma 4. *Let $y(t) \in C^{(n)}(I)$ be such that $y(t) > 0$ in $\langle b, \infty)$ where $a \leqq b < \infty$. Then there is a c, $b \leqq c < \infty$, such that*

either

(i) *there is an l, $0 \leqq l \leqq n$, with the following property. If $l > 0$, then $y^{(i)}(t) > 0$, $c \leqq t < \infty$, $i = 0, 1, \ldots, l-1$; if $l \leqq n-1$, then $(-1)^{l+j}\, y^{(j)}(t) > 0$ for $c \leqq$ $\leqq t < \infty$, $j = l, l+1, \ldots, n-1$; and $(-1)^{l+n}\, y^{(n)}(t) \geqq 0$ in $\langle c, \infty)$, $y^{(n)}(t) \equiv 0$ holds in no subinterval $\langle d, \infty) \subset \langle c, \infty)$,*

or

(ii) *there is a k, $1 \leqq k \leqq n$, with the property*

$$y^{(k)}(t) \equiv 0 \quad \textit{in} \quad \langle c, \infty)$$

and $y^{(i)}(t) > 0$ in $\langle c, \infty)$ for $i = 0, 1, \ldots, k-1$,

or

(iii) *there is a k, $1 \leqq k \leqq n$, and an l, $0 \leqq l \leqq k-1$, such that if $l > 0$, then $y^{(i)}(t) > 0$ in $\langle c, \infty)$, $i = 0, 1, \ldots, l-1$; if $l \leqq k-2$, then $(-1)^{l+j}\, y^{(j)}(t) > 0$ in $\langle c, \infty)$, $j = l, l+1, \ldots, k-2$; further $(-1)^{l+k-1}\, y^{(k-1)}(t) \geqq 0$ in $\langle c, \infty)$, $y^{(k-1)}(t) \equiv 0$ holds in no subinterval $\langle d, \infty) \subset \langle c, \infty)$ and*

$y^{(k)}(t)$ is strictly oscillatory in $\langle c, \infty)$, i.e. it changes its sign in each subinterval $\langle d, \infty) \subset \langle c, \infty)$ infinitely many times.

Proof. Consider the function $y^{(n)}(t)$ in $\langle b, \infty)$. Three cases may occur.

1. $y^{(n)}$ is of constant sign in $\langle b, \infty)$, $y^{(n)}(t) \equiv 0$ holds in no subinterval $\langle d, \infty) \subset \subset \langle c, \infty)$. Then the first two Kiguradze lemmas are applicable. They give the statement (i).

2. $y^{(n)}(t) \equiv 0$ in an interval $\langle d, \infty) \subset \langle b, \infty)$. We denote by k, $1 \leqq k \leqq n$, the smallest integer i for which $y^{(i)}(t) \equiv 0$ in an interval $\langle d_1, \infty) \subset \langle b, \infty)$. Clearly $y^{(k-1)}(t) \equiv \text{const} > 0$ in $\langle d_1, \infty)$ and by integration we get the statement (ii).

3. $y^{(n)}(t)$ is strictly oscillatory in an interval $\langle d, \infty) \subset \langle b, \infty)$. Then we again consider the smallest integer i for which $y^{(i)}(t)$ is strictly oscillatory. If we denote it by k, then $y^{(k)}$ is strictly oscillatory, but $y^{(k-1)}(t)$ is of constant sign in an interval $\langle d_1, \infty)$ and $y^{(k-1)}(t) \equiv 0$ holds on no subinterval $\langle d_2, \infty)$ of $\langle d_1, \infty)$. Again the statement (iii) follows from the Kiguradze lemmas.

The next lemma is similar to a result proved in [7] by M. Medveď under stronger conditions $(p_k(t) \in C^{(n-k)}((a, b)))$. The same result has been given in Corollary 5.1, [6], p. 90. Here another proof is constructed.

Lemma 5. *Suppose that the equation* (E) *is disconjugate in* $\langle t_0, \infty)$ *where* $t_0 \in I$, *and the function* $f(t) \in C(\langle t_0, \infty))$ *does not change its sign in* $\langle t_0, \infty)$. *Then the differential equation*

$$L[y] = f(t) \tag{10}$$

is nonoscillatory in $\langle t_0, \infty)$, *i.e. for each solution* $y(t)$ *of* (10) *there exists an interval* $\langle t_1, \infty)$, $t_0 \leqq t_1$, *such that either* $y(t) \equiv 0$ or $y(t) \neq 0$ *for* $t_1 < t < \infty$. *If the former case occurs then* $f(t) \equiv 0$ *in* $\langle t_1, \infty)$.

Proof. According to G. Mammana (see [6], p. 45, or [7], pp. 102–103), if (E) is disconjugate, then there exist real continuous functions $g_1(t), \ldots, g_n(t)$ in $\langle t_0, \infty)$ such that the operator L can be decomposed into factors

$$L[y] = \left(\frac{\mathrm{d}}{\mathrm{d}t} - g_n(t)\right) \ldots \left(\frac{\mathrm{d}}{\mathrm{d}t} - g_1(t)\right) y .$$

Hence, if we put

$$y = y_1, \quad y_i' - g_i(t)\, y_i = y_{i+1}, \quad i = 1, \ldots, n-1, \quad y_n' - g_n(t)\, y_n = f(t), \tag{11}$$

then the equation (10) is equivalent to the system

$$\begin{aligned} y_i' &= g_i(t)\, y_i + y_{i+1}, \quad i = 1, \ldots, n-1, \\ y_n' &= g_n(t)\, y_n + f(t) \end{aligned} \tag{12}$$

in the following sense. If $y(t)$ is a solution of (10), then the vector function $(y_1(t), \ldots \ldots, y_n(t))$ determined by (11) is a solution of the system (12) and conversely, if $(y_1(t), \ldots, y_n(t))$ is a solution of (12), then $y(t) = y_1(t)$ satisfies (10).

Suppose that $f(t) \geqq 0$ in $\langle t_0, \infty)$. In the case $f(t) \leqq 0$ we would proceed similarly. First we show that any solution of the equation $y_n' = g_n(t)\, y_n + f(t)$ is nonoscillatory. If such a solution y_n is negative in a neighbourhood of ∞, then it is nonoscillatory.

If there is a point $t_1 \geqq t_0$ such that $y_n(t_1) \geqq 0$, then it can be written in the form

(13) $$y_n(t) = u_n(t) + \int_{t_1}^{t} K(t, s) f(s) \, ds,$$

where $K(t, s)$, $t_1 \leqq s \leqq t$, is the Cauchy function for the equation

(14) $$y' - g_n(t) \, y = 0$$

and hence it is positive for $t_1 \leqq s < t$, while $u_n(t)$ is the solution of (14) determined by the condition $u_n(t_1) = y_n(t_1) \geqq 0$ and thus $y_n(t) \geqq u_n(t) \geqq 0$ for all $t > t_1$. Further, (13) yields that either $y_n(t) \equiv 0$ in a neighbourhood of ∞ or $y_n(t) > 0$ in an interval $\langle t_2, \infty)$. Therefore the equation $y_n' = g_n(t) \, y_n + f(t)$ is nonoscillatory. By finite induction we can show that all equations in the system (12) are nonoscillatory. This implies that (10) is nonoscillatory. Clearly $y(t) \equiv 0$ in $\langle t_1, \infty)$ implies that $f(t) \equiv 0$ in the same interval.

In the next lemma the notation $f(t) \ll g(t)$ for $t \to \infty$ (taken from [6], p. 57) will mean that both functions f and g are positive in a neighbourhood of ∞ and $f(t) = o(g(t))$ for $t \to \infty$. For the sake of completeness we state Lemma 2.1 from [6], p. 58 as

Lemma 6. *Let V be an n-dimensional vector space of functions continuous in I. Then the following two statements are equivalent*:

1. *Each function $y(t) \in V$, $y(t) \not\equiv 0$, is different from 0 in a neighbourhood of ∞.*
2. *There exists a basis $\{y_i(t)\}_{i=1}^{n}$ for V such that*

$$y_1(t) \ll y_2(t) \ll \ldots \ll y_n(t) \quad \textit{for} \quad t \to \infty .$$

Using this lemma we prove the following result.

Lemma 7. *Suppose that the equation* (E) *is nonoscillatory, the function $f(t) \in C(I)$ is nonoscillatory and let $y_1(t) \ll y_2(t) \ll \ldots \ll y_n(t)$ for $t \to \infty$ be a hierarchical system of solutions of* (E). *Then the nonhomogeneous equation* (10) *is nonoscillatory, i.e., all its solutions are nonoscillatory, iff there is a nonoscillatory solution $y_0(t)$ of* (10) *such that $|y_0|, y_1, y_2, \ldots, y_n$ form a hierarchical system of functions, i.e.*

either

$$|y_0(t)| \ll y_1(t) \ll y_2(t) \ll \ldots \ll y_n(t) \quad \textit{for} \quad t \to \infty$$

or there is a $j \in \{1, 2, \ldots, n - 1\}$ such that

$$y_1(t) \ll \ldots \ll y_j(t) \ll |y_0(t)| \ll y_{j+1}(t) \ll \ldots \ll y_n(t) \quad \textit{for} \quad t \to \infty ,$$

or

$$y_1(t) \ll y_2(t) \ll \ldots \ll y_n(t) \ll |y_0(t)| \quad \textit{as} \quad t \to \infty .$$

Proof. If $|y_0|, y_1, y_2, \ldots, y_n$ form a hierarchical system in the sense given above, then it is clear that each solution

$$y(t) = \sum_{i=1}^{n} c_i \, y_i(t) + y_0(t) , \quad t \in I ,$$

of (10) is nonoscillatory.

Conversely, suppose that each solution of (10) is nonoscillatory. Then for each $i = 1, 2, \ldots, n$ and any solution y_0 of (10) the function $y_i(t)/y_0(t)$ is continuous in a neighbourhood of ∞ and moreover, for each $c \in R$, $y_i(t)/y_0(t) \neq c$ in a neighbourhood of ∞. In fact, the roots of the equation $y_i(t)/y_0(t) = c$ are either the roots of $y_i(t)$ (when $c = 0$) or the zeros of the solution $-(1/c)\, y_i(t) + y_0(t)$ of (10). The inequality $y_i(t)/y_0(t) \neq c$ for each $c \in R$ and the continuity of $y_i(t)/y_0(t)$ in a neighbourhood of ∞ ensure the existence of a finite or infinite $\lim_{t\to\infty} y_i(t)/y_0(t)$. This implies that for each $i = 1, 2, \ldots, n$ and any solution $y_0(t)$ of (10) one and only one of the following cases may arise: $y_i(t) \ll |y_0(t)|$, $|y_0(t)| \ll y_i(t)$, $y_0(t) \sim c\, y_i(t)$ for $t \to \infty$ with $0 < |c| < \infty$.

Let us fix a solution y_0 of (10). Then either $|y_0|, y_1, y_2, \ldots, y_n$ form a hierarchical system in the sense given in the statement of the lemma, or there exists a j, $1 \leqq j \leqq n$, such that $y_0(t) \sim c\, y_j(t)$ for $t \to \infty$. In the former case the proof of the lemma is complete. In the latter case we consider the function $y_0^1(t) = y_0(t) - c\, y_j(t)$, $t \in I$, which is a nonoscillatory solution of (10). Again two cases may occur. Either y_0^1, $y_1, y_2, \ldots, y_n$ form a hierarchical system and the proof is done or there exists a k, $1 \leqq k \leqq n$, such that $y_0^1(t) \sim c_1\, y_k(t)$ for $t \to \infty$. As $\lim_{t\to\infty} y_0^1(t)/y_j(t) = 0$ and thus $|y_0^1(t)| \ll y_j(t)$, we must have $k < j$. We now consider the solution of (10), $y_0^2(t) = y_0^1(t) - c_1 y_k(t) = y_0(t) - c\, y_j(t) - c_1\, y_k(t)$, $t \in I$. As concerns this solution, either $|y_0^2|, y_1, y_2, \ldots, y_n$ is a hierarchical system or there is an l, $1 \leqq l < k < j$ such that $y_0^l(t) \sim c_2\, y_l(t)$ for $t \to \infty$. After at most j steps we come to a solution $y_0^m(t)$ of (10) such that $|y_0^m|, y_1, y_2, \ldots, y_n$ form a hierarchical system of functions.

The next lemma is interesting in itself and will play an important role in the investigation of a nonoscillatory equation (E). It has been given by U. Elias in [2], p. 269 as Theorem 1.

Lemma 8. *Suppose that the equation* (E) *is disconjugate on* I, $p(t)$ *is a continuous function of a fixed sign on* I, *and the perturbed equation*

$$(\mathrm{E}) \qquad L[y] + p(t)\, y = 0$$

is nonoscillatory on I. *Then the equation* (E) *is eventually disconjugate.*

3. THE EXISTENCE OF MONOTONIC SOLUTIONS

Theorem 1. *Suppose that* (A) *holds. Then there exists a solution* $y(t)$ *of* (E) *such that*

$$(15) \qquad y^{(i)}(t) > 0 \quad \text{for all} \quad t > a\,, \quad i = 0, 1, \ldots, n-1\,.$$

Proof. Let $y(t)$ be the solution of (E) which satisfies the initial conditions $y^{(i)}(a) = 0$, $i = 0, 1, \ldots, n-2$, $y^{(n-1)}(a) = 1$. Then, by Lemma 3, the inequalities (15) follow.

Denote by $z_0(t), z_1(t), \ldots, z_{n-1}(t)$ the solutions of (E) defined on I which are determined by the initial conditions

$$z_i^{(j)}(a) = \delta_{ij} = \begin{cases} 0, & i \neq j \quad \text{for} \quad i, j = 0, 1, \ldots, n-1. \\ 1, & i = j \end{cases}$$

Theorem 2. *Suppose that* (A) *holds. Then there exists a solution* $z(t)$ *of* (E) *such that*

either

(16) $$(-1)^i z^{(i)}(t) > 0 \quad \textit{for all} \quad t \in I, \quad i = 0, 1, \ldots, n-2, \\ (-1)^{n-1} z^{(n-1)}(t) \geqq 0 \quad \textit{in} \quad I,$$

or

$z(t) > 0$ *for all* $t \in I$ *and there exists a* $t_0 \in I$ *such that* $z^{(i)}(t) \equiv 0$ *for all* $t \geqq t_0$, $i = 1, 2, \ldots, n-1$.

Proof. We shall apply a construction similar to that given in [4], p. 15. For each natural number $k > a$ let $c_{0,k}, c_{1,k}, \ldots, c_{n-1,k}$ be numbers satisfying

(17) $$\sum_{i=0}^{n-1} c_{i,k}^2 = 1, \\ \sum_{i=0}^{n-1} c_{i,k}\, z_i^{(j)}(k) = a_j, \quad j = 0, 1, \ldots, n-1,$$

where a_j, $j = 0, 1, \ldots, n-1$, are such that $a_0 = a_1 = \ldots = a_{n-2} = 0$, $(-1)^n a_{n-1} < 0$.

Since $z_0(t), z_1(t), \ldots, z_{n-1}(t)$ are linearly independent and a_{n-1} can be arbitrarily chosen, the numbers $c_{0,k}, c_{1,k}, \ldots, c_{n-1,k}$ do exist.

Denote $\tilde{z}_k(t) = \sum_{i=0}^{n-1} c_{i,k}\, z_i(t)$. Then $\tilde{z}_k(t)$ is a nontrivial solution of (E). In view of the first condition in (17) and by Lemma 3,

(18) $$(-1)^j \tilde{z}_k^{(j)}(t) > 0, \quad j = 0, 1, \ldots, n-1, \quad a \leqq t < k.$$

For each natural number i, $0 \leqq i \leqq n-1$, the sequence $\{c_{i,k}\}$ is bounded, thus there exists a sequence of natural numbers $\{k(l)\}$ such that the subsequences $\{c_{i,k(l)}\}$ converge to numbers c_i, $i = 0, 1, \ldots, n-1$, as $l \to \infty$. From (17) we see that $\sum_{i=0}^{n-1} c_i^2 = 1$. The sequences $\{\tilde{z}_{k(l)}(t)\}, \{\tilde{z}'_{k(l)}(t)\}, \ldots, \{\tilde{z}_{k(l)}^{(n-1)}(t)\}$ converge uniformly on any compact subinterval of I to the functions $z(t), z'(t), \ldots, z^{(n-1)}(t)$, respectively, where

$$z(t) = \sum_{i=0}^{n-1} c_i\, z_i(t)$$

is a nontrivial solution of (E). The inequalities (18) imply that

(19) $$(-1)^i z^{(i)}(t) \geqq 0 \quad \text{for all} \quad t \in I \quad \text{and} \quad i = 0, 1, \ldots, n-1.$$

If there existed a point $t_0 \in I$ and a $j \in \{0, 1, \ldots, n-2\}$ such that $z^{(j)}(t_0) = 0$, we would consider the smallest j with the mentioned property. Then, by (19), $z^{(j)}(t) \equiv 0$

would hold for all $t \geqq t_0$. As $z(t)$ is a nontrivial solution of (E), $z(t) > 0$ in I and hence, $j \geqq 1$. Denote $l = j - 1$. Then $z^{(l)}(t) \equiv c_l \neq 0$ in $\langle t_0, \infty)$ and, since $(-1)^l z^{(l)}(t) \geqq 0$ and c_l cannot be negative (this would imply that $z(t)$ is negative in a neighbourhood of ∞), l must be even. If $l > 0$, then $z^{(l)}(t) \equiv c_l > 0$ and $z^{(l-1)}(t) = c_l t + q$ and hence $z^{(l-1)}(t) > 0$ for all sufficiently great t, which contradicts (19), because $l - 1$ is odd. This contradiction shows that $l = 0$ and the theorem is proved.

Remark. If $p_n(t) \not\equiv 0$ in any neighbourhood of ∞ (and hence $n = 2m$), then in the alternative (16) only the first statement can hold. If $p_n(t) \equiv 0$, then solutions of both types in (16) can occur, as the example of the equation $y^{(5)} - y' = 0$ shows. This equation has the following fundamental system of solutions: $y_1(t) = 1$, $y_2(t) = e^t$, $y_3(t) = e^{-t}$, $y_4(t) = \sin t$, $y_5(t) = \cos t$.

Corollary. *If* (C) *holds, then there exists a solution* $z(t)$ *of* (E) *such that*

$$(-1)^i z^{(i)}(t) > 0 \text{ for all } t \in I\,, \quad i = 0, 1, \ldots, n - 1\,.$$

Proof. First, we know that there exists a solution $z(t)$ of the equation (E) with $(-1)^i z^{(i)}(t) > 0$ for all $t \in I$, $i = 0, 1, \ldots, n - 2$, and $(-1)^{n-1} z^{(n-1)}(t) \geqq 0$ in I. By Lemma 3″, $(-1)^{n-1} z^{(n-1)}(t) > 0$ must hold in the whole interval I.

4. COMPARISON THEOREMS

The fundamental property of the equation (E) under the assumption (A) is given by the following theorem which is stronger than the Čaplygin comparison theorem ([6], p. 46).

Theorem 3. *Suppose that* (A) *holds. Let* $t_0 \in I$ *and let* $u(t), v(t) \in C^n(I)$ *be two functions such that*

$$u^{(i)}(t_0) = v^{(i)}(t_0)\,, \quad i = 0, 1, \ldots, n - 1\,, \quad \text{and} \tag{20}$$
$$L[u](t) \geqq L[v](t) \quad \text{for all} \quad t \in I\,.$$

Then

(i) $u^{(i)}(t) \geqq v^{(i)}(t)$ *for all* $t \geqq t_0$, $i = 0, 1, \ldots, n - 1$;

(ii) *If* $a < t_0$, *then*

$$(-1)^{n-i} u^{(i)}(t) \geqq (-1)^{n-i} v^{(i)}(t) \quad \text{for all} \quad t\,, \quad a \leqq t \leqq t_0\,,$$
$$i = 0, 1, n - 1\,.$$

Moreover, if there is a $t_1, t_0 \leqq t_1$ $(a < t_1 \leqq t_0)$ *such that* $L[u](t_1) > L[v](t_1)$, *then*

$$u^{(i)}(t) > v^{(i)}(t) \quad \text{for all} \quad t > t_1\,, \quad i = 0, 1, \ldots, n - 1$$
$$((-1)^{n-i} u^{(i)}(t) > (-1)^{n-i} v^{(i)}(t) \quad \text{for all } t\,, \quad a \leqq t < t_1\,,$$
$$i = 0, 1, \ldots, n - 1)\,.$$

Proof. Denote by $K(t, s)$ the Cauchy function for the equation (E), i.e., $K(\cdot, s)$ is the solution $y(t)$ of (E) such that $y^{(i)}(s) = 0$, $i = 0, 1, \ldots, n-2$, $y^{(n-1)}(s) = 1$. By Lemma 3,

(21) $$\frac{\partial^i K(t, s)}{\partial t^i} > 0 \quad \text{for all} \quad t > s \gtreqless t_0\,, \quad i = 0, 1, \ldots, n-1\,,$$

and

(22) $$(-1)^{n-i+1} \frac{\partial^i K(t, s)}{\partial t^i} > 0 \quad \text{for all} \quad a \leqq t < s \leqq t_0\,, \quad i = 0\,, \; 1, \ldots, n-1\,.$$

Denote by $y_1(t)$ the solution of (E) satisfying $y_1^{(i)}(t_0) = u^{(i)}(t_0) = v^{(i)}(t_0)$, $i = 0, 1, \ldots, n-1$. Then

$$u^{(i)}(t) = y_1^{(i)}(t) + \int_{t_0}^{t} \frac{\partial^i K(t, s)}{\partial t^i} L[u](s)\, ds\,,$$

$$v^{(i)}(t) = y_1^{(i)}(t) + \int_{t_0}^{t} \frac{\partial^i K(t, s)}{\partial t^i} L[v](s)\, ds\,, \quad t \in I\,, \quad i = 0, 1, \ldots, n-1\,.$$

Hence

(23) $$u^{(i)}(t) - v^{(i)}(t) = \int_{t_0}^{t} \frac{\partial^i K(t, s)}{\partial t^i} \{L[u](s) - L[v](s)\}\, ds \quad \text{for all} \quad t \in I\,, \quad i = 0, 1, \ldots, n-1\,.$$

By (20), (21), (22) and (23) the result follows.

If the condition (B) is fulfilled, a stronger result holds.

Theorem 3'. *Suppose that* (B) *holds. Let $t_0 \in I$ and let $u(t), v(t) \in C^n(I)$ be two functions such that*

(20') $$u^{(i)}(t_0) \geqq v^{(i)}(t_0)\,, \quad i = 0, 1, \ldots, n-1\,, \quad \textit{and} \quad L[u](t) \geqq L[v](t) \quad \textit{for all} \quad t \geqq t_0\,.$$

Then

$$u^{(i)}(t) \geqq v^{(i)}(t) \quad \textit{for all} \quad t \geqq t_0\,, \quad i = 0, 1, \ldots, n-1\,.$$

Proof. Let $K(t, s)$ have the same meaning as in the proof of Theorem 3 and let $y_i(t)$, $i = 1, 2$, be the solutions of (E) determined by the conditions

$$y_1^{(i)}(t_0) = u^{(i)}(t_0)\,, \quad y_2^{(i)}(t_0) = v^{(i)}(t_0)\,, \quad i = 0, 1, \ldots, n-1\,.$$

Then instead of (23) we get

(23') $$u^{(i)}(t) - v^{(i)}(t) = y_1^{(i)}(t) - y_2^{(i)}(t) + \int_{t_0}^{t} \frac{\partial^i K(t, s)}{\partial t^i} \{L[u](s) - L[v](s)\}\, ds \quad \text{for all} \quad t \in I\,, \quad i = 0, 1, \ldots, n-1\,.$$

Since $y_1 - y_2$ is a solution of (E) with $(y_1 - y_2)^{(i)}(t_0) \geqq 0$, Lemma 3′ implies that $(y_1 - y_2)^{(i)}(t) \geqq 0$ for all $t \geqq t_0$, $i = 0, 1, \ldots, n - 1$. By these inequalities as well as by (21), on the basis of (20′) and (23′), we get the result.

If we apply Lemma 3″ instead of Lemma 3′, we get the following theorem.

Theorem 3″. *Suppose that* (C) *holds. Let* $a < t_0$, *let* $u(t), v(t) \in C^n(I)$ *be two functions such that*

$$(20'') \qquad (-1)^i u^{(i)}(t_0) \geqq (-1)^i v^{(i)}(t_0), \quad i = 0, 1, \ldots, n - 1, \quad \textit{and}$$
$$L[u](t) \geqq L[v](t) \quad \textit{for all } t, \quad a \leqq t \leqq t_0.$$

Then

$$(-1)^i u^{(i)}(t) \geqq (-1)^i v^{(i)}(t) \quad \textit{for all } t, \quad a \leqq t \leqq t_0, \quad i = 0, 1, \ldots, n - 1.$$

5. REGULARITY OF BUNDLES

Let $t_0 \in I$. Denote by $y_0(t), y_1(t), \ldots, y_{n-1}(t)$ the solutions of (E) defined on I which are determined by the initial conditions

$$y_i^{(j)}(t_0) = \delta_{ij}, \quad i, j = 0, 1, \ldots, n - 1.$$

It is clear that for each $j \in \{0, 1, \ldots, n - 1\}$ each solution $y(t)$ of (E) such that $y^{(j)}(t_0) = 0$ is a linear combination $\sum_{\substack{k=0 \\ k \neq j}}^{n-1} c_k y_k(t)$. The set of all such solutions will be called the *bundle of solutions of* (E) *of the j-th kind at the point* t_0. If the wronskian $W(y_0, \ldots, y_{j-1}, y_{j+1}, \ldots, y_{n-1})(t)$ does not vanish on a subinterval $J \subset I$, then we say that this bundle is regular on J.

The following theorem is true.

Theorem 4. *Suppose that* (A) *holds. Then for each point* $t_0 \in I$, *each* $j \in \{0, 1, \ldots, n - 1\}$, *the bundle of solutions of* (E) *of the j-th kind at* t_0 *is regular in* $I - \{t_0\}$ *and hence the functions* $y_0(t), \ldots, y_{j-1}(t), y_{j+1}(t), \ldots, y_{n-1}(t)$ *form a fundamental set of solutions for a certain homogeneous linear differential equation of the* $(n - 1)$*-st order in* $I - \{t_0\}$.

Proof. Consider the wronskian $W(y_0, \ldots, y_{j-1}, y_{j+1}, \ldots, y_{n-1})(t)$. If there were a point $t_1 \in I$, $t_1 \neq t_0$, such that $W(y_0, \ldots, y_{j-1}, y_{j+1}, \ldots, y_{n-1})(t_1) = 0$, then the system (in unknowns $c_0, \ldots, c_{j-1}, c_{j+1}, \ldots, c_{n-1}$)

$$\sum_{\substack{k=0 \\ k \neq j}}^{n-1} c_k y_k^{(i)}(t_1) = 0, \quad i = 0, 1, \ldots, n - 2,$$

would have a nontrivial solution, and hence there would exist an $(n - 1)$-tuple $(\tilde{c}_0, \ldots, \tilde{c}_{j-1}, \tilde{c}_{j+1}, \ldots, \tilde{c}_{n-1})$ such that $\sum_{\substack{k=0 \\ k \neq j}}^{n-1} \tilde{c}_k y_k^{(i)}(t_1) =: \tilde{y}^{(i)}(t_1) = 0$, $i = 0, 1, \ldots$

$\ldots, n-2$. If $\tilde{y}(t) \equiv 0$ in I, then $0\, y_j(t) + \sum_{\substack{k=0 \\ k \neq j}}^{n-1} c_k\, y_k(t) \equiv 0$ which would contradict the fact that $y_0(t), \ldots, y_{n-1}(t)$ are linearly independent. Thus $\tilde{y}^{(n-1)}(t_1) \neq 0$ and $\tilde{y}(t)$ should be a solution of (E) with an $(n-1)$-tuple zero at t_1 and with $\tilde{y}^{(j)}(t_0) = 0$, which cannot occur if (E) satisfies (A) (a contradiction with Lemma 3).

Corollary. *Suppose that* (A) *holds. Then for each pair of different numbers* $t_0, t_1 \in I$, *any* j, $0 \leqq j \leqq n-1$, *and any* $(n-1)$-*tuple* y_1^i, $i = 0, 1, \ldots, n-2$, *of real numbers there exists a unique solution* y *of* (E) *which satisfies the conditions*

$$y^{(j)}(t_0) = 0\,, \quad y^{(i)}(t_1) = y_1^i\,, \quad i = 0, 1, \ldots, n-2\,.$$

In particular, the homogeneous boundary value problem

$$L[y] = 0\,, \quad y^{(j)}(t_0) = 0\,, \quad y^{(i)}(t_1) = 0\,, \quad i = 0, 1, \ldots, n-2\,,$$

has only the trivial solution and thus, there exists a unique Green's function for that problem.

6. NONOSCILLATORY EQUATIONS

Suppose that the equation (E) is nonoscillatory. Then, by Lemma 6, there exists a hierarchical fundamental system of solutions $y_i(t)$, $i = 1, 2, \ldots, n$, of the equation (E), which means that

$$y_1(t) \ll y_2(t) \ll \ldots \ll y_n(t) \quad \text{for} \quad t \to \infty\,. \tag{24}$$

Moreover, from the proof of the lemma it follows that for any two nontrivial solutions y, z of (E) there exists a finite or infinite $\lim_{t \to \infty} y(t)/z(t)$. We shall call these two solutions *equivalent*, notation $y(t) \approx z(t)$, when this limit is finite and different from 0. The relation to be equivalent is reflexive, symmetric and transitive and hence by this relation the set of all nontrivial solutions of (E) is decomposed into classes of equivalent solutions. In view of (24) and of the representation of the solution $y(t)$ of (E) in the form $y(t) = \sum_{j=1}^{n} c_j\, y_j(t)$, $t \in I$, the following statements hold:

1. *For each nontrivial solution* y *of* (E) *there exists one and only* $j \in \{1, 2, \ldots, n\}$ *such that* $y(t) \approx y_j(t)$, *and thus there are exactly* n *classes* U_j, $j = 1, 2, \ldots, n$, *of equivalent solutions of* (E), *possessing* y_j *as their representatives.*

2. *The class* U_j *consists of the solutions* $\sum_{k=1}^{j} c_k\, y_k(t)$ *of* (E), *where* $c_1, \ldots, c_{j-1}, c_j \neq 0$ *are arbitrary numbers.* Hence, the class U_1 is a one dimensional vector subspace, without the trivial solution, of the space of all solutions of (E), and the solution $y_1(t)$ is unique up to multiplication by positive constants.

By Theorem 7.1_n in [3], p. 329, we get the following statement.

3. *If the equation* (E) *is eventually disconjugate, i.e. it is disconjugate on an*

interval $(b, \infty) \subset I$, *then the solution* $y_1(t)$ (*with the smallest growth*) *has two properties*:

a) $y_1(t) > 0$ *in* (b, ∞).

b) *When* $\tilde{z}_k(t)$ *is the solution of* (E) (*from the proof of Theorem* 2) *satisfying the initial conditions*

$$\tilde{z}_k^{(j)}(k) = 0\,, \quad j = 0, \dots, n-2\,, \quad (-1)^n\, \tilde{z}_k^{(n-1)}(k) < 0$$

for each $k > b$, *and the normalization condition*

$$\sum_{j=0}^{n-1} \tilde{z}_k^{(j)2}(a) = 1$$

(i.e. *the conditions* (17)), *then* $y_1(t) = \lim_{k\to\infty} \tilde{z}_k(t)$ *in the sense that for each* $j = 0, 1, \dots, n-1$, $\tilde{z}_k^{(j)}(t)$ *converges uniformly to* $y_1^{(j)}(t)$ *on every compact subinterval of* (b, ∞) *as* $k \to \infty$. Hence, Corollary to Theorem 2 implies that, provided that the equation (E) satisfies the condition (C) and is eventually disconjugate, then the solution $z(t)$ of the equation with

$$(25) \qquad (-1)^i\, z^{(i)}(t) > 0 \quad \text{for all} \quad t \in I\,, \quad i = 0, 1, \dots, n-1\,,$$

belongs to the class U_1 (with the smallest growth) and is uniquely determined up to multiplication by positive constants.

Remark. If $p_n(t) \not\equiv 0$ in any neighbourhood of ∞, the equation (E) fulfils the condition (A) and is eventually disconjugate, then the same result holds, only (25) is replaced by (16). If $p_n(t) \equiv 0$ in (b, ∞) and there exists a solution $z_1(t)$ satisfying (16) as well as a solution $z_2(t) \equiv \text{const.} > 0$, then $z_1(t) \in U_1$ and $\lim_{t\to\infty} z_1(t) = 0$.

Suppose now that the equation (E) satisfies the condition (B). By Lemma 3′, for a point $t_0 \geqq a$ and for two different solutions $y(t)$, $z(t)$ of (E) such that $y^{(i)}(t_0) \leqq z^{(i)}(t_0)$, $i = 0, 1, \dots, n-1$, we have $y^{(i)}(t) < z^{(i)}(t)$ for all $t > t_0$ and $i = 0, 1, \dots, n-1$. Hence a nontrivial solution $y(t)$ of (E) such that

$$(26) \qquad y^{(i)}(t_0) \geqq 0\,, \quad i = 0, 1, \dots, n-1\,,$$

has the property that at any fixed point $t_1 > t_0$ all values $y^{(i)}(t_1)$ are positive, $i = 0, 1, \dots, n-1$. Comparing this solution with $y_n \in U_n$ we see that there exists a $k > 0$ such that $y_n^{(i)}(t_1) < k\, y^{(i)}(t_1)$, $i = 0, 1, \dots, n-1$, and thus, by Lemma 3′, $\lim_{t\to\infty} k\, y(t)/y_n(t) \geqq 1$. Since y_n belongs to the class of the highest growth, we have the following statement:

4. *If the equation* (E) *satisfies the condition* (B), *then each nontrivial solution* y *of* (E) *satisfying* (26) *at a point* $t_0 \geqq a$ *is equivalent to* y_n *and thus* $y \in U_n$. *In particular, any solution* y *of* (E) *with a zero of multiplicity* $n-1$ *at* t_0 *belongs to* U_n.

Remark. The last statement concerning the solution with an $(n-1)$-tuple zero is also true under the condition (A).

Let $t_0 \geqq a$ be an arbitrary but fixed point. As the bundle $B(t_0)$ of solutions of (E) of the 0 kind at the point t_0 is an $(n-1)$-dimensional vector space of solutions of (E) which are all nonoscillatory with the exception of the trivial one, by Lemma 6 there exists a system $z_1, \ldots, z_{n-1} \in B(t_0)$ such that

$$z_1(t) \ll z_2(t) \ll \ldots \ll z_{n-1}(t) \quad \text{for} \quad t \to \infty .$$

Denote by V_j the class of all solutions from $B(t_0)$ which are equivalent to z_j. Then for each $j \in \{1, \ldots, n-1\}$ there is a unique $k(j) = k \in \{1, \ldots, n\}$ such that $V_j \subset U_{k(j)}$, and for $j_1 < j_2$ we have $k(j_1) < k(j_2)$. Therefore $k(j) \geqq j$ for each $j = 1, 2, \ldots, n-1$. Now we prove the statement.

5. *If the equation* (E) *satisfies the condition* (B) *and is eventually disconjugate, then*

$$V_j \subset U_{j+1} \quad \textit{for each} \quad j = 1, 2, \ldots, n-1 .$$

Proof. By the statement 4, the solution with the $(n-1)$-tuple zero at t_0 has a maximal growth and therefore belongs to $U_n \cap V_{n-1}$. Hence $V_{n-1} \subset U_n$. On the other hand, both classes U_1, V_1 are one dimensional vector subspaces without the null solution and hence, if $U_1 \cap V_1 \neq \emptyset$, then $U_1 = V_1$. But, in view of the statement 3, U_1 contains a solution z satisfying (25) which has no zeros, and thus, z cannot belong to any of V_j. Hence $k(1) > 1$. However, $k(1) > 2$ cannot hold. Therefore $k(1) = 2$ and proceeding step by step we obtain that $k(j) = j + 1$, $j = 1, 2, \ldots$ $\ldots, n-1$.

Suppose now that the equation (E) satisfies the condition (C). Then the equation (E) generates a chain of equations

$$(\mathrm{E}_{2j}) \qquad y^{(2j)} + \sum_{k=2}^{2j} p_k(t)\, y^{(2j-k)} = 0\,, \quad j = 1, 2, \ldots, \tfrac{1}{2}n\,.$$

Here (E_n) means the equation (E). Each of the equations (E_{2j}), $j = 1, 2, \ldots, \frac{1}{2}n$, fulfils the condition (A), and if $p_{2j}(t)$ is not identically zero in any subinterval of I, it also satisfies the condition (C). Moreover, the equation (E_2) is disconjugate on I. The remaining equations of the chain are dealt with in the following lemma.

Lemma 9. *Let the equation* (E) *satisfy the condition* (C) *and let* $n \geqq 4$. *Then the equations* $(\mathrm{E}_4), (\mathrm{E}_6), \ldots, (\mathrm{E}_{n-2}), (\mathrm{E}_n)$ *are all eventually disconjugate if and only if they are all nonoscillatory.*

Proof. Since the eventual disconjugacy implies the nonoscillation, we shall only show that if the equations $(\mathrm{E}_4), (\mathrm{E}_6), \ldots, (\mathrm{E}_{n-2}), (\mathrm{E}_n)$ are nonoscillatory, then they are eventually disconjugate. Denote

$$(\tilde{\mathrm{E}}_{2j}) \qquad y^{(2j)} + \sum_{k=2}^{2j-2} p_k(t)\, y^{(2j-k)} = 0\,, \quad j = 2, 3, \ldots, \tfrac{1}{2}n\,,$$

which differs from (E_{2j}) by the term $p_{2j}(t)\, y$ and let L_{2j} $(\tilde{L}_{2j})$ be the operator standing on the left-hand side of (E_{2j}) $((\tilde{\mathrm{E}}_{2j}))$. As (E_2) is disconjugate on I and $\tilde{L}_4[y] = L_2[\mathrm{d}^2 y/\mathrm{d}t^2]$, $(\tilde{\mathrm{E}}_4)$ is disconjugate on I as well. As (E_4) is nonoscillatory and $p_4(t) \leqq$

$\leqq 0$ in I, by Lemma 8 we get that (E_4) is eventually disconjugate. Proceeding in this way, step by step we derive the eventual disconjugacy of the equations $(E_6), \ldots, (E_{n-2})$, (E_n) and the lemma is proved.

Further we shall need a lemma which extends a result by T. Čanturija in [1], p. 33.

Lemma 10. *If the equation* (E) *satisfies the condition* (C) *and is oscillatory, then for each* $c \geqq a$ *there are two numbers* $c < c_1 < c_2$, *an* $l_0 \in \{2, 4, \ldots, n-2\}$ *and a solution* $v(t)$ *of* (E) *such that*

(27) $$v^{(j)}(c_1) = 0\,, \quad j = 0, 1, \ldots, l_0 - 1\,,$$

(28) $$v^{(j)}(c_2) = 0\,, \quad j = l_0, l_0 + 1, \ldots, n-1\,,$$

(29) $$v^{(j)}(t) \neq 0 \quad \textit{for} \quad t \in (c_1, c_2)\,, \quad j = 0, 1, \ldots, n-1\,.$$

More precisely,

(30) $$v(t)\, v^{(j)}(t) > 0\,, \quad j = 0, 1, \ldots, l_0 - 1\,, \quad \textit{in} \quad (c_1, c_2\rangle \quad \textit{and}$$
$$(-1)^{j+l}\, 0\, v(t)\, v^{(j)}(t) > 0\,, \quad j = l_0, l_0 + 1, \ldots, n-1\,, \quad \textit{in} \quad \langle c_1, c_2)\,.$$

Proof. Let the equation (E) be oscillatory. Then there is an oscillatory solution $y(t)$ of this equation with a zero $c_1 > c$. Consider the set S of all nontrivial solutions $y(t)$ of (E) such that there is a k with $1 \leqq k \leqq n-1$ and a $d > c_1$, $d = d(y)$, with the following properties: (a) y has a k-tuple zero at c_1; (b) $y^{(k)}(t)$ has $n-k$ zeros in $\langle c_1, d\rangle$ counting each zero according to its multiplicity. By Corollary to Theorem 4, $1 \leqq k \leqq n-2$. Put $c_2 = \inf_{y \in S} d(y)$. Then $c_1 \leqq c_2$ and there is a sequence $y_m \in S$ and a fixed k, $1 \leqq k \leqq n-2$, such that all y_m possess a zero at c_1 of the same multiplicity k and $y_m^{(k)}(t)$ have $n-k$ zeros in $\langle c_1, c_2 + 1/m\rangle$. When we normalize the solutions $y_m(t)$ by $\sum_{i=0}^{n-1} y_m^{(i)^2}(c_1) = 1$, the resulting family contains a subsequence which we again denote by $y_m(t)$ and which is locally uniformly convergent on I to a nontrivial solution $v(t)$ of (E). Moreover, $y_m^{(j)}(t)$ locally uniformly converge to $v^{(j)}(t)$ for $j = 0, 1, \ldots, n-1$.

Clearly v enjoys the property (a). By Rolle's theorem the property (b) of $y_m(t)$ implies that the statement (c) $y_m^{(j)}(t)$ have $n-j$ zeros in $\langle c_1, c_2 + 1/m\rangle$, $j = k, k+1, \ldots, n-1$, $m = 1, 2, \ldots$, is true. Hence, on the basis of the convergence properties of $\{y_m^{(j)}(t)\}$ the equality $c_2 = c_1$ would imply $v^{(k)}(c_1) = \ldots = v^{(n-1)}(c_1) = 0$ and thus, $v(t) \equiv 0$ which contradicts the fact that v is a nontrivial solution of (E). Therefore $c_2 > c_1$. Further, denote by $c_1 \leqq t_{1,m} \leqq t_{2,m} \leqq \ldots \leqq t_{n-k,m} \leqq c_2 + 1/m$ the set of zeros of the function $y_m^{(k)}(t)$ in $\langle c_1, c_2 + 1/m\rangle$. Here each term stands so many times as its multiplicity indicates. By the compactness of the cubes in R^n as well as by the locally uniform convergence of $y_m^{(j)}(t)$ to $v^{(j)}(t)$ we get that for each i, $1 \leqq i \leqq n-k$, there is a subsequence $t_{i,m_p} \to t_i$ such that $y_{m_p}^{(k)}(t_{i,m_p}) \to v^{(k)}(t_i)$ as $p \to \infty$, whereby $c_1 \leqq t_1 \leqq \ldots \leqq t_{n-k} \leqq c_2$. Again by Rolle's theorem, the equality

$t_i = t_{i+1} = \ldots = t_{i+r-1} < t_{i+r}$ means that $v^{(k)}(t_i) = v^{(k+1)}(t_i) = \ldots = v^{(k+r-1)}(t_i) = 0$, and thus, $v(t)$ has the property (b), too. Therefore $v \in S$.

Let $1 \leqq k \leqq l_0 \leqq n - 2$ be such that $v(t)$ satisfies (27), but $v^{(l_0)}(c_1) \neq 0$. Then $v^{(l_0)}(t)$ has $n - l_0$ zeros in $(c_1, c_2\rangle$. Proceeding in a similar way as in the proof of Theorem 3.3 in [6], p. 75, using the method of perturbation of zeros, we can show that $v^{(l_0)}(t) \neq 0$ in (c_1, c_2) and hence the conditions (28) are fulfilled, too. At the same time, on the basis of (27), the inequalities (29) are true for $j = 0, 1, \ldots, l_0$.

Suppose that $v^{(l_0)}(t) > 0$ in (c_1, c_2). Then $v^{(j)}(t) > 0$ in (c_1, c_2) for $j = 0, 1, \ldots, l_0$. Two cases may occur: (i) l_0 is even, and hence $2 \leqq l_0 \leqq n - 2$. The function $v^{(l_0)}(t) = u(t)$ is a solution of the equation

$$u^{(n-l_0)} + \sum_{k=2}^{n-l_0} p_k(t)\, u^{(n-l_0-k)} = - \sum_{k=n-l_0+2}^{n} p_k(t)\, v^{(n-k)}(t) \tag{31}$$

which satisfies the conditions

$$u^{(j)}(c_2) = 0\,, \quad j = 0, 1, \ldots, n - l_0 - 1\,. \tag{32}$$

By the condition (C) the right-hand side of (31) is nonnegative in (c_1, c_2) and attains positive values in any subinterval of (c_1, c_2). Hence in virtue of (32), Theorem 3 implies that $(-1)^{n-l_0-j}\, u^{(j)}(t) = (-1)^{l_0+j}\, v^{(l_0+j)}(t) > 0$ in $\langle c_1, c_2)$ for $j = 0, 1, \ldots \ldots, n - l_0 - 1$. Thus (30) is true for all $j = 0, 1, \ldots, n - 1$.

Finally we show that the case (ii) l_0 is odd, cannot occur and this will complete the proof of the lemma. In this case we put $v^{(l_0+1)}(t) = u(t)$, which implies that u satisfies the initial value problem

$$u^{(n-l_0-1)} + \sum_{k=2}^{n-l_0-1} p_k(t)\, u^{(n-l_0-1-k)} = - \sum_{k=n-l_0+1}^{n} p_k(t)\, v^{(n-k)}(t)$$

$$u^{(j)}(c_2) = 0\,, \quad j = 0, 1, \ldots, n - l_0 - 2\,.$$

Again by Theorem 3 we get the inequalities $(-1)^{n-l_0-1-j}\, u^{(j)}(t) = (-1)^{l_0+j+1} \,.\, v^{(l_0+j+1)}(t) > 0$ in $\langle c_1, c_2)$, for $j = 0, 1, \ldots, n - l_0 - 2$. Hence $v^{(l_0+1)}(t) > 0$ in $\langle c_1, c_2)$ and thus $v^{(l_0)}(t)$ is increasing in the interval. But $v^{(l_0)}(c_2) = 0$ which leads to contradiction with $v^{(l_0)}(t) > 0$ in (c_1, c_2).

Theorem 5. *Suppose that the equation* (E) *satisfies the condition* (C) *and that* $n \geqq 4$. *Then the following statements are true*:

1. *If the equation* (E) *is nonoscillatory and, in the case* $n \geqq 6$ *the equations* $(E_4), (E_6), \ldots, (E_{n-2})$ *are eventually disconjugate, then for each nontrivial solution* y *of the equation* (E)

either

there exists an even number $l \in \{0, 2, \ldots, n\}$ *and a point* $c \geqq a$ *such that*

(33) *if* $l \geqq 2$, *then* $y(t)\, y^{(j)}(t) > 0$ *for* $c \leqq t < \infty$, $j = 0, 1, \ldots, l - 1$,
if $l \leqq n - 2$, *then* $(-1)^{l+j}\, y(t)\, y^{(j)}(t) > 0$ *for*
$c \leqq t < \infty$, $j = l, l + 1, \ldots, n - 1$

and
$(-1)^{l+n}\, y(t)\, y^{(n)}(t) \geqq 0$ *in* $\langle c, \infty)$, $y^{(n)}(t) \equiv 0$ *holds in no subinterval* $\langle d, \infty) \subset \langle c, \infty)$,
or
there exists an odd $l \in \{1, 3, \ldots, n-3\}$ *and a point* $c \geqq a$ *such that*

(34) $y(t)\, y^{(j)}(t) > 0$ *for* $c \leqq t < \infty$, $j = 0, 1, \ldots, l-1$,
$(-1)^{j+1}\, y(t)\, y^{(j)}(t) > 0$ *in* $\langle c, \infty)$, $j = l, l+1$.

Moreover, if

(35) $$\int^{\infty} x_1(t)\, \mathrm{d}t = \infty$$

for the first solution $x_1(t)$ *of the hierarchical fundamental system for each of the equations* $(E_2), (E_4), \ldots, (E_{n-2})$, *then for* y *only the possibility* (33) *arises.*

2. *If for each* $l \in \{2, 4, \ldots, n-2\}$ *there exists a solution* $y(t)$ *of the equation* (E) *with the property* (33), *then the equation* (E) *is nonoscillatory.*

Proof. 1. If y is a nontrivial solution of (E), then there is an interval $\langle b, \infty)$ in which $y(t) \neq 0$. Two cases may occur. In the first case the inequalities

(36) $$y(t)\, y''(t) > 0\,, \quad y(t)\, y^{(4)}(t) > 0, \ldots, y(t)\, y^{(n-2)}(t) > 0$$

hold in an interval $\langle c, \infty) \subset \langle b, \infty)$. Since the equation (E) satisfies (C), by this equation we get that $y(t)\, y^{(n)}(t) \geqq 0$ in $\langle c, \infty)$ and $y^{(n)}(t) \equiv 0$ holds in no subinterval $\langle d, \infty) \subset \langle c, \infty)$. Then by Lemma 4 there is an l such that the solution $y(t)$ fulfils (33). In view of (36), l must be an even number and hence $l \in \{0, 2, 4, \ldots, n\}$.

In the second case there is an even number l_0, $0 \leqq l_0 \leqq n-4$, such that for $l_0 \geqq 2$ the inequalities

(37) $$y(t)\, y''(t) > 0, \ldots, y(t)\, y^{(l_0)}(t) > 0$$

hold in an interval $\langle c, \infty) \subset \langle b, \infty)$ and $y(t)\, y^{(l_0+2)}(t) > 0$ holds in no subinterval $\langle d, \infty) \subset \langle c, \infty)$. Hence, by (37), the right-hand side of the equation

(38) $$v^{(n-l_0-2)} + p_2(t)\, v^{(n-l_0-4)} + \ldots + p_{n-l_0-2}(t)\, v = \\ = -p_n(t)\, y(t) - p_{n-2}(t)\, y''(t) - \ldots - p_{n-l_0}(t)\, y^{(l_0)}(t)$$

is of a constant sign in $\langle c, \infty)$ and thus, by Lemma 5, each solution of the equation is eventually different from 0. The function $y^{(l_0+2)}(t)$ is one of these solutions and therefore $y(t)\, y^{(l_0+2)}(t)$ is eventually negative. Clearly $y(t)\, y^{(l_0+1)}(t) > 0$ in a neighbourhood of ∞ and thus $l = l_0 + 1$, l is odd and $l \in \{1, 3, \ldots, n-3\}$.

On the other hand, since (E_{n-l_0-2}) is eventually disconjugate, by [13], p. 322, there exist continuous and positive functions $p_0(t), p_1(t), \ldots, p_{n-l_0-2}(t)$ in an interval $\langle d, \infty) \subset \langle c, \infty)$ such that

$$\int_d^{\infty} \frac{1}{p_i(t)}\, \mathrm{d}t = \infty \quad \text{for} \quad 1 \leqq i \leqq n-l-3$$

and

$$L_{n-l_0-2}[y](t) = p_{n-l_0-2}(t)\frac{\mathrm{d}}{\mathrm{d}t}p_{n-l_0-3}(t)\dots\frac{\mathrm{d}}{\mathrm{d}t}p_1(t)\frac{\mathrm{d}}{\mathrm{d}t}p_0(t)\,y(t) \quad \text{in} \quad \langle d, \infty)$$

$$\text{for all} \quad y(t) \in C^{n-l_0-2}(\langle d, \infty)).$$

By (37), (38), $L_{n-l_0-2}[y^{(l_0+2)}(t)]\,y(t) \geqq 0$, $L_{n-l_0-2}[y^{(l_0+2)}(t)] \equiv 0$ holds in no subinterval of $\langle d, \infty)$, while $y^{(l_0+2)}(t)\,y(t) < 0$ in $\langle d, \infty)$. Hebce, by the first Kiguradze lemma, [12], p. 94, there is an odd k, $1 \leqq k \leqq n - l_0 - 2$ and a δ, $d \leqq \delta < \infty$ such that

$$y(t)\,\tilde{L}_j[y^{(l_0+2)}](t) < 0\,, \quad j = 0, 1, \dots, k-1\,, \quad \text{and} \quad (-1)^{k+j}\,.$$
$$\tilde{L}_j[y^{(l_0+2)}](t) < 0\,, \quad j = k, k+1, \dots, n - l_0 - 3\,, \quad t \in \langle \delta, \infty)$$

where

$$\tilde{L}_0[y](t) = p_0(t)\,y(t)\,, \quad \tilde{L}_j[y](t) = p_j(t)\,[\tilde{L}_{j-1}[y(t)]]'\,,$$
$$j = 0, 1, \dots, n - l_0 - 2\,.$$

Hence $y(t)\,p_0(t)\,y^{(l_0+2)}(t) < 0$ and $y(t)\,p_1(t)\,[p_0(t)\,y^{(l_0+2)}(t)]' < 0$ in $\langle \delta, \infty)$. Suppose that $y(t) > 0$ in $\langle c, \infty)$. Then $p_0(t)\,y^{(l_0+2)}(t)$ is a decreasing and negative function in $\langle \delta, \infty)$. Therefore there is a $c_1 > 0$ such that $y^{(l_0+2)}(t) \leqq -c_1/p_0(t)$ in that interval. As by [13], p. 321, we have $p_0(t) = 1/x_1(t)$ where $x_1(t) \neq 0$ is the first solution in the hierarchical system for the equation (E_{n-l_0-2}), (35) implies that $\lim_{t\to\infty} y^{(l_0+1)}(t) = -\infty$, which contradicts the inequality $y^{(l_0+1)}(t) > 0$ and thus the statement 1 is proved.

2. Suppose that for each $l \in \{2, 4, \dots, n-2\}$ there is a solution $y_l(t)$ of (E) with the property (33) in the same interval $\langle c, \infty)$ and that the equation (E) is oscillatory. Then, by Lemma 10, there is a solution $v(t)$ of (E), two numbers $c < c_1 < c_2$ and an $l_0 \in \{2, 4, \dots, n-2\}$ such that (30) is true. Without loss of generality we may assume that both solutions $y_{l_0}(t)$, $v(t)$ are positive in (c_1, c_2). Let $\varepsilon > 0$ and consider the solution w_ε of (E), $w_\varepsilon(t) = y_{l_0}(t) - \varepsilon\,v(t)$, $t \in \langle c_1, c_2\rangle$. Since $w_0^{(j)}(t) > 0$ for $t \in \langle c_1, c_2\rangle$, $j = 0, 1, \dots, l_0 - 1$, and $(-1)^{l_0+j}\,w_0^{(j)}(t) > 0$ for $t \in \langle c_1, c_2\rangle$, $j = l_0, l_0 + 1, \dots, n-1$, there exists a maximal $\varepsilon_0 > 0$ such that for all ε, $0 \leqq \varepsilon \leqq \varepsilon_0$ we have

$$(39) \qquad w_\varepsilon^{(j)}(t) \geqq 0 \quad \text{for} \quad t \in \langle c_1, c_2\rangle\,, \quad j = 0, 1, \dots, l_0 - 1\,,$$
$$(-1)^{l_0+j}\,w_\varepsilon^{(j)}(t) \geqq 0 \quad \text{for} \quad t \in \langle c_1, c_2\rangle\,, \quad j = l_0, l_0+1, \dots, n-1\,.$$

Then at least one inequality in (39) is nonstrict for $\varepsilon = \varepsilon_0$. On the other hand, on the basis of (E), the inequalities (39) lead to the inequality

$$w_{\varepsilon_0}^{(n)}(t) = -\sum_{k=2}^{n} p_k(t)\,w_{\varepsilon_0}^{(n-k)}(t) \geqq 0 \quad \text{in} \quad \langle c_1, c_2\rangle\,,$$

whereby in each subinterval of $\langle c_1, c_2\rangle$ there are points t at which $w_{\varepsilon_0}^{(n)}(t) > 0$. Therefore the function $w_{\varepsilon_0}^{(n-1)}(t)$ is increasing in $\langle c_1, c_2\rangle$ and hence, in view of (28), $(-1)^{l_0+n-1}\,w_{\varepsilon_0}^{(n-1)}(t) > 0$ in $\langle c_1, c_2\rangle$. Using the inequalities (27), (28), (30) and (33), we get step by step that

$$(-1)^{l_0+j}\, w_{\varepsilon 0}^{(j)}(t) > 0 \quad \text{for} \quad t \in \langle c_1, c_2 \rangle\,, \quad j = n-1, n-2, \ldots, l_0+1, l_0$$

and

$$w_{\varepsilon 0}^{(j)}(t) > 0 \quad \text{for} \quad t \in \langle c_1, c_2 \rangle\,, \quad j = l_0 - 1, l_0 - 2, \ldots, 1, 0\,.$$

The obtained contradiction with (39) shows that the equation (E) is nonoscillatory.

Corollary. *Suppose that the equation* (E) *satisfies the condition* (C) *and* $n = 4$. *Then the following statements are true*:

1. *The equation* (E) *is eventually disconjugate if and only if it is nonoscillatory.*

2. *If the equation* (E) *is nonoscillatory, then for each nontrivial solution* y *of* (E) *either there exists an even number* $l \in \{0, 2, 4\}$ *and a point* $c \geqq a$ *such that*

$$(33') \quad \begin{aligned} &\text{if} \quad l \geqq 2\,, \quad \text{then} \quad y(t)\, y^{(j)}(t) > 0 \quad \text{for} \quad c \leqq t < \infty\,, \quad j = 0, 1, \ldots, l-1\,, \\ &\text{if} \quad l \leqq 2\,, \quad \text{then} \quad (-1)^{l+j}\, y(t)\, y^{(j)}(t) > 0 \quad \text{for} \quad c \leqq t < \infty\,, \\ &j = l, l+1, \ldots, 3\,, \\ &\text{and} \quad (-1)^{l+4}\, y(t)\, y^{(4)}(t) \geqq 0 \quad \text{in} \quad \langle c, \infty)\,, \quad y^{(4)}(t) \equiv 0 \\ &\text{holds in no subinterval } \langle d, \infty) \subset \langle c, \infty), \end{aligned}$$

or there is a point $c \geqq a$ *such that*

$$(34') \qquad y(t)\, y'(t) > 0 \quad \text{in} \quad \langle c, \infty)\,, \quad y(t)\, y''(t) < 0 \quad \text{in} \quad \langle c, \infty)\,.$$

Moreover, if for the first solution $x_1(t)$ *of the hierarchical fundamental system for the equation* (E_2) (35) *holds, then only the possibility* (33′) *can arise for* $y(t)$.

3. *If for* $l = 2$ *there exists a solution* $y(t)$ *of the equation* (E) *with the property* (33′), *then the equation* (E) *is nonoscillatory.*

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