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# SMOOTHNESS AS AN INVARIANT PROPERTY OF COEFFICIENTS OF LINEAR DIFFERENTIAL EQUATIONS

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*Dedicated to Academician Otakar Borůvka with best wishes on his 90th birthday*

## 1. INTRODUCTION

Each ordinary homogeneous second order linear differential equation

$$y'' + p_1(x)\, y' + p_0(x)\, y = 0\,,$$

where $p_1$ and $p_0$ are real continuous functions on an (open) interval $I \subset \mathbb{R}$, can be transformed on the whole interval $I$ by means of a transformation of the form

$$z(t) = f(t)\, y(h(t)) \tag{1}$$

into the same type of equation with (real) analytic coefficients, in particular, into the equation

$$z'' + z = 0$$

on a suitable interval $J \subset \mathbb{R}$, see [1].

A natural question arises whether for a given ordinary homogeneous linear differential equation of the $n$-th order

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_0(x)\, y = 0\,, \quad p_i \in C^0(I)\,, \tag{2}$$

$i = 0, \ldots, n - 1$, a transformation exists which converts this equation into an equation of the same type with more regular coefficients, e.g., belonging to $C^k$ for some $k > 0$, or $k = \infty$ (infinitely differentiable functions), or even $k = \omega$ (real analytic functions).

The above result for the second order equations is misleading in some respect. We shall prove, e.g., that the smoothness of the coefficient $p_{n-1}$ in (2) cannot be improved by a transformation if $p_{n-1} \in C^k(I) \setminus C^{k+1}(I)$ and $k < n - 2$ (the number $k$ is invariant), whereas if $k = n - 2$ then after a suitable transformation the transformed coefficient can be even (real) analytic. Of course, for $n = 2$ this critical case occurs for the class $C^0$ $(= C^{n-2})$.

## II. NOTATION, DEFINITIONS AND SOME BASIC FACTS

In 1892 P. Stäckel [5] proved that under certain regularity conditions the form (1) is the most general pointwise transformation that converts the set of all solutions of every linear differential homogeneous equation of the $n$-th order ($n \geqq 2$) into the set of all solutions of an equation of the same type, i.e.,

$$(3) \quad z^{(n)} + q_{n-1}(t)\, z^{(n-1)} + \ldots + q_0(t)\, z = 0\,, \quad q_i \in C^0(J)\,, \quad i = 0, \ldots, n-1\,.$$

M. Čadek in [2] has proved the same assertion under a weaker assumption involving only continuity instead of differentiability.

However, for our purpose, we shall need the following result that guarantees a certain smoothness of the functions $f$ and $h$ in (1) if only one equation of the type (2) is transformed by means of (1) into an equation (3).

**Lemma 1** (see [4]). *Let $n$ be an integer, $n \geqq 2$, and let $I \subset \mathbb{R}$, $J \subset \mathbb{R}$ be open intervals. Suppose $y_i\colon I \to \mathbb{R}$, and $z_i\colon J \to \mathbb{R}$, $i = 1, \ldots, n$, are two $n$-tuples of linearly independent solutions of equations* (2) *and* (3), *respectively.*

*Let*

$$z_i(t) = f(t)\, y_i(h(t))\,, \quad t \in J\,, \quad i = 1, \ldots, n\,,$$

*be satisfied for real continuous functions $f$ and $h$ defined on $J$ such that $h(J) = I$.*

*Then*

$$(4) \qquad f \in C^n(J)\,, \quad f(t) \neq 0 \quad \textit{for all} \quad t \in J\,,$$

$$h \in C^n(J)\,, \quad \mathrm{d}h(t)/\mathrm{d}t \neq 0 \quad \textit{for all} \quad t \in J\,,$$

*i.e., $h$ is a $C^n$-diffeomorphism of $J$ onto $I$.*

With respect to the above mentioned results it is reasonable to say that (1) globally transforms the equation (2) into the equation (3) if $f$ and $h$ satisfy (4), and for each solution $y\colon I \to \mathbb{R}$ of (2) the function $z\colon J \to \mathbb{R}$ given by (1) is a solution of (3), cf. [3].

**Lemma 2** (see e.g. [6]). *Let* (1) *globally transform* (2) *into* (3). *Then the following identities hold*:

$$(5) \qquad q_{n-1}(t) = p_{n-1}(h(t))\, h'(t) - n\, f'(t)/f(t) - \binom{n}{2} h''(t)/h'(t)\,,$$

and

$$(6) \quad q_{n-2}(t) = \binom{n+1}{3}\left(\tfrac{1}{2}h'''/h' - \tfrac{3}{4}h''^2/h'^2\right) + \left(p_{n-2}(h) - \frac{n-1}{2n}\, p_{n-1}^2(h)\right) h'^2 +$$

$$+ \frac{n-1}{2}\, p_{n-1}(h)\, h'' + \frac{n-1}{2n}\, q_{n-1}^2 + \frac{n-1}{2}\left(p_{n-1}(h)\, h' - q_{n-1}\right)',$$

$h = h(t)$, $t \in J$.

**Corollary 1.** *If $p_{n-1} = 0$ on $I$ then $q_{n-1} = 0$ on $J$ if and only if*

$$f(t) = c|h'(t)|^{(1-n)/2}\,, \quad t \in J\,, \tag{7}$$

$c = \text{const.} \neq 0$.

*If $p_{n-1} = 0$ on $I$ and $q_{n-1} = 0$ on $J$, then*

$$q_{n-2}(t) = \binom{n+1}{3}\left(\tfrac{1}{2}\, h'''(t)/h'(t) - \tfrac{3}{4}\, h''^2(t)/h'^2(t)\right) + \\ + p_{n-2}(h(t))\, h'^2(t)\,,$$

$t \in J$.

Proof is a direct consequence of relations (5) and (6).

## III. AUXILIARY RESULTS

**Lemma 3.** *Let $n \geqq 2$ be an integer, and let*

$$z(t) = f(t)\, y(h(t))\,, \tag{1}$$

*where $z \in C^n(J)$, $y \in C^n(I)$, the functions $f$ and $h$ satisfy*

$$f \in C^r(J)\,, \quad f(t) \neq 0 \text{ on } J\,,$$
$$h \in C^r(J)\,, \quad \mathrm{d}h(t)/\mathrm{d}t \neq 0 \text{ on } J\,, \quad h(J) = I\,, \quad \text{and} \quad r \geqq n\,.$$

*Then*

$$\begin{aligned}
y(x) &= A_{00} z(k(x))\,,\\
y'(x) &= A_{10} z(k(x)) + A_{11} z'(k(x))\,,\\
y''(x) &= A_{20} z(k(x)) + A_{21} z'(k(x)) + A_{22} z''(k(x))\,,\\
&\dots\\
y^{(i)}(x) &= A_{i0} z(k(x)) + A_{i1} z'(k(x)) + \dots + A_{ij} z^{(j)}(k(x)) + \dots + A_{ii} z^{(i)}(k(x))\,,\\
&\dots\\
y^{(n)}(x) &= A_{n0} z(k(x)) + A_{n1} z'(k(x)) + \dots + A_{nn} z^{(n)}(k(x))\,,
\end{aligned}$$

*where $k$ is the inverse function to $h$ and $A_{ij}$, $0 \leqq i \leqq n$, $0 \leqq j \leqq i$, are rational expressions in $f$ and $h$ and their derivatives, such that $x \mapsto A_{ij}(x)$ are functions of the class $C^{r-(i-j)-1}(I)$ for $j > 0$, and $C^{r-i}(I)$ for $j = 0$. Moreover, at most the $(i-j)$th order of the derivatives of $f$ occurs in $A_{ij}$, and $A_{ii}(x) \neq 0$ for all $x \in I$ and $i$, $0 \leqq i \leqq n$.*

Proof. From (1) we have $y(x) = 1/f(k(x))\, z(k(x))$, thus $A_{00} = 1/f(k) \in C^r(I) = C^{r-i}(I)$ for $i = 0$ and $j = 0$. Also $A_{00}(x) \neq 0$ on $I$ and $A_{00}$ contains no derivatives of $f$. Further,

$$y'(x) = A'_{00}(x)\, z(k(x)) + A_{00}(x)\, k'(x)\, z'(k(x))\,, \quad \text{or}$$
$$y'(x) = A_{10}(x)\, z(k(x)) + A_{11}(x)\, z'(k(x))\,,$$

where

$$A_{10} \in C^{r-1}(I) = C^{r-i}(I) \quad \text{for} \quad i = 1 \quad \text{and} \quad j = 0\,, \quad \text{and}$$
$$A_{11} \in C^{r-1}(I) = C^{r-(i-j)-1} \quad \text{for} \quad i = 1 \quad \text{and} \quad j = 1\,.$$

Also $A_{11}(x) \neq 0$ on $I$ and $A_{1i}$ contains derivatives of $f$ of orders $\leqq 1 - i$, $i = 0, 1$.

Proceeding by induction, suppose that

$$y^{(i)}(x) = A_{i0} z(k(x)) + \ldots + A_{ij} z^{(j)}(k(x)) + \ldots + A_{ii} z^{(i)}(k(x)),$$
$$A_{ij} \in C^{r-(i-j)-1}(I) \quad \text{for} \quad j > 0,$$
$$A_{i0} \in C^{r-i}(I) \; A_{ii}(x) \neq 0 \text{ on } I, \quad \text{and}$$
$$A_{ij} \text{ contains derivatives of } f \text{ of orders } \leqq i - j, \quad 0 \leqq j \leqq i < n.$$

Then
$$\begin{aligned} y^{(i+1)}(x) = A'_{i0} z(k(x)) &+ (A_{i0} k' + A'_{i1}) z'(k(x)) \\ &+ (A_{i1} k' + A'_{i2}) z''(k(x)) \\ &+ \ldots \\ &+ (A_{i,j-1} k' + A'_{ij}) z^{(j)}(k(x)) \\ &+ (A_{ij} k' + A'_{i,j+1}) z^{(j+1)}(k(x)) \\ &+ \ldots \\ &+ (A_{i,i-1} k' + A'_{ii}) z^{(i)}(k(x)) \\ &+ (A_{ii} k') z^{(i+1)}(k(x)) = \end{aligned}$$
$$= A_{i+1,0}\, z(k) + A_{i+1,1}\, z'(k) + \ldots + A_{i+1,j}\, z^{(j)}(k) + A_{i+1,j+1}\, z^{(j+1)}(k) + \ldots$$
$$\ldots + A_{i+1,i}\, z^{(i)}(k) + A_{i+1,i+1}\, z^{(i+1)}(k).$$

Evidently $A_{i+1,0} \in C^{r-(i+1)}(I)$, and for $j > 0$ we have

$$A_{i+1,j} \in C^{r-(i-(j-1))-1}(I) \cap C^{r-1}(I) \cap C^{r-(i-j)-2}(I) =$$
$$= C^{r-(i-j)-2}(I) = C^{r-(i+1-j)-1}(I),$$
$$A_{i+1,j+1} \in C^{r-(i-j)-1}(I) \cap C^{r-1}(I) \cap C^{r-(i-(j+1))-2}(I) =$$
$$= C^{r-((i+1)-(j+1))-1}(I).$$

Moreover,

$A_{i+1,j}$ contains derivatives of $f$ of orders $\leqq \max\{i - (j - 1), i - j + 1\} = (i + 1) - j$,

$A_{i+1,j+1}$ contains derivatives of $f$ of orders $\leqq \max\{i - j, i - (j + 1) + 1\} = i - j = (i + 1) - (j + 1)$.

Finally, $A_{i+1,i+1}(x) = A_{ii}(x)\, k'(x) \neq 0$ on $I$. Q.E.D.

**Lemma 4.** *Let* (2) *be globally transformed into* (3) *by means of* (1), *where $f$ and $h$ satisfy the assumptions of Lemma* 3.

*Then the coefficients $q_i$, $i = 0, \ldots, n - 1$, of the equation* (3) *are expressible in the following way*:

(8) $(q_n = 1)$
$$q_{n-1}(t) = B_{n-1,n} + B_{n-1,n-1} p_{n-1}(h(t)),$$
$$q_{n-2}(t) = B_{n-2,n} + B_{n-2,n-1} p_{n-1}(h(t)) + B_{n-2,n-2} p_{n-2}(h(t))$$
$$\ldots$$

$$q_i(t) = B_{i,n} + B_{i,n-1}p_{n-1}(h(t)) + B_{i,n-2}p_{n-2}(h(t)) + \dots$$
$$\dots + B_{i,j}p_j(h(t)) + \dots + B_{i,i}p_i(h(t)),$$
$$\dots$$
$$q_0(t) = B_{0,n} + B_{0,n-1}p_{n-1}(h(t)) + \dots + B_{ij}p_j(h(t)) + \dots + B_{0,0}p_0(h(t)),$$

*where* $t \mapsto B_{ij}(t)$ *for* $0 \leqq i \leqq n-1$ *and* $i \leqq j \leqq n$ *are functions of the class* $C^{r+i-j-1}(J)$ *for* $i > 0$ *and of the class* $C^{r-j}(J)$ *for* $i = 0$. *Moreover,* $B_{ii}(t) \neq 0$ *on* $J$ *for all* $i = 0, \dots, n-1$ *and* $B_{ij}$ *is expressible in terms of the derivatives of* $f$ *of orders* $\leqq j - i$.

Proof. We may write

$$y^{(n)}(x) + p_{n-1}(x)\,y^{(n-1)}(x) + \dots + p_0(x)\,y(x) = \sum_{j=0}^{n} p_j(x)\,y^{(j)} =$$
$$= \sum_{j=0}^{n}\sum_{i=0}^{j} p_j(x)\,A_{ji}(x)\,z^{(i)}(k(x)).$$

Hence the coefficient $q_i(t)$ of $z^{(i)}$ is

$$q_i(t) = (\sum_{j=i}^{n} A_{ji}(h(t))\,p_j(h(t))/A_{nn}(h(t)).$$

Thus

$$B_{ij}(t) = A_{ji}(h(t))/A_{nn}(h(t)).$$

For $i > 0$, $A_{ji} \in C^{r-(j-i)-1}(I)$. Since $h \in C^r(J)$, $r \geqq n$, and $A_{nn} \in C^{r-1}(I)$, $A_{nn}(x) \neq 0$ for $x \in I$, we have $B_{ij} \in C^{r-(j-i)-1}(J)$. For $i = 0$, $A_{j0} \in C^{r-j}(I)$, hence $B_{0j} \in C^{r-j}(J)$ for $j = 0, \dots, n$. Moreover, $B_{ii}(t) = A_{ii}(h(t))/A_{nn}(h(t)) \neq 0$ for $t \in J$. Q.E.D.

**Lemma 5.** *For a given* $p \in C^{n-2}(I)$ *and a positive constant* $d$ *there exist an interval* $J \subset \mathbb{R}$ *and a* $C^{n+1}$*-diffeomorphism* $h$ *of* $J$ *onto* $I$ *satisfying*

$$d = \tfrac{1}{2}\,h'''(t)/h'(t) - \tfrac{3}{4}\,h''^2(t)/h'^2(t) + p(h(t))\,h'^2(t), \quad t \in J. \tag{9}$$

Proof. The relation (9) is the so-called Kummer equation for the second order equations

$$y'' + p(x)\,y = 0, \quad p \in C^{n-2}(I), \tag{10}$$

and

$$z'' + dz = 0 \quad \text{on} \quad J\ (d \in C^{\omega}(J)), \tag{11}$$

see also Corollary 1. According to O. Borůvka [1], a solution $h$ of (9) satisfying $h(J) = I$, $h \in C^3(J)$, $\mathrm{d}h(t)/\mathrm{d}t \neq 0$ on $J$, exists if and only if the equations (10) and (11) are of the same character. This can be always achieved for any constant $d > 0$, when a suitable interval $J \subset \mathbb{R}$ is taken. Then all such solutions $h$ are obtained as compositions of the so-called phases and their inverses of the equations (10) and (11). These phases are expressible as anti-derivatives of the expressions

$$(y_1^2(x) + y_2^2(x))^{-1} \quad \text{and} \quad (z_1^2(t) + z_2^2(t))^{-1},$$

where $y_1, y_2$ and $z_1, z_2$ are linearly independent solutions of the equations (10) and (11), respectively. Hence these phases are of the classes $C^{n+1}(I)$ and $C^{\omega}(J)$, respectively, and $h \in C^{n+1}(J)$. Q.E.D.

## IV. MAIN RESULTS

**Theorem 1.** *Let $n \geqq 2$ be an integer, and let a transformation (1) globally transform (2) into (3).*

*If $p_{n-1} \in C^k(I) \setminus C^{k+1}(I)$ for $0 \leqq k \leqq n-3$, then $q_{n-1} \in C^k(J) \setminus C^{k+1}(J)$, i.e. this $k$ is an invariant with respect to transformations (1).*

*If $p_{n-1} \in C^k(I)$, $k \geqq n-2$, then there exists a transformation (1) with $f, h \in C^{k+2}(J)$ that globally transforms (2) into (3) with $q_{n-1} \in C^{\omega}(J)$, especially with $q_{n-1} = 0$ on $J$.*

*If $p_{n-1} \in C^k(I)$ and $p_{n-2} \in C^{k-1}(I)$, $k \geqq n-1$, then (2) can be transformed into (3) with $q_{n-1} = 0$ on $J$ and $q_{n-2} \in C^{k-1}(J)$.*

Proof. Let $p_{n-1} \in C^k(I)$ for $0 \leqq k \leqq n-3$. Since $f'/f \in C^{n-1}(J)$ and $h''/h' \in C^{n-2}(J)$ according to Lemma 1, the relation (5) gives $q_{n-1} \in C^s(J)$, where $s = \min\{k, n-2\} = k$. However, if $p_{n-1} \bar{\in} C^{k+1}(I)$ for $0 \leqq k \leqq n-3$ then $q_{n-1} \bar{\in} C^{k+1}(J)$. In fact, if $q_{n-1} \in C^{k+1}(J)$ then

$$p_{n-1}(x) = \frac{1}{h'(t)} \left( q_{n-1}(t) + n f'(t)/f(t) + \binom{n}{2} h''(t)/h'(t) \right)_{t=h^{-1}(x)}$$

belongs to $C^s(I)$, where $s = \min\{k+1, n-2\} = k+1$, contrary to our assumption.

Now, let $p_{n-1} \in C^k(I)$, $k \geqq n-2$. Consider a function $q_{n-1}$ of the class $C^{\omega}(\mathbb{R})$. Choose also an arbitrary function $f \in C^{k+2}(\mathbb{R})$ satisfying (4) for $J = \mathbb{R}$. According to Lemma 2 we have

$$\binom{n}{2} h''(t)/h'(t) - p_{n-1}(h(t))\, h'(t) = -q_{n-1}(t) - n f'(t)/f(t),$$

or

$$h'(t) = c|f(t)|^{(1-n)/2} \exp\left\{\frac{-2}{n(n-1)} \int_{t_0}^{t} q_{n-1}(s)\, ds\right\} \times \exp\left\{\frac{2}{n(n-1)} P(h(t))\right\},$$

where $P \in C^{k+1}(I)$ is an anti-derivative of $p_{n-1}$, and $c$ is a non-zero constant. For

$$F(h) := \exp\left\{\frac{-2}{n(n-1)} P(h)\right\} \in C^{k+1}(I)$$

and

$$g(t) := c|f(t)|^{(1-n)/2} \exp\left\{\frac{-2}{n(n-1)} \int_{t_0}^{t} q_{n-1}(s)\, ds\right\} \in C^{k+2}(\mathbb{R})$$

we get

(12) $$F(h(t))\, h'(t) = g(t),$$

where $F(h) > 0$ for $h \in I$, and $g(t) \neq 0$ for $t \in \mathbb{R}$. By integrating (12) from $t_0 \in I$ to $t$, we obtain

(13) $$S(h(t)) - S(h(t_0)) = \int_{t_0}^{t} g(s)\, ds =: G(t),$$

where $S'(h) = F(h) > 0$, $S \in C^{k+2}(I)$, $G'(t) = g(t)$, $G \in C^{k+3}(\mathbb{R})$. Denote $\hat{J} := S(I)$. For the inverse function $S^{-1}$ to $S$ we have $S^{-1} \in C^{k+2}(\hat{J})$ and $S^{-1}(\hat{J}) = I$.

Now, we shall suppose that the functions $f$ and $q_{n-1}$ are chosen in such a way that $G$ maps $\mathbb{R}$ onto $\mathbb{R}$. This can be achieved e.g. by taking $f \equiv 1$ and $q_{n-1} \equiv 0$, when

$$G(t) = \int_{t_0}^{t} g(s)\, ds = \int_{t_0}^{t} c\, ds = c(t - t_0), \quad c \neq 0.$$

Then

$$S^{-1}(G(t) + S(h(t_0)))$$

is defined for those $t$ ($\in J$) for which $G(t) + S(h(t_0)) \in \hat{J}$. Thus we can write

$$h(t) := S^{-1}(G(t) + S(h(t_0)))$$

for $t \in J$. We can see that $h \in C^{k+2}(J)$, $dh(t)/dt = g(t)/S(h(t)) \neq 0$ on $J$, $h(J) = I$, and, due to (13), $h$ satisfies the relation (5) with the given $p_{n-1} \in C^k(I)$, $q_{n-1} \in C^\omega(J)$, and $f \in C^{k+2}(J)$ complying with (4) (e.g. $q_{n-1} = 0$ and $f = 1$).

If $p_{n-1} \in C^k(I)$, $k \geqq n - 1$, then there exists $h \in C^{k+2}(J)$, $k + 2 \geqq n + 1$, such that $f = 1$ and $q_{n-1} = 0$ on $J$, i.e. $f \in C^\omega(J)$ and $q_{n-1} \in C^\omega(J)$. Moreover, due to Lemma 4 with $k + 2$ instead of $r$, $q_{n-2} \in C^{k+2+n-2-n-1}(J) \cap C^{k+2-2}(J) \cap C^k(J) \cap \cap C^{k+2-1}(J) \cap C^{k-1}(J) = C^{k-1}(J)$. Q.E.D.

**Theorem 2.** *Let* (2) *be globally transformable into* (3), *let $i$ be a positive integer,* $1 \leqq i \leqq n - 1$.

*Then* $p_{n-1} \in C^{i-1}(I)$, $p_{n-2} \in C^{i-1}(I)$, ..., $p_{i+1} \in C^{i-1}(I)$ *if and only if*

$$q_{n-1} \in C^{i-1}(J),\ q_{n-2} \in C^{i-1}(J) \ldots, q_{i+1} \in C^{i-1}(I).$$

*In this case*

$$p_i \in C^{i-1}(I) \quad \textit{if and only if} \quad q_i \in C^{i-1}(J),$$

*and, moreover, for* $0 \leqq k \leqq i - 2$ *we have*

$$p_i \in C^k(I) \setminus C^{k+1}(I) \quad \textit{if and only if}$$
$$q_i \in C^k(J) \setminus C^{k+1}(J), \quad \textit{i.e. the pair } (i, k) \textit{ is an invariant}.$$

Proof. Due to Lemma 1, we may put $r = n$ in Lemma 4. If $p_{n-1} \in C^{i-1}(I), \ldots$ $\ldots, p_{i+1} \in C^{i-1}(I)$ for some $i$, $1 \leqq i \leqq n - 1$, then owing to (8),

$$q_{n-1} = B_{n-1,n} + B_{n-1,n-1}p_{n-1}(h) \in C^{n-2}(J) \cap C^{n-1}(J) \cap C^{i-1}(J) = C^{i-1}(J),$$
$$q_{n-2} = B_{n-2,n} + B_{n-2,n-1}\, p_{n-1}(h) + B_{n-2,n-2}\, p_{n-2}(h) \in$$
$$\in C^{n-3}(J) \cap C^{n-2}(J) \cap C^{i-1}(J) \cap C^{n-1}(J) \cap C^{i-1}(J) =$$
$$= C^{i-1}(J) \quad \text{because} \quad n - 2 \geqq i + 1,$$
$$\ldots$$

$$q_{i+1} = B_{i+1,n} + B_{i+1,n-1}\,p_{n-1}(h) + B_{i+1,n-2}\,p_{n-2}(h) + \dots$$
$$\dots + B_{i+1,j}\,p_j(h) + \dots + B_{i+1,i+1}\,p_{i+1}(h) \in$$
$$\in C^{n-(n-i-1)-1}(J) \cap C^{n-(n-1-i-1)-1}(J) \cap C^{i-1}(J) \cap$$
$$\cap C^{n-(n-2-i-1)-1}(J) \cap C^{i-1}(J) \cap \dots$$
$$\dots \cap C^{n-(j-i-1)-1}(J) \cap C^{i-1}(J) \cap \dots$$
$$\dots \cap C^{n-1}(J) \cap C^{i-1}(J) = C^{i-1}(J),$$

because $0 < i \leqq n - 1$. Under these assumptions, if also $p_i \in C^k(I)$ for some $k$, $0 \leqq k \leqq i - 1$, then $q_i \in C^k(J)$. In fact,

$$q_i = B_{in} + B_{i,n-1}\,p_{n-1}(h) + B_{i,n-2}\,p_{n-2}(h) + \dots + B_{ii}\,p_i(h) \in$$
$$\in C^{n-(n-i)-1}(J) \cap C^{n-(n-1-i)-1}(J) \cap C^{i-1}(J) \cap \dots \cap C^{n-1}(J) \cap C^k(J) =$$
$$= C^k(J), \text{ since } k \leqq i - 1\,.$$

However, if $p_{n-1} \in C^{i-1}(I), \dots, p_{i+1} \in C^{i-1}(I)$ and $p_i \,\bar{\in}\, C^{k+1}(I)$ for some $k$, $0 \leqq k \leqq \leqq i - 2$, then also $q_i \,\bar{\in}\, C^{k+1}(J)$. Otherwise, due to the fact that $B_{ii} \neq 0$, $p_i(h)$ can be expressed as a linear combination of functions of the class $C^{k+1}(J)$ with ocef-ficients of the class $C^{i-1}$, where $k + 1 \leqq i - 1$. Hence $p_i \in C^{k+1}(I)$ which contradicts our assumption.

The converse is true because of the symmetry of our assumptions on the equations (2) and (3). Q.E.D.

**Corollary 2.** *If* (2) *is globally transformable into* (3) *and* $p_{n-1} \in C^{n-2}(I)$, $p_{n-2} \in \in C^{n-3}(I), \dots, p_j \in C^{j-1}(I)$, *then* $q_{n-1} \in C^{n-2}(J)$, $q_{n-2} \in C^{n-3}(J), \dots, q_j \in C^{j-1}(J)$ *for* $0 < j \leqq n - 1$, *and conversely. Of course, we always have* $p_0 \in C^0(I)$ *and* $q_0 \in C^0(J)$.

This follows immediately from Theorem 2 if we put successively $i = n - 1$, $i = n - 2, \dots, i = j$.

Example 1. $y''' + p_2(x)\,y'' + p_1(x)\,y' + p_0(x)\,y = 0$ on $I$ with $p_2 \in C^0(I) \setminus C^1(I)$ cannot be globally transformed into

$$z''' + q_2(t)\,z'' + q_1(t)\,z' + q_0(t)\,z = 0 \quad \text{on } J \text{ with } \quad q_2 \in C^1(J)\,.$$

**Theorem 3.** *Let* $p_{n-1} \equiv 0$ *and* $p_{n-2} \in C^{n-2}(I)$ *in equation* (2). *Let equation* (3) *with* $q_{n-1} \equiv 0$ *on* $J$ *be globally equivalent to* (2). *Then* $q_{n-2} \in C^{n-2}(J)$. *Moreover, there exists a global transformation of* (2) *into* (3) *with* $q_{n-2} \in C^{\omega}(J)$, *in particular with* $q_{n-2} = 1$ *on* $J$.

Proof. Due to Corollary 1, if $p_{n-1} \equiv 0$ and $q_{n-1} \equiv 0$ then (7) holds for a global transformation (1) of (2) into (3). According to Lemma 1, we have $f \in C^n(J)$. Hence $h \in C^{n+1}(J)$. Corollary 1 gives

$$q_{n-2}(t) \Big/ \binom{n+1}{3} = \tfrac{1}{2}\,h'''(t)/h'(t) - \tfrac{3}{4}\,h''^2(t)/h'^2(t) + p_{n-2}(h(t))\,h'^2(t) \Big/ \binom{n+1}{3}\,.$$

Since $p_{n-2} \in C^{n-2}(I)$, $h''' \in C^{n-2}(J)$, we have $q_{n-2} \in C^{n-2}(J)$. Moreover, for $d := 1/\binom{n+1}{3}$ and $p = p_{n-2}$, Lemma 5 guarantees the existence of an interval $J \subset \mathbb{R}$ and a function $h \in C^{n+1}(J)$ satisfying the above equation for $q_{n-2} = 1$. If we put $f := |h'|^{(1-n)/2}$, then the transformation (1) globally transforms equation (2) into equation (3) with $q_{n-1} \equiv 0$ and $q_{n-2} \equiv 1$ on $J$. Q.E.D.

Example 2. $y''' + p_1(x)\, y' + p_0(x)\, y = 0$ on $I$ with $p_1 \bar{\in} C^1(I)$ cannot be globally transformed into $z''' + q_1(t)\, z' + q_0(t)\, z = 0$ on $J$ with $q_1 \in C^1(J)$.

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