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ON SPECIAL PLANE NETS

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The local projective differential geometry of nets has been studied extensively; see [1]–[3]. In the paper, I prove a global result.

1. Let $P^2(\mathbb{R})$ be the projective plane over reals. Let N be a net of curves given on a domain $D \subset P^2(\mathbb{R})$; let $D_0 \subset \mathbb{R}^2$ be a domain with coordinates (x, y) and $m\colon D_0 \to D$ a diffeomorphism mapping the lines $x = \text{const.}$ and $y = \text{const.}$ into the lines of our net. The points $m(x, y)$, $m_x(x, y)$, $m_y(x, y)$ being linearly independent (here $m_x = \partial m/\partial x$, etc.), the homogeneous coordinates of the point $m(x, y)$ satisfy hyperbolic partial differential equation

$$m_{xy} = am_x + bm_y + cm\,; \quad a = a(x, y), \ldots, c = c(x, y) \tag{1.1}$$

on D_0. Of course, we may choose other coordinates $\tilde{x} = \tilde{x}(x)$, $\tilde{y} = \tilde{y}(y)$ and another analytic point $\tilde{m} = \varrho(x, y)\, m$; the net N determines thus the equation (1.1) up to these changes.

The theory of the equation (1.1) is well known. The *Laplace transform* of our net N given as above is a mapping $m'\colon D_0 \to P^2(\mathbb{R})$ such that there is a tangent field $t(x, y)$ on D_0 satisfying $t(x, y)\, m'(x, y) \in \{m(x, y), m'(x, y)\}$ for each $(x, y) \in D_0$; by $\{z_1, z_2\}$, we denote the subspace through $z_1, z_2 \in P^2(\mathbb{R})$. It is known that our net has exactly two Laplace transforms

$$m_1 = m_y - am\,, \quad m_{-1} = m_x - bm\,; \tag{1.2}$$

indeed,

$$(\partial/\partial x)\, m_1 = bm_1 + hm\,, \quad (\partial/\partial y)\, m_{-1} = am_{-1} + km \tag{1.3}$$

with

$$h = c + ab - a_x\,, \quad k = c + ab - b_y\,. \tag{1.4}$$

The functions h, k are the so-called *Laplace-Darboux invariants*. In fact, they are not invariants, but the quadratic *point forms*

$$\varphi_1 = h\, \mathrm{d}x\, \mathrm{d}y\,, \quad \varphi_{-1} = k\, \mathrm{d}x\, \mathrm{d}y \tag{1.5}$$

are invariants of (1.1) with respect to the changes $x \to \tilde{x}(x)$, $y \to \tilde{y}(y)$, $m \to \varrho m$, and are thus invariants of our net N. The point m_1 satisfies, if $h \neq 0$ on D_0,

$$(1.6) \qquad m_{1xy} = a_1 m_{1x} + b_1 m_{1y} + c_1 m_1 \quad \text{with} \quad a_1 = a + (\log h)_y\,, \quad b_1 = b\,,$$
$$c_1 = c + h - k - b(\log h)_y\,; \quad (\log h)_y := h^{-1}h_y\,;$$

and the Laplace transforms of the net N_1 are

$$(1.7) \qquad m_2 := (m_1)_1 = m_{1y} - a_1 m_1\,, \quad (m_1)_{-1} = m_{1x} - b_1 m_1 = hm\,;$$

similarly for N_{-1}.

Let us consider the complexification $P^2_{\mathbb{C}}(\mathbb{R})$ of $P^2(\mathbb{R})$ and of D_0. An *elliptic net* N on a domain $D \subset P^2(\mathbb{R})$ is a diffeomorphism $f\colon D_0 \to D$ carrying the lines $x \pm \mathrm{i}y =$ = const. of D_0 into the lines of N. Let us introduce the complex coordinate $z =$ $= x + \mathrm{i}y$ and the usual operators $\partial/\partial z = \frac{1}{2}(\partial/\partial u - \mathrm{i}\partial/\partial v)$, $\partial/\partial \bar z = \frac{1}{2}(\partial/\partial u + \mathrm{i}\partial/\partial v)$. As in the real case, an elliptic net induces an equation of the type

$$(1.8) \qquad m_{z\bar z} = \mathscr{A} m_z + \bar{\mathscr{A}} m_{\bar z} + \mathscr{C}_m\,; \quad \mathscr{C} = \bar{\mathscr{C}}$$

on D_0; it may be rewritten as

$$(1.9) \qquad m_{xx} + m_{yy} = 2(\bar{\mathscr{A}} + \mathscr{A})\, m_x + 2\mathrm{i}(\bar a - a)\, m_y + 4\mathscr{C} m\,.$$

Then the Laplace transforms are

$$(1.10) \qquad m_1 = m_{\bar z} - \mathscr{A} m\,, \quad m_{-1} = \bar m_1 = m_z - \bar{\mathscr{A}} m$$

with

$$(1.11) \qquad m_{1z} = \bar{\mathscr{A}} m_1 + Hm\,, \quad m_{-1\bar z} = \mathscr{A} m_{-1} + Km\,;$$
$$H = \mathscr{C} + \mathscr{A}\bar{\mathscr{A}} - \mathscr{A}_z\,, \quad K = \bar H\,;$$

the associated invariant point forms are then

$$(1.12) \qquad \varphi_1 = H\,\mathrm{d}z\,\mathrm{d}\bar z\,, \quad \varphi_{-1} = \bar\varphi_1 = K\,\mathrm{d}z\,\mathrm{d}\bar z\,.$$

2. In this section we introduce a certain elliptic net N^ε, $\varepsilon = \pm 1$, on $P^2(\mathbb{R})$ (or, as the case may be, a part of it). Consider the domain

$$(2.1) \qquad D_\varepsilon = \{z \in \bar{\mathbb{C}} \equiv \mathbb{C} \cup \{\infty\}\,;\ 1 + \varepsilon z \bar z > 0\}\,;$$

of course, $D_{+1} = \bar{\mathbb{C}}$. With each point $z \in D_\varepsilon$ let us associate the point

$$(2.2) \qquad m(z) = (1 + \varepsilon z\bar z)^{-1}\,(\bar z + z,\ \mathrm{i}(\bar z - z),\ 1 - \varepsilon z \bar z) \in P^2(\mathbb{R})\,;$$

the elliptic net N^ε on $m(D_\varepsilon)$ is formed by the images of the lines $z =$ const. and $\bar z =$ = const. It is easy to see that

$$(2.3) \qquad m_{z\bar z} = -2\varepsilon(1 + \varepsilon z \bar z)^{-2}\, m$$

and the point forms (1.12) are

$$(2.4) \qquad \varphi_1 = \varphi_{-1} = -2\varepsilon(1 + \varepsilon z\bar z)^{-2}\,\mathrm{d}z\,\mathrm{d}\bar z\,.$$

Consider an affine space A^3 over reals with a fixed basis $\{O; e_1, e_2, e_3\}$ and the coordinates (X, Y, Z) defined by $P = O + Xe_1 + Ye_2 + Ze_3$. Let $\iota\colon D_\varepsilon \to A^3$ be an inclusion map given by

$$(2.5)\qquad \iota(z) = O + \tfrac{1}{2}(\bar{z} + z)\,e_1 + \tfrac{1}{2}\mathrm{i}(\bar{z} - z)\,e_2\,.$$

Further, consider the point $S = O - e_3$ and the quadric Q_ε

$$(2.6)\qquad X^2 + Y^2 + \varepsilon Z^2 = \varepsilon$$

in A^3; of course, $S \in Q_\varepsilon$. Let the mapping $\mu_\varepsilon\colon D_\varepsilon \to Q_\varepsilon$ be defined as follows: $\mu_\varepsilon(z)$ is the intersection of the line $\{\iota(z), S\}$ with Q_ε and $\mu_1(\infty) = S$. It is easy to see that

$$(2.7)\qquad \mu_\varepsilon(z) = O + X(z)\,e_1 + Y(z)\,e_2 + Z(z)\,e_3 \quad \text{with}$$
$$X(z) = (1 + \varepsilon z\bar{z})^{-1}(\bar{z} + z)\,, \quad Y(z) = \mathrm{i}(1 + \varepsilon z\bar{z})^{-1}(\bar{z} - z)\,,$$
$$Z(z) = (1 + \varepsilon z\bar{z})^{-1}(1 - \varepsilon z\bar{z})\,.$$

In A^3, let us introduce the scalar product (for $\varepsilon = -1$ non-definite) by

$$(2.8)\qquad \langle e_1, e_1\rangle_\varepsilon = \langle e_2, e_2\rangle_\varepsilon = 1\,, \quad \langle e_3, e_3\rangle_\varepsilon = \varepsilon\,; \quad \langle e_i, e_j\rangle_\varepsilon = 0 \ \text{ otherwise}\,.$$

It is easy to see that

$$(2.9)\qquad \mathrm{d}X(z) = (1 + \varepsilon z\bar{z})^{-2}\{(1 - \varepsilon\bar{z}^2)\,\mathrm{d}z + (1 - \varepsilon z^2)\,\mathrm{d}\bar{z}\}\,,$$
$$\mathrm{d}Y(z) = -\mathrm{i}(1 + \varepsilon z\bar{z})^{-2}\{1 + \varepsilon\bar{z}^2)\,\mathrm{d}z - (1 + \varepsilon z^2)\,\mathrm{d}\bar{z}\}\,,$$
$$\mathrm{d}Z(z) = -2\varepsilon(1 + \varepsilon z\bar{z})^{-2}(\bar{z}\,\mathrm{d}z + z\,\mathrm{d}\bar{z})\,,$$

and the mapping μ_ε induces, on D_ε, the metric

$$(2.10)\qquad \mathrm{d}s_\varepsilon^2 = (\mathrm{d}X(z))^2 + (\mathrm{d}Y(z))^2 + \varepsilon(\mathrm{d}Z(z))^2 =$$
$$= 4(1 + \varepsilon z\bar{z})^{-2}\,\mathrm{d}z\,\mathrm{d}\bar{z} = -2\varepsilon\varphi_1\,.$$

Let us remark that $\mathrm{d}s_{-1}^2$ is exactly the complete Caley metric on D_{-1}, and $\mathbb{H}^2 = (D_{-1}, \mathrm{d}s_{-1}^2)$ is the hyperbolic plane.

Consider the „sphere" $S^2 \subset A^3$ given by $X^2 + Y^2 + Z^2 = 1$; let $\nu_\varepsilon\colon Q_\varepsilon \to S^2$ be the projection from the origin O and $\pi\colon S^2 \to P^2(\mathbb{R})$ the usual identification mapping. Comparing (2.2) and (2.7), we see that *our net N^ε is induced by the map $\pi \circ \nu_\varepsilon \circ \mu_\varepsilon\colon D_\varepsilon \to P^2(\mathbb{R})$, and the lines of N^ε are the images of the isotropic lines of the metric* $\mathrm{d}s_\varepsilon^2$. In this way, the geometric construction of N^ε is fully described.

3. On a domain $D \subset P^2(\mathbb{R})$ let an elliptic net N be given. With each point $m \in D$ let us associate a moving frame $\{m, M, \overline{M}\}$ such that $M = m_1$ and $\overline{M} = m_{-1}$ are the Laplace transforms; let the analytic points m, M be chosen in such a way that

$$(3.1)\qquad m = \overline{m}\,, \quad [m, M, \overline{M}] = \mathrm{i}\,.$$

Then we may write

$$(3.2)\qquad \mathrm{d}m = \tau_0^0 m + \tau M + \bar{\tau}\overline{M}\,, \quad \mathrm{d}M = \tau_1^0 m + \tau_1^1 M + \tau_1^2\overline{M}\,,$$
$$\mathrm{d}\overline{M} = \tau_2^0 m + \tau_2^1 M + \tau_2^2\overline{M}$$

with

$$(3.3)\qquad \tau_2^0 = \bar{\tau}_1^0\,, \quad \tau_2^1 = \bar{\tau}_1^2\,, \quad \tau_2^2 = \bar{\tau}_1^1\,; \quad \tau_0^0 = \bar{\tau}_0^0\,, \quad \tau_0^0 + \tau_1^1 + \bar{\tau}_1^1 = 0\,;$$

the last two identities result from (3.1). Further, we have to take into account the integrability conditions

$$(3.4)\qquad d\tau_i^j = \sum_{k=0}^{2} \tau_i^k \wedge \tau_k^j \quad \text{with} \quad \tau_0^1 := \tau,\ \tau_0^2 := \bar{\tau}\,.$$

Obviously, the lines of N are given by $\tau\bar{\tau} = 0$. The point M being the Laplace transform, we have

$$(3.5)\qquad \tau_1^2 = a\tau\,.$$

The exterior differentiation yields

$$(3.6)\qquad \{da + a(\tau_0^0 - 2\tau_1^1 + \bar{\tau}_1^1)\} \wedge \tau - \tau_1^0 \wedge \bar{\tau} = 0\,.$$

According to Cartan's lemma, there are functions such that

$$(3.7)\qquad da + a(\tau_0^0 - 2\tau_1^1 + \bar{\tau}_1^1) = b_1\tau - b_2\bar{\tau}\,, \quad \tau_1^0 = b_2\tau + b_3\bar{\tau}\,.$$

Thus

$$(3.8)\qquad dM = \tau_1^1 M + \tau(b_2 m + a\overline{M}) + \bar{\tau} b_3 m\,.$$

This means that the second Laplace transorm m_2 either does not exist (in the case $a = b_2 = 0$) or is situated on the straight line $\{M, b_2 m + a\overline{M}\}$.

Definition. The elliptic net N will be called *special* if, at each point $m \in D$, the second Laplace transform m_2 either does not exist or is situated on the straight line $\{m_1, m_{-1}\}$.

It is easy to see that N is special if and only if

$$(3.9)\qquad b_2 = 0\,,$$

and this is equivalent to the condition that m_{-2} either does not exist or is situated on the same line $\{m_1, m_{-1}\}$. *From now on, let N be a special net.*

Let us choose other analytic points

$$(3.10)\qquad m^* = \alpha m\,, \quad M^* = \beta M\,; \quad \alpha\beta\bar{\beta} = 1\,;$$

the last relation arising from (3.1_2). Writting down the equations (3.2) with (3.5) + + (3.7_2) + (3.9) and the similar equations (3.2*), we easily find

$$(3.11)\qquad \tau^* = \alpha\beta^{-1}\tau\,, \quad a^* = \alpha^{-1}\beta^2\bar{\beta}^{-1}a\,, \quad b_3^* = \alpha^{-2}\beta\bar{\beta}b_3\,.$$

Thus the forms

$$(3.12)\qquad \varphi_1 = b_3\tau\bar{\tau}\,, \quad \varphi_{-1} = \bar{\beta}_3\tau\bar{\tau}$$

are invariant; they are exactly the point forms (1.12) of N.

Theorem. *Let N be an elliptic special net on $P^2(\mathbb{R})$. Let us suppose $\varphi_1 = \varphi_{-1}$ to be an $\mathbb{R}$-valued definite form. If it is positive definite, let it have positive curvature. Then $N = N^{+1}$.*

Proof. We have $b_3 = \bar{b}_3 \neq 0$ on $P^2(\mathbb{R})$; it follows from (3.10_3) + (3.11_3) that we may choose $b_3 = -\varepsilon = \mp 1$, i.e., our fundamental equations are

(3.13) $\qquad \tau_1^2 = a\tau\,, \quad \tau_1^0 = -\varepsilon\bar{\tau}\,.$

The differential consequences are

(3.14) $\qquad \{\mathrm{d}a + a(\tau_0^0 - 2\tau_1^1 + \bar{\tau}_1^1)\} \wedge \tau = 0\,, \quad (2\tau_0^0 - \tau_1^1 - \bar{\tau}_1^1) \wedge \bar{\tau} = 0\,.$

The complex conjugate of (3.14_2) being $(2\tau_0^0 - \tau_1^1 - \bar{\tau}_1^1) \wedge \tau = 0$, we have $2\tau_0^0 - - \tau_1^1 - \bar{\tau}_1^1 = 0$. Taking into regard (3.3_5), we get

(3.15) $\qquad \tau_0^0 = 0\,, \quad \tau_1^1 + \bar{\tau}_1^1 = 0\,.$

Thus we get, from (3.14_1) and Cartan's lemma, the existence of a function b such that

(3.16) $\qquad \mathrm{d}a - 3a\tau_1^1 = b\tau\,.$

The exterior differentiation yields

(3.17) $\qquad (\mathrm{d}b - 4b\tau_1^1) \wedge \tau = -3a(a\bar{a} + \varepsilon)\,\tau \wedge \bar{\tau}$

and the existence of a new function c such that

(3.18) $\qquad \mathrm{d}b - 4b\tau_1^1 = c\tau + 3a(a\bar{a} + \varepsilon)\,\tau \wedge \bar{\tau}\,.$

We have $\varphi_1 = \varphi_{-1} = -\varepsilon\tau\bar{\tau}$. On $P^2(\mathbb{R})$, consider the metric

(3.19) $\qquad \mathrm{d}s^2 := |\varphi_1| = \tau\bar{\tau} = (\omega^1)^2 + (\omega^2)^2$

with

(3.20) $\qquad \tau = \omega^1 + \mathrm{i}\omega^2\,,$

ω^1 and ω^2 being $\mathbb{R}$-valued 1-forms. Considering the Hodge $*$-operator with respect to $\mathrm{d}s^2$, we have $*\omega^1 = \omega^2$, $*\omega^2 = -\omega^1$, i.e.,

(3.21) $\qquad *\tau = -\mathrm{i}\tau\,.$

Further,

(3.22) $\qquad \tau \wedge \bar{\tau} = -2\mathrm{i}\,\mathrm{d}o\,,$

$\mathrm{d}o := \omega^1 \wedge \omega^2$ being the area element with respect to $\mathrm{d}s^2$. Let us calculate the Laplacian of the $\mathbb{R}$-valued function $a\bar{a}$. We have

(3.23) $$\mathrm{d}(a\bar{a}) = \bar{a}b\tau + a\bar{b}\bar{\tau}\,, \quad *\mathrm{d}(a\bar{a}) = -\mathrm{i}(\bar{a}b\tau - a\bar{b}\bar{\tau})\,,$$
$$\mathrm{d} * \mathrm{d}(a\bar{a}) = \Delta(a\bar{a})\,.\,\mathrm{d}o = 4\{b\bar{b} + 3a\bar{a}(a\bar{a} + \varepsilon)\}\,\mathrm{d}o\,.$$

We now have to evaluate the curvature of (3.19). It is well known that there exists exactly one 1-form ω such that $\mathrm{d}\omega^1 = -\omega^2 \wedge \omega$, $\mathrm{d}\omega^2 = \omega^1 \wedge \omega$, and the curvature $\varkappa$ is given by $\mathrm{d}\omega = -\varkappa\omega^1 \wedge \omega^2$. Now, let $\mathrm{d}\tau = \tau \wedge \varrho$, $\mathrm{d}\varrho = k\tau \wedge \bar{\tau}$. Then $\varrho = \mathrm{i}\omega$, and we get $\varkappa = 2k$ from the second equation. In our particular case, $\varrho = \tau_1^1$ and $\mathrm{d}\varrho = (a\bar{a} + \varepsilon)\,\tau \wedge \bar{\tau}$. Thus

(3.24) $\qquad \varkappa = 2(a\bar{a} + \varepsilon)\,.$

In the case $\varepsilon = 1$, we have $a\bar{a} + \varepsilon > 0$, and (3.23_3) yields, via the maximum principle,

(3.25) $\qquad a = 0\,.$

In the case $\varepsilon = -1$, $\varkappa > 0$ implies the same equation (3.25). However, then (3.24) yields $\varkappa = -2$, a contradiction.

The equations (3.18) are now reduced to

$$\tau_1^2 = 0\,, \quad \tau_1^0 = -\bar{\tau}\,, \tag{3.26}$$

and we have

$$d\tau = \tau \wedge \tau_1^1\,, \quad d\tau_1^1 = \tau \wedge \bar{\tau}\,. \tag{3.27}$$

It is easy to check that we are in position to satisfy them by taking

$$\tau = \sqrt{2}\,.\,(1 + z\bar{z})^{-1}\,dz\,, \quad \tau_1^1 = -(1 + z\bar{z})^{-1}\,(\bar{z}\,dz - z\,d\bar{z})\,. \tag{3.28}$$

The equations (3.2) may be written as

$$\begin{aligned} m_z &= \sqrt{2}\,.\,(1 + z\bar{z})^{-1}\,M\,, \quad M_z = -(1 + z\bar{z})^{-1}\,\bar{z}M\,, \\ M_{\bar{z}} &= -(1 + z\bar{z})^{-1}\,(\sqrt{2}\,.\,m - zM) \end{aligned} \tag{3.29}$$

and the complex conjugate equation; here we take into account (3.3), (3.15) and (3.26). From (3.29_2) we get the existence of a function $\varphi(\bar{z})$ such that

$$M = (1 + z\bar{z})^{-1}\,\varphi(\bar{z})\,. \tag{3.30}$$

Calculating then m from (3.29_3) and $\overline{M}$ from $(\overline{3.29_1})$, we get

$$\begin{aligned} m &= \tfrac{1}{2}\sqrt{2}\,.\,(1 + z\bar{z})^{-1}\,\{2z\,\varphi(\bar{z}) - (1 + z\bar{z})\,\varphi'(\bar{z})\}\,, \\ \overline{M} &= -(1 + z\bar{z})^{-1}\,z^2\,\varphi(\bar{z}) + z\,\varphi'(\bar{z}) - \tfrac{1}{2}(1 + z\bar{z})\,\varphi''(\bar{z})\,, \end{aligned} \tag{3.31}$$

respectively. Inserting this into $(\overline{3.29_2})$, we obtain

$$\varphi'''(\bar{z}) = 0\,, \quad \text{i.e.,} \quad \varphi(\bar{z}) = B_0 + B_1\bar{z} + B_2\bar{z}^2 \quad \text{with} \quad B_i \in \mathbb{C}\,. \tag{3.32}$$

Thus

$$m = \tfrac{1}{2}\sqrt{2}\,.\,(1 + z\bar{z})^{-1}\,\{2B_0z + 2B_2\bar{z} + B_1(z\bar{z} - 1)\}\,. \tag{3.33}$$

The condition $m = \overline{m}$ yields $B_0 + \overline{B}_2 = 0$, $B_1 = \overline{B}_1$. Put $2B_0 = A_1 - \mathrm{i}A_2$, $B_1 = -A_3$ with $A_i \in \mathbb{R}$. Then

$$m = \tfrac{1}{2}\sqrt{2}\,.\,(1 + z\bar{z})^{-1}\,\{A_1(\bar{z} + z) + A_2\,\mathrm{i}(\bar{z} - z) + A_3(1 - z\bar{z})\}\,. \tag{3.34}$$

Thus we get the general solution of (3.29) + $(\overline{3.29})$. Now, it is sufficient to compare it with (2.2) for $\varepsilon = 1$. QED.

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