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MAXIMAL ANTICHAINS IN A PARTIALLY ORDERED SET

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1. INTRODUCTION

All partially ordered sets dealt with in the present paper are assumed to be finite.

Antichains in a partially ordered set are also called *Sperner families* in the literature. A thorough investigation of combinatorial questions concerning antichains was performed in [3] (including applications in the theory of Boolean functions, data bases, and in other fields).

For a partially ordered set X we denote by $A(X)$ the system of all antichains in X. Next, let $MA(X)$ be the set of all $B \in A(X)$ having the property that for each $C \in A(X)$ with $B \subseteq C$ the relation $B = C$ is valid. The elements of $MA(X)$ are said to be maximal antichains in X.

Each nonempty subset of a partially ordered set is considered to be partially ordered by the inherited relation of partial order.

Let $B_1, B_2 \in A(X)$. We put $B_1 \leqq B_2$ if for each $b_1 \in B_1$ there exists $b_2 \in B_2$ with $b_1 \leqq b_2$. Then $A(X)$ turns out to be a partially ordered set. Hence $MA(X)$ is a partially ordered set as well.

In [1] it has been proved that $MA(X)$ is a lattice and that for each lattice L there exists a partially ordered set Y such that L is isomorphic to $MA(Y)$.

The results on $MA(X)$ were applied in [2] for studying cut-sets of the partially ordered set X.

Let S and S' be the partially ordered set in Fig. 1 and Fig. 2, respectively. It is easy to verify that the lattice $MA(S)$ is non-modular and that $MA(S')$ is distributive.

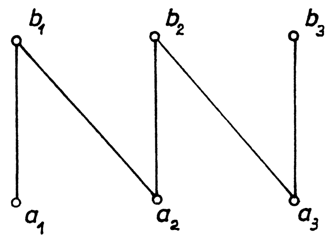

Fig. 1.

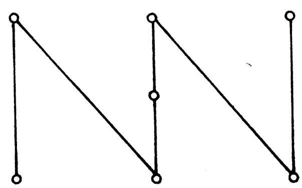

Fig. 2.

(These examples were given in [1].) The partially ordered set S' possesses a subset S_1 which is isomorphic to S, but S_1 fails to be a convex subset of S'.

In [1] it is also pointed out that no internal characterization is known of those partially ordered sets X for which the lattice $MA(X)$ is modular.

In view of the above examples the natural question arises what are the relations between the following conditions for a partially ordered set X:

(α) $MA(X)$ is non-modular.

(β) There exists a convex subsystem S_1 of X such that S_1 is isomorphic to S.

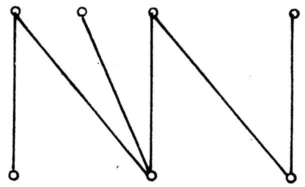

Fig. 3.

Let S_2 be the partially ordered set in Fig. 3. Then S_2 satisfies the condition (β). It can be easily verified that the lattice $MA(S_2)$ is distributive. (This example is due to M. Ploščica.) Hence the implication $(\beta) \Rightarrow (\alpha)$ is not valid in general.

In the present paper it will be proved that the implication $(\alpha) \Rightarrow (\beta)$ always holds.

A convex subset of X which is isomorphic to the partially ordered set in Fig. 1 will be said to be a *serpentine subset of X*.

Let $\mathscr{C}(X)$ be the set of all chains in X. We put $l(X) = \max\{\text{card } Y: Y \in \mathscr{C}(X)\}$. Next, let $\mathscr{S}(X)$ be the system of all subsets X_1 of X which have the following properties: (i) there exist A and B in $MA(X)$ with $A \leqq B$ such that $X_1 = \{x \in X: \text{there are } a \in A \text{ and } b \in B \text{ with } a \leqq x \leqq b\}$; (ii) $l(X_1) \leqq 2$. The elements of $\mathscr{S}(X)$ will be called *short subsets of X*.

It will be shown that the following conditions are equivalent:

(γ_1) The lattice $MA(X)$ is modular.

(γ_2) For each short subset X_1 of X, the lattice $MA(X_1)$ is modular.

Let $y \in X$ and $P \subseteq X$. We shall write $y <_1 P$ if (a) there exists $p \in P$ with $y < p$, and (b) whenever $p_1 \in P$ and the elements p_1, y are comparable, then y is covered by p_1.

Let us denote by $\mathscr{N}(X)$ the set of all triples (P_1, P_2, P_3) of mutually disjoint subsets of X such that

(i) $P_2 \neq \emptyset \neq P_3$ and each element of P_2 is covered by each element of P_3;

(ii) both the sets $P_1 \cup P_2$ and $P_1 \cup P_3$ belong to $MA(X)$.

A serpentine subset S of X will be said to be regular if there exist (B_1, B_2, A_2) and (B'_1, B'_2, A'_2) in $\mathscr{N}(X)$ with $B_1 \neq B'_1$ and $B_1 \cup B_2 = B'_1 \cup B'_2$ such that (under the notation as in Fig. 1) we have

(i) $a^1 \in A_2'$, $a^2 <_1 B_1$, $a^2 <_1 B_1'$, $a^3 \in A_2$,

(ii) $b^1 \in B_1$, $b^2 \in B_1'$, $b^3 \in B_1'$,

(iii) a^2 is incomparable with all elements of $A_2 \cup A_2'$.

In a dual way we define the notion of a dually regular serpentine subset in X.

It will be proved that the lattice $MA(X)$ fails to be modular if and only if X possesses either a regular serpentine subset or a dually regular serpentine subset.

2. THE COVERING RELATION

Let X be a partially ordered set. If $x_1, x_2 \in X$ and x_1 si covered by x_2, then we write $x_1 \prec x_2$. The same notation will be used for the covering relation in $MA(X)$.

In this section we shall investigate pairs (A, B) of elements of $MA(X)$ such that $A \prec B$.

Let $A_0 \in A(X)$, $B \in MA(X)$, $A_0 \leqq B$. Let us denote by $\mathscr{A}(A_0, B)$ the set of all $A_1 \in A(X)$ such that $A_0 \subseteq A_1 \leqq B$.

2.1. Lemma. *Let $C \in \mathscr{A}(A_0, B)$. Assume that for each $C_1 \in \mathscr{A}(A_0, B)$ with $C \subseteq C_1$ the relation $C = C_1$ is valid. Then $C \in MA(X)$.*

Proof. By way of contradiction, suppose that C does not belong to $MA(X)$. Then there exists $x \in X$ such that $x \notin C$ and x is incomparable with each element of C. Put $C_1 = C \cup \{x\}$. Hence $C \subset C_1 \in A(X)$. Thus $C_1 \notin \mathscr{A}(A_0, B)$. Therefore for each element $b \in B$ the relation $x \nleqq b$ holds.

If x is incomparable with each element of B then $x \in B$ (since $B \in MA(x)$), which is a contradiction. Thus there is $b_1 \in B$ with $b_1 < x$. We distinguish the following cases:

(i) There exists $c \in C$ with $b_1 < c$. Since there is $b_2 \in B$ with $c \leqq b_2$, we obtain $b_1 < b_2$, which is impossible.

(ii) The element b_1 is incomparable with all elements of C. Then $b_1 \in C$ and thus $b_1 < x$ cannot hold.

(iii) There exists $c \in C$ with $c \leqq b_1$. Hence $c < x$, which is a contradiction.

The proof is complete.

Now let $A_0 \in A(X)$, $B_1 \in MA(X)$, $B_2 \in MA(X)$, $B_1 \leqq A_0 \leqq B_2$. We denote by $\mathscr{A}(A_0, B_1, B_2)$ the set of all $A_1 \in A(X)$ such that $A_0 \subseteq A_1$ and $B_1 \leqq A_1 \leqq B_2$.

The proof of the following lemma is analogous to that of 2.1; it will be omitted.

2.2. Lemma. *Let $C \in \mathscr{A}(A_0, B_1, B_2)$. Assume that for each $C_1 \in \mathscr{A}(A_0, B_1, B_2)$ with $C \subseteq C_1$ the relation $C = C_1$ is valid. Then $C \in MA(X)$.*

2.3. Lemma. (Cf. [2].) *Let $A, B \in MA(X)$, $A \leqq B$, $b \in B$. Then there exists $a \in A$ such that $a \leqq b$.*

2.4. Lemma. *Let $A, B \in MA(X)$, $A \prec B$. Let $b \in B \setminus A$ and let a be as in 2.3. Then $a \prec b$.*

Proof. By way of contradiction, assume that the relation $a \prec b$ does not hold. Hence there is $a_0 \in X$ with $a < a_0 < b$. Put $A_0 = \{a_0\}$. There exists $C \in \mathscr{A}(A_0, A, B)$ such that, whenever $C_1 \in \mathscr{A}(A_0, A, B)$ and $C \subseteq C_1$, then $C = C_1$. Thus in view of 2.2, C belongs to $MA(X)$. Since $a_0 \notin A$ and $a_0 \notin B$ we obtain that $C \neq A$ and $C \neq B$. Hence $A < C < B$, which is a contradiction.

2.5. Lemma. *Let us apply the same assumptions and notation as in* 2.4. *Let* $a_1 \in A \setminus B$, *then* $a_1 \prec b$.

Proof. In view of 2.4, it suffices to verify that $a_1 < b$. Since $a_1 \notin B$, we have $a_1 \neq b$. Suppose that $a_1 > b$; there exists $b_1 \in B$ with $a_1 < b_1$, and then $b < b_1$, which is a contradiction. Next, suppose that a_1 is incomparable with b. Put $A_0 = \{a_1, b\}$. Applying the same argument as in the proof of 2.4 we infer that A fails to be covered by B, which is a contradiction. Hence $a_1 < b$.

2.6. Lemma. *Let the same assumptions as in* 2.4 *be valid and let us apply the same notation. Let* $b_1 \in B \setminus A$. *Then* $b_1 \succ a$.

Proof. According to 2.4 it suffices to show that $b_1 > a$. The relation $b_1 \leqq a$ is obviously impossible. If b_1 is incomparable with a, then we put $A_0 = \{a, b_1\}$ and proceed as in the proof of 2.4.

2.7. Lemma. *Let* $A, B \in MA(X)$, $A \neq B$. *Then the following conditions are equivalent*:

(i) $A \prec B$;

(ii) $a \prec b$ *for each* $a \in A \setminus B$ *and each* $b \in B \setminus A$.

Proof. The implication (ii) $\Rightarrow$ (i) is obvious. From 2.4, 2.5 and 2.6 we infer that (i) $\Rightarrow$ (ii) holds.

2.8. Corollary. *Let* $A, B \in MA(X)$, $A \neq B$. *Then* A *is covered by* B *if and only if* $(A \cap B, A \setminus B, B \setminus A)$ *belongs to the set* $\mathscr{N}(X)$.

3. SHORT SUBSETS OF X

Again, let X be a partially ordered set. In this section we shall deal with elements A, A' and B in $MA(X)$ such that $X \neq A'$, $A \prec B$ and $A' \prec B$. Let such elements A, A' and B be fixed.

Let X_1 be the set of all elements x_1 of X having the property that there exists $b \in B$ with $x_1 \leqq b$. Then we have

3.1. Lemma. $MA(X_1)$ *is a principal ideal of the lattice* $MA(X)$ *with the greatest element* B.

Next, since A and A' are subsets of X_1, we obtain

3.2. Lemma. *Assume that* $A \wedge A'$ *fails to be covered by* A *in* $MA(X)$. *Then the lattice* $MA(X_1)$ *is non-modular.*

Denote $B_1 = B \setminus A$, $B_2 = B \setminus B_1$, $B'_1 = B \setminus A'$, $B'_2 = B \setminus B'_1$. In view of 2.7, the relation $A \neq A'$ yields that $B_1 \neq B'_1$.

Put $A_2 = A \setminus B$ and $A'_2 = A' \setminus B$.

3.3. Lemma. *$A_2 \cap A'_2 = \emptyset$ and $A_2 \neq \emptyset \neq A'_2$.*

Proof. In view of $B_1 \neq B'_1$ we have either $B_1 \setminus B'_1 \neq \emptyset$ or $B'_1 \setminus B_1 \neq \emptyset$. In the first case there exists $b_1 \in B_1 \setminus B'_1$. Assume that $a \in A_2 \cap A'_2$. Since $a \in A_2$, it is incomparable with b_1. On the other hand, b_1 belongs to B'_2 and $a \in A'_2$; thus $a \prec b_1$, which is a contradiction. The case $B'_1 \setminus B_1 \neq \emptyset$ is analogous.

If we had $A_2 = \emptyset$, then $A \subseteq B$ and thus $A = B$, which is a contradiction. Therefore $A_2 \neq \emptyset$. Similarly we obtain $A'_2 \neq \emptyset$.

3.4. Lemma. *Let $a_2 \in A_2$ and $a'_2 \in A'_2$. Then a_2 and a'_2 are incomparable.*

Proof. In view of 3.3 we have $a_2 \neq a'_2$. By way of contradiction assume that, e.g., $a_2 < a'_2$. There exists $b \in B'_2$ with $a'_2 < b$. Then $a_2 < b$ and thus $b \in B_2$. Hence according to 2.7 we have $a_2 \prec b$, which is a contradiction.

Let us denote by Y the set of all elements y of X_1 such that the following conditions are satisfied:

(i) y is incomparable with all elements of the set $(B_1 \cap B'_1) \cup (A_2 \cup A'_2)$;

(ii) if $b \in B$ and $y \leqq b$, then $y \prec b$.

If $y \in Y$ and if A is as in (ii), then (i) yields that $b \in B_2 \cup B'_2$. From this we infer (by applying the same argument as in the proof of 3.4) that either $Y = \emptyset$ or $Y \in A(X_1)$. Hence $C \in A(X_1)$ according to (i), where $C = Y \cup (B_1 \cap B'_1) \cup (A_2 \cap A'_2)$.

3.5. Lemma. $C \in MA(X_1)$.

Proof. We have already observed that $C \in A(X_1)$. By way of contradiction, assume that C does not belong to $MA(X_1)$. Hence there exists $x_1 \in X_1 \setminus C$ such that x_1 is incomparable with each element of C. Since $x_1 \in X_1$, there is $b \in B$ with $x_1 \leqq b$.

Since x_1 is incomparable with all elements of $B_1 \cap B'_1$, the element b must belong to $B_2 \cup B'_2$. If $x_1 = b$, then x_1 is comparable with some element of A_2 or with some element of A'_2, which is a contradiction. Thus $x_1 < b$. Hence there exists $y \in X_2$ such that $x_1 \leqq y \prec b$. This implies that y satisfies both the conditions (i) and (ii). Therefore $y \in Y \subseteq C$ and so x_1 is incomparable with y, which is a contradiction.

3.6. Lemma. *$C = A \wedge A'$ in $MA(X_1)$.*

Proof. Denote $I(A) = \{x_1 \in X_1 : \{x_1\} \leqq A\}$ and let $I(B)$ be defined analogously. Let C_1 be the system of all maximal elements of the partially ordered set $I(A) \cap I(B)$. In [2] it has been proved that the relation

$$C_1 = A \wedge A'$$

is valid in $MA(X_1)$. Thus we have to verify that $C = C_1$. Since both C and C_1 are maximal chains in X_2 it suffices to show that $C \subseteq C_1$.

Let $y \in Y$. We have already observed above that there is $b \in B_2 \cup B'_2$ such that $y \prec b$. If y is incomparable with all elements of B_1, then it is incomparable with all elements of A, which is a contradiction (since it is clear that y cannot belong to A). Hence there is $b_1 \in B_1$ such that $y \prec b_1$. Analogously there is $b'_1 \in B'_1$ with $y \prec b'_1$. Thus $y \in I(A) \cap I(B)$. Next, if $t \in X_2$ such that $y \prec t$, then either $t \notin I(A)$ or $t \notin I(B)$. Therefore $y \in C_1$.

It is obvious that each element of the set $B_1 \cap B'_1$ is maximal in $I(A) \cap I(B)$.

Let $a_2 \in A_2$. There exists $b_2 \in B_2$ with $a_2 \leqq b_2$. Since $B_2 \subseteq B'_1 \subseteq A'$ we obtain that $a_2 \in I(A) \cap I(A')$. Let $t \in I(A) \cap I(A')$ and $t \geqq a_2$. Then $t \in I(A)$; but a_2 is a maximal element in $I(A)$ and hence $t = a_2$. Thus $a_2 \in C_1$ and so $A_2 \subseteq C_1$. Similarly, $A'_2 \subseteq C_1$, which completes the proof.

The following assertion which was shown to be valid in the above proof will be applied in the next section.

3.6.1. Lemma. *Let $y \in Y$. Then there are elements $b_1 \in B_1$ and $b'_1 \in B'_1$ such that $y \prec b_1$ and $y \prec b'_1$.*

3.7. Lemma. $C = A \wedge A'$ *in* $MA(X)$.

Proof. This is a consequence of 3.6 and 3.1.

Let X_2 be the set of all elements $x_1 \in X_1$ such that there is $c \in C$ with $x_1 \geqq c$. Then we have

3.8. Lemma. *$MA(X_2)$ is a principal filter of $MA(X_1)$ with the least element c.*

From 3.8 and 3.2 we infer

3.9. Lemma. *Assume that $A \wedge A'$ fails to be covered by A in $MA(X)$. Then the lattice $MA(X_2)$ is non-modular.*

Also, the construction of C yields

3.10. Lemma. *Let P be a chain in X_2. Then* card $P \leqq 2$.

3.11. Theorem. *Let X be a partially ordered set. Then the following conditions are equivalent:*

(i) *$MA(X)$ is a modular lattice.*

(ii) *For each short subsystem Z of X, the lattice $MA(Z)$ is modular.*

Proof. The implication (i) $\Rightarrow$ (ii) is obvious. Next, (ii) $\Rightarrow$ (i) is a consequence of 3.9, 3.10 and of the corresponding dual results.

4. FURTHER RESULTS ON A, A' AND B

Let A, A' and B be as in the previous section. Also, the other notation introduced above will be applied here.

Most of the results of the present section have an auxiliary character; they will be used in Section 5 below.

Let us consider the following condition:

(c) Both A and A' cover $A \wedge A'$ in the lattice $MA(X_1)$.

It is obvious that (c) is equivalent to the condition which we obtain from (c) if X_1 is replaced by X.

4.1. Lemma. *Let $Y = \emptyset$. Then the condition* (c) *holds.*

Proof. We have

$$C = Y \cup (B_1 \cap B'_1) \cup (A_2 \cup A'_2)$$

and $C = A \wedge A'$ (cf. 3.6).

The relation $Y = \emptyset$ yields that $C = (B_1 \cap B'_1) \cup (A_2 \cup A'_2)$. Hence by 2.7 we infer that $C \prec A$ and $C \prec A'$.

4.2. Lemma. $B_2 \neq \emptyset \neq B'_2$.

Proof. By way of contradiction, assume that $B_2 = \emptyset$. Hence $B_1 = B$ and thus $A = B$, which is impossible. Therefore $B_2 \neq \emptyset$. Similarly, $B'_2 \neq \emptyset$.

Put $X_3 = X_2 \setminus (B_1 \cap B'_1)$. Then X_3 is a convex subset of X_2 and $X_3 \neq \emptyset$. For each $D \in MA(X_2)$ let $p(D) = D \cap X_3$. Next, for each $D_1 \in MA(X_3)$ put $p'(D_1) = (B_1 \cap B'_1) \cup D_1$. The following result is easy to verify.

4.3. Lemma. *For each $D \in MA(X_2)$ and each $D_1 \in MA(X_3)$ we have $p(D) \in MA(X_3)$ and $p'(D_1) \in MA(X_2)$. Next, p is an isomorphism of $MA(X_2)$ onto $MA(X_3)$, and p' is an isomorphism of $MA(X_3)$ onto $MA(X_2)$ which is inverse to p.*

The above lemma shows that, when investigating the lattice-theoretic properties of $MA(X_2)$, it suffices to assume that the relation

$$B_1 \cap B'_1 = \emptyset$$

is valid. In the present section this relation will be always supposed to hold.

4.4. Lemma. $B_1 \neq \emptyset \neq B'_1$.

Proof. In view of the symetry it suffices to verify that $B_1 \neq \emptyset$. By way of contradiction, assume that $B_1 = \emptyset$. Then $B'_1 \neq \emptyset$. Next, $B_2 = B$ and thus $A_2 = A$.

According to 4.2 and 3.3 we have $B'_2 \neq \emptyset$ and $A'_2 \neq \emptyset$, thus there are $a'_2 \in A'_2$ and $b'_2 \in B'_2$ with $a'_2 \prec b'_2$. If $a \in A$, then $a \prec b'_2$, hence the elements a'_2 and a are either equal or incomparable. Lemma 3.3 yields that $a \neq a'_2$; therefore a'_2 is incomparable with each element of A. Hence A fails to be a maximal antichain in X_2, which is a contradiction.

Now, 4.2 and 4.4 yield

4.5. Corollary. card $B \geqq 2$.

4.6. Proposition. *Let* card $B = 2$. *Then the condition* (c) *holds.*

Proof. Let $B = \{b_1, b_2\}$. In view of 4.2 and 4.4 we can assume that $B_1 = \{b_1\}$ and $B_2 = \{b_2\}$. Similarly, both B'_1 and B'_2 are one-element sets. If $B'_1 = B_1$, then $A = A'$, which is a contradiction. Hence $B'_1 = \{b_2\}$ and $B'_2 = \{b_1\}$.

The set A_2 consists of all elements of X_2 which are covered by b_2 and are incomparable with b_1; the set A'_2 has analogous properties (with b_1 and b_2 interchanged).

Next, Y is the set of all elements of X_2 which are covered by both b_1 and b_2 (cf. 3.6.1). By 3.6,

$$A \wedge A' = Y \cup A_2 \cup A'_2 .$$

Now from 2.7 it follows that $C \prec A$ and $C \prec B$, which completes the proof.

By applying a dual argument we obtain the following result.

4.7. Lemma. *Let A_1, A'_1, B_1 be elements of $MA(X)$ such that $A_1 \neq A'_1$, $B_1 \prec A_1$ and $B_1 \prec A'_1$. Assume that* card $B = 2$. *Then both A_1 and A'_1 are covered by $A_1 \vee A'_1$ in $MA(X)$.*

Let C be as in Section 2; i.e., $C = A \wedge A'$. Since A and A' are incomparable, there exist A_1 and A'_1 in $MA(X_2)$ such that $C \prec A_1 \leqq A$ and $C \prec A'_1 \leqq A'$. Let such A_1 and A'_1 be fixed.

4.9. Lemma. card $C \geqq 2$.

Proof. This can be obtained from 4.5 by applying duality (if we consider the elements A_1, A'_1 and C instead of A, A' and B).

4.9. Proposition. *Let* card $C = 2$. *Then* (c) *holds.*

Proof. Clearly $Y \cap A_2 = Y \cap A'_2 = \emptyset$. Hence according to 3.3 we have also $A_2 \cap A'_2 = \emptyset$. Thus 4.2 and 3.3 yield that card A_2 = card $A'_2 = 1$. Therefore $Y = \emptyset$ and by 4.1, the condition (c) is valid.

5. NON-MODULARITY

Assume that A, A' and B are as above. We also suppose that the relation $B_1 \cap B'_1 = \emptyset$ is valid.

5.1. Lemma. *Assume that $y \prec b_1$ for each $y \in Y$ and each $b_1 \in B_1$. Then $C \prec A$.*

Proof. Let $C_1 \in MA(X)$, $C < C_1 \leqq A$. Let $a_2 \in A_2$. Hence $a_2 \in C$ and thus there exists $c_1 \in C_1$ with $a_2 \leqq c_1$. Next, there is $a \in A$ with $c_1 \leqq a$. Hence $a_2 \leqq a$, which implies that $a_2 = a$. Therefore $A_2 \subseteq C_1$.

There exists $c_2 \in C_1 \setminus C$. Thus we must have $c_2 \in B$. Next, c_2 must be incomparable with all elements of A_2 and hence $c_2 \in B_1$. This implies that $c_2 \succ y$ for each $y \in Y$; therefore $Y \cap C_1 = \emptyset$.

Assume that $C_1 < A$. Thus there exists $a \in A \setminus C_1$. Hence $a \in B_1$. There exists $c'_1 \in C_1$ with $c'_1 \prec a$. The element c'_1 cannot belong to $Y \cup A_2$, thus $c'_1 \in A'_2$. Then c'_1 is covered by each element of B'_2. In particular, c'_1 is covered by c_2, which is a contradiction. Therefore $C \prec A$.

For each $y \in Y$ let $B_1(y)$ be the set of all elements $b_1 \in B_1$ such that y is not covered by b_1. Let $B'_1(y)$ be defined analogously.

5.2. Lemma. *Assume that* (c) *does not hold. Then there exists $y \in Y$ such that either $B_1(y) \neq \emptyset$ or $B'_1(y) \neq \emptyset$.*

Proof. According to 4.1 we have $Y \neq \emptyset$. If $B_1(y) = B'_1(y) = \emptyset$ for each $y \in Y$, then from 5.1 we infer that (c) holds, which is a contradiction.

In 5.3 and 5.4 we suppose that the condition (c) does not hold. Hence in view of 5.2 we can assume without loss of generality that $B'_1(y_1) \neq \emptyset$ for some $y_1 \in Y$.

5.3. Lemma. *There exist distinct elements $a^1 \in A'_2$, $a^2 \in Y$, $a^3 \in A_2$, $b^1 \in B'_1$ and $b^3 \in B'_1$ such that the relations*

$$(*) \qquad a^1 \prec b^1 \succ a^2 \prec b^2 \succ a^3 \prec b^3$$

are valid.

Proof. As we already mentioned above we assume that there is $a^2 \in Y$ such that $B'_1(a^2) \neq \emptyset$; thus there is $b^3 \in B'_1(a^2)$. In view of 3.6.1 there are $b^1 \in B_1$ and $b^2 \in B'_1$ with $a^2 \prec b^1$ and $a^2 \prec b^2$. Thus $b^2 \neq b^3$. Next, the relation $B_1 \cap B'_1 = \emptyset$ yields that $b^2 \neq b^1 \neq b^3$.

From 3.3 we infer that $A_2 \neq \emptyset \neq A'_2$. Hence there are $a^1 \in A_2$ and $a^3 \in A'_2$. Then the elements a^1, a^2, a^3 are distinct. It is clear that $a^i \neq b^j$ for each $i, j \in \{1, 2, 3\}$.

Since $B_1 \cap B'_1 = \emptyset$, we have $B_1 \subseteq B'_2$ and thus $a^1 \prec b^1$. Similarly $B'_1 \subseteq B_2$ and hence $a^3 \prec b^2$, $a^3 \prec b^3$. Therefore the relations $(*)$ hold.

If u and v are incomparable elements of X, then we write $u \parallel v$.

5.4. Lemma. *Let a^i and b^i $(i = 1, 2, 3)$ be as in 5.3 and let S be the set consisting of these elements. Then S is a regular serpentine set in X.*

Proof. It is obvious that S is a convex subset of X. From $b^3 \in B'_1(a^2)$ we obtain that $a^2 \parallel b^3$. Next, from $a^1 \in A'_2$ and $b^2 \in B'_1$ it follows that $a^1 \parallel b^2$ holds. Hence S is a serpentine subset of X. Thus by 5.3 and 3.6.1, S is a regular serpentine subset of X.

5.5. Corollary. *Assume that the condition (c) does not hold. Then X possesses a regular serpentine subset.*

Let (c′) be the condition dual to (c). From 5.5 we obtain by duality:

5.6. Corollary. *Assume that the condition (c′) does not hold. Then X possesses a dually regular serpentine subset.*

5.7. Corollary. *Assume that the lattice $MA(X)$ is not modular. Then X possesses either a regular serpentine subset or a dually regular serpentine subset.*

5.8. Lemma. *Let S be a regular serpentine subset of X. Under the notation as in Section 1, let $B = B_1 \cup B_2$, $A = B_1 \cup A_2$ and $A' = B'_1 \cup A'_2$. Then the condition (c) fails to be valid in $MA(X)$.*

Proof. Let us apply the notation from the definition of the regular serpentine subset. We also the other notation concerning A, A' and B which was introduced above. According to 1.7, the relations $A \prec B$ and $A' \prec B$ hold. We have to verify that (c) fails to be valid in the lattice $MA(X_2)$. Similarly as in the above investigation it suffices to assume that $B_1 \cap B'_1 = \emptyset$.

Let Y_1 be the set of all $y \in Y$ such that y is incomparable with all elements belonging to $B'_1(a^2)$. Denote

$$C_1 = A'_2 \cup Y_1 \cup B'_1(a^2).$$

Then $Y_1 \neq \emptyset$ (since $a^2 \in Y_1$), and also $B'_1(a^2) \neq \emptyset$ (since $b^3 \in B'_1(a^2)$). Next, $C_1 \in A(X)$ and $C \leqq C_1 \leqq A'$. Finally, each element of c_1 belongs either to C or to A'.

Suppose that $C_1 \notin MA(X_2)$. Thus there exists $z \in X_2 \setminus C_1$ such that z is incomparable with all elements of C_1, and there are $z_1 \in C$, $z_2 \in A'$ with $z_1 \leqq z \leqq z_2$.

First suppose that $z_1 = z_2$. Then $z \in B'_1$. The case $z \in B'_1(a^2)$ is impossible, since $B'_1(a^2) \subseteq C_1$. Thus $z \in B'_1 \setminus B'_1(a^2)$ and hence $z \succ a^2 \in C_1$, which is a contradiction.

Hence $z_1 < z_2$. Thus $z_1 \prec z_2$, $z_2 \in B'_1$ and $z_1 \in Y \cup A_2$. Next, either $z = z_1$ or $z = z_2$. We have already observed that $z \in B'_1$, hence $z \neq z_2$. Thus $z = z_1$. If $z \in A_2$, then $z \prec b^3 \in B'_1 \subseteq A_2$, which is impossible, since $b^3 \in C_1$. Therefore $z \in Y \setminus Y_1$. But in this case z is covered by some element belonging to $B'_1(a^2) \subseteq C_1$, which is a contradiction. Thus $C_1 \in MA(X)$. Now, since $C \neq C_1 \neq A'$, we obtain that $C < C_1 < A'$. Hence the condition (c) fails to be valid.

The following result can be proved by a dual investigation.

5.9. Lemma. *Let S be a dually regular serpentine subset of X. Then (under the notation analogous to those in 5.8) the condition (c') fails to be valid in $MA(X)$.*

Summarizing 5.7, 5.8 and 5.9 we conclude:

5.10. Theorem. *Let X be a finite partially ordered set. Then the following conditions are equivalent:*

(i) *The lattice $MA(X)$ fails to be modular.*

(ii) *X possesses either a regular serpentine subset or a dually regular serpentine subset.*

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