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VARIETIES HAVING DISTRIBUTIVE LATTICES OF QUASIORDERS

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Let A be an algebra with a set of (fundamental) operations F. A binary relation R in A is called *compatible* if it has the *substitution property*, i.e. if for each n-ary operation $f \in F$ and any elements a_i, b_i of A $(i = 1, \ldots, n)$, the following implication holds:

$$\langle a_i, b_i \rangle \in R \ (i = 1, \ldots, n) \quad \text{implies} \quad \langle f(a_1, \ldots, a_n), f(b_1, \ldots, b_n) \rangle \in R .$$

By a *quasiorder* on an algebra A we mean a reflexive and transitive compatible binary relation on A. It is almost evident that the set of all quasiorders on A forms a complete lattice with respect to set inclusion; we denote it by $\mathcal{Q}(A)$. Hence for each two elements a, b of A there exists the least quasiorder on A containing the pair $\langle a, b \rangle$; we denote it by $Q(a, b)$. By [2], the congruence lattice Con A is a sublattice of $\mathcal{Q}(A)$ and, trivially, $Q(a, b) \subseteq \Theta(a, b)$ for each $a, b \in A$.

Analogously, the set $\mathcal{R}(A)$ of all reflexive and compatible (so called *diagonal*) binary relations on an algebra A forms a complete lattice with respect to set inclusion. On the other hand, neither Con A nor $\mathcal{Q}(A)$ is a sublattice of $\mathcal{R}(A)$ in the general case, see [2]. Hence for each a, b of A there exists the least reflexive and compatible binary relation containing the pair $\langle a, b \rangle$; we denote it by $R(a, b)$. Evidently, $R(a, b) \subseteq Q(a, b)$ for each a, b of A.

B. Jónsson [3] gave a Mal'cev type characterization of varieties of algebras whose congruence lattices are distributive. A certain polynomial characterization of varieties whose members have distributive lattices of reflexive and symmetrical compatible relations (so called *tolerances*) is contained in [1]. The aim of this paper is to characterize varieties of algebras whose lattices of quasiorders are distributive.

Lemma 1. *Let A be an algebra and $a, b, x, y \in A$. Then $\langle a, b \rangle \in Q(x, y)$ if and only if there exist an integer $k \geqq 0$ and elements $d_0, \ldots, d_k \in A$ such that $a = d_0$, $b = d_k$ and $\langle d_i, d_{i+1} \rangle \in R(x, y)$ for $i = 0, \ldots, k-1$.*

Proof. Evidently, $Q(x, y)$ is a transitive closure of $R(x, y)$, thus $\langle a, b \rangle \in Q(x, y)$ if and only if

$$\langle a, b \rangle \in \underbrace{R(x, y) \circ R(x, y) \circ \ldots \circ R}_{k-\text{times}}(x, y) . \quad \square$$

Lemma 2. *Let A be an algebra and $a, b, x, y, z, v \in A$. Then*

$$\langle a, b\rangle \in Q(x, y) \vee Q(z, v)$$

(in the lattice $Q(A)$) if and only if there exist an integer $n \geqq 0$ and elements $c_0, \ldots, c_n \in A$ such that $c_0 = a$, $c_n = b$ and

$$\langle c_i, c_{i+1}\rangle \in R(x, y) \quad \text{for } i \text{ even}, \quad \text{and}$$

$$\langle c_i, c_{i+1}\rangle \in R(z, v) \quad \text{for } i \text{ odd}.$$

Proof. By virtue of the reflexivity of $Q(x, y)$, $Q(z, v)$, $R(x, y)$, $R(z, v)$, the sequence $c_0, \ldots, c_n$ of elements of A can be assembled in a way that $\langle c_i, c_{i+1}\rangle \in R(x, y)$ for i even and $\langle c_i, c_{i+1}\rangle \in R(z, v)$ for i odd. The rest of the assertion follows directly from Lemma 1 and the fact that $Q(x, y) \vee Q(z, v)$ is the least quasiorder containing $Q(x, y) \cup Q(z, v)$. □

Lemma 3. *Let A be an algebra and $a, b, x, y \in A$. Then $\langle a, b\rangle \in R(x, y)$ if and only if there exist an algebraic function φ over A and elements $c_1, \ldots, c_n \in A$ such that*

$$a = \varphi(x, c_1, \ldots, c_n)\,, \quad b = \varphi(y, c_1, \ldots, c_n)\,.$$

The proof is straightforward (see e.g. [2]). □

Theorem. *For a variety $\mathscr{V}$, the following conditions are equivalent:*

(1) *$Q(A)$ is distributive for each $A \in \mathscr{V}$;*

(2) *there exist ternary terms $p_0, \ldots, p_n$ and 4-ary terms t_i, q_i, r_i $(i = 1, \ldots, n-1)$ such that $x = p_0(x, y, z)$, $z = p_n(x, y, z)$ and*

$$p_i(x, y, z) = t_i(x, x, y, z)\,, \quad p_{i+1}(x, y, z) = t_i(z, x, y. z) \quad \text{for}$$
$$i = 0, \ldots, n-1\,,$$
$$p_i(x, y, z) = q_i(x, x, y, z)\,, \quad p_{i+1}(x, y, z) = q_i(y, x, y, z) \quad \text{for } i \text{ even}\,,$$
$$p_i(x, y, z) = r_i(y, x, y, z)\,, \quad p_{i+1}(x, y, z) = r_i(z, x, y, z) \quad \text{for } i \text{ odd}\,.$$

Proof. (1) ⇒ (2): Let $A = F_V(x, y, z)$ be a free algebra of $\mathscr{V}$ with three free generators x, y, z. By Lemma 2, we have

$$\langle x, z\rangle \in Q(x, z) \wedge (Q(x, y) \vee Q(y, z))\,.$$

Distributivity of $\mathscr{Q}(A)$ implies

$$\langle x, z\rangle \in [Q(x, z) \wedge Q(x, y)] \vee [Q(x, z) \wedge Q(y, z)]\,,$$

thus, by Lemma 2, there exist elements $c_0, \ldots, c_n \in A$ such that $c_0 = x$, $c_n = z$ and

$$\langle c_i, c_{i+1}\rangle \in R(x, z) \wedge R(x, y) \quad \text{for } i \text{ even}\,,$$

$$\langle c_i, c_{i+1}\rangle \in R(x, z) \wedge R(y, z) \quad \text{for } i \text{ odd}\,,$$

thus

$$\langle c_i, c_{i+1}\rangle \in R(x, z) \quad \text{for} \quad i = 0, \ldots, n-1\,,$$

$$\langle c_i, c_{i+1}\rangle \in R(x, y) \quad \text{for } i \text{ even}\,,$$

$$\langle c_i, c_{i+1}\rangle \in R(y, z) \quad \text{for } i \text{ odd}\,.$$

Since $c_i \in F_V(x, y, z)$, there exist ternary terms $p_i(x, y, z)$ such that $c_i = p_i(x, y, z)$. By Lemma 3, there exist 4-ary terms t_i, q_i, r_i $(i = 0, \ldots, n-1)$ with

$$c_i = t_i(x, x, y, z)\,, \quad c_{i+1} = t_i(z, x, y, z) \quad \text{for} \quad i = 0, \ldots, n-1\,,$$
$$c_i = q_i(x, x, y, z)\,, \quad c_{i+1} = q_i(y, x, y, z) \quad \text{for } i \text{ even}\,,$$
$$c_i = r_i(y, x, y, z)\,, \quad c_{i+1} = r_i(z, x, y, z) \quad \text{for } i \text{ odd}\,.$$

(2) $\Rightarrow$ (1): Suppose $A \in \mathscr{V}$ and Q, R, T are reflexive and compatible binary relations on A. We prove

$$(*) \qquad Q \wedge (R \circ T) \subseteq (Q \wedge R) \circ (Q \wedge T) \circ \ldots \circ (Q \wedge R) \circ (Q \wedge T)$$

where the relational product on the right side of $(*)$ contains n factors for some integer $n \geqq 2$.

Suppose $\langle a, b\rangle \in Q \wedge (R \circ T)$. Then $\langle a, b\rangle \in Q$ and there exists an element $d \in A$ such that $\langle a, d\rangle \in R$ and $\langle d, b\rangle \in T$. Put $c_i = p_i(a, d, b)$ for the polynomials p_i in (2). Since Q, R, T are reflexive and compatible, we obtain by (2) also

$$\langle c_i, c_{i+1}\rangle = \langle t_i(a, a, d, b), t_i(b, a. d, b)\rangle \in Q \quad \text{for each} \quad i = 1, \ldots, n\,;$$

moreover $c_0 = a$, $c_n = b$, and

$$\langle c_i, c_{i+1}\rangle = \langle q_i(a, a, d, b),\ q_i(d, a, d, b)\rangle \in R \quad \text{for } i \text{ even} \quad \text{and}$$
$$\langle c_i, c_{i+1}\rangle = \langle r_i(d, a, d, b),\ r_i(b, a, d, b)\rangle \in T \quad \text{for } i \text{ odd}\,.$$

Hence

$$\langle a, b\rangle \in (Q \wedge R) \circ (Q \wedge T) \circ \ldots \circ (Q \wedge R) \circ (Q \wedge T)$$

which proves $(*)$.

Now, suppose Q, R, S are quasiorders on A and

$$\langle a, b\rangle \in Q \wedge (R \vee S)\,.$$

Then $\langle a, b\rangle \in Q$ and there exists an integer $m \geqq 2$ such that

$$\langle a, b\rangle \in \underbrace{R \circ S \circ R \circ S \circ \ldots \circ R \circ S}_{m \text{ times}}\,.$$

Put $T = \underbrace{S \circ R \circ S \circ \ldots \circ R \circ S}_{m-1 \text{ times}}$. Then T is reflexive and compatible and

$$\langle a, b\rangle \in Q \wedge (R \circ T)\,.$$

By $(*)$, we have

$$\langle a, b\rangle \in (Q \wedge R) \circ (Q \wedge T) \circ \ldots \circ (Q \wedge R) \circ (Q \wedge T)\,.$$

Now, put $T_1 = \underbrace{R \circ S \circ \ldots \circ R \circ S}_{m-2 \text{ times}}$. Then $T = S \circ T_1$ and, by $(*)$,

$$Q \wedge T = Q \wedge (S \circ T_1) \subseteq$$
$$\subseteq (Q \wedge S) \circ (Q \wedge T_1) \circ \ldots \circ (Q \wedge S) \circ (Q \wedge T_1)\,.$$

Applying this method (using (*)) successively, we obtain after a finite number of steps

$$\langle a, b\rangle \in (Q \wedge R) \circ (Q \wedge S) \circ \ldots \circ (Q \wedge R) \circ (Q \wedge S) .$$

Since Q, R, S are quasiorders, it means that

$$\langle a, b\rangle \in (Q \wedge R) \vee (Q \wedge S)$$

which proves (1).

Example 1. Let L be a *variety of lattices*. We can put $n = 2$,

$$p_0(x, y, z) = x ,$$
$$p_1(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (x \wedge z) ,$$
$$p_2(x, y, z) = z .$$

Moreover, put

$$q_0(w, x, y, z) = (x \wedge w) \vee (w \wedge z) \vee (x \wedge z) ,$$
$$r_1(w, x, y, z) = (x \wedge w) \vee (w \wedge z) \vee (x \wedge z) ,$$
$$t_0(w, x, y, z) = (x \wedge y) \vee (y \wedge w) \vee (x \wedge w) ,$$
$$t_1(w, x, y, z) = (w \wedge y) \vee (y \wedge z) \vee (w \wedge z) .$$

Then

$$p_0(x, y, z) = x = (x \wedge y) \vee (y \wedge x) \vee (x \wedge x) = t_0(x, x, y, z) ,$$
$$p_1(x, y, z) = t_0(z, x, y, z)$$

and

$$p_1(x, y, z) = t_1(x, x, y, z) ,$$
$$p_2(x, y, z) = z = (z \wedge y) \vee (y \wedge z) \vee (z \wedge z) = t_1(z, x, y, z) .$$

Moreover, for i even we have

$$p_0(x, y, z) = x = (x \wedge x) \vee (x \wedge z) = q_0(x, x, y, z) ,$$
$$p_1(x, y, z) = q_0(y, x, y, z)$$

and for i odd we obtain

$$p_1(x, y, z) = r_1(y, x, y, z) ,$$
$$p_2(x, y, z) = z = r_1(z, x, y, z) .$$

Hence every lattice variety has distributive lattices of quasiorders.

Example 2. As was mentioned above, Con A is a sublattice of $\mathcal{Q}(A)$ for any algebra A. Hence, distributivity of $\mathcal{Q}(A)$ implies also distributivity of Con A, i.e. the Mal'cev condition (2) of Theorem ought to imply the existence of the *Jónsson terms* for distributivity of congruences. However, this is easy, since (2) immediately gives

$$p_i(x, y, x) = t_i(x, x, y, x) = p_{i+1}(x, y, x) \quad \text{for each } i ,$$

$$p_i(x, x, y) = q_i(x, x, x, y) = p_{i+1}(x, x, y) \quad \text{for } i \text{ even},$$

$$p_i(x, y, y) = r_i(y, x, y, y) = p_{i+1}(x, y, y) \quad \text{for } i \text{ odd},$$

thus the terms $p_0(x, y, z), \ldots, p_n(x, y, z)$ in (2) of Theorem are the Jónsson terms, see [3].

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