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DECOMPOSITION OF SMOOTH FUNCTIONS OF TWO MULTIDIMENSIONAL VARIABLES

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1. INTRODUCTION

The aim of this paper is to find necessary and sufficient conditions in terms of partial differential equations for a smooth function $H: X \times Y \to \boldsymbol{K}$ to be in the form

$$H(x, y) = \sum_{i=1}^{m} f_i(x)\, g_i(y)\,, \tag{1.1}$$

where X and Y are n-dimensional and p-dimensional real or complex smooth manifolds, respectively, and $\boldsymbol{K}$ is the field of all real or complex numbers. The history of this decomposition began in the year 1904 when at the 3rd International Congress of Mathematicians S. Cyparissos announced a criterion for an analytic function of two real or complex variables to be written in the form (1.1) (see [3]). This was rediscovered and proved by F. Neuman in [5] and [6], where the case of arbitrary (even non-continuous) functions was also solved. The original statement of [3] was discussed by T. M. Rassias in [8]. In all these papers, the fundamental role was played by the determinants

$$\det\left(\frac{\partial^{i+j} H}{\partial x^i\, \partial y^j}\right)_{i,j=0,1,\ldots,q}. \tag{1.2}$$

Later on, in 1988, the same results and many others concerning this decomposition (e.g. extended separation of variables in PDE's) were obtained by H. Gauchman and L. A. Rubel in [4]. They also raised a similar question of when a function of three real or complex variables permits a representation of the form

$$H(x, y, z) = \sum_{i=1}^{m} f_i(x)\, h_i(y)\, k_i(z)\,. \tag{1.3}$$

This problem was investigated and completely solved in [7] and [2]. The decomposition (1.3) is substantially based on the possibility of separating one variable from the other two variables, i.e. on the decompositions of the form

$$H(x, y, z) = \sum_{i=1}^{m} f_i(x)\, g_i(y, z)\,, \quad x \in \boldsymbol{K}\,, \quad (y, z) \in \boldsymbol{K}^2\,.$$

So the problem we investigate in this paper is a natural generalization of the previous one to multidimensional variables. Although we believe the problem is interesting in itself, we recall one of its well-known motivations: consider an integral equation

$$u(x) = a(x) + \int_Y H(x, y)\, u(y)\, \mathrm{d}y \quad (x \in X)\,, \tag{1.4}$$

where $u: X \to \boldsymbol{K}$ is an unknown function. If a decomposition (1.1) of the kernel H is available, we can reduce (1.4) to an algebraic system of m linear equations, because any solution of (1.4) is then of the form

$$u(x) = a(x) + \sum_{i=1}^{m} c_i f_i(x)\,,$$

with some constants c_i $(1 \leqq i \leqq m)$.

In the present paper we will give a solution of the problem of decomposition (1.1) for arbitrary smooth manifolds X and Y. Moreover, we will also consider (1.1) as a functional equation for unknown functions $f_1, f_2, \ldots, f_m$ and $g_1, g_2, \ldots, g_m$. Under the conditions of Theorem 9.1 we can explicitly compute all m-tuples $f_1, f_2, \ldots, f_m$ and $g_1, g_2, \ldots, g_m$ that satisfy (1.1). This result is new even if $n = p = 1$. Replacing the global assumptions of Theorem 9.1 by local ones, Theorem 10.1 substantially weakens the sufficient conditions for the decomposition (1.1) if $\max\,(n, p) > 1$. Moreover, the assumptions of Theorem 10.1 can be verified more easily than those of Theorem 9.1.

Our approach to the problem is based on the following idea. If (1.1) holds, then for each fixed $y \in Y$, the function $x \mapsto H(x, y)$ is an element of the linear subspace of $C^\infty(X)$ generated by the functions $f_1, f_2, \ldots, f_m$. This is why we find a class of linear partial differential systems the solutions of which form a finite-dimensional linear subspace of $C^\infty(X)$ (see Theorem 7.1). It will be a generalization of ordinary differential equations

$$f^{(m)} + a_{m-1}(x) f^{(m-1)} + \ldots + a_0(x) f = 0$$

for functions of several variables. After finding such a differential system, we derive sufficient conditions (symmetrical in x and y) for the family of functions $x \mapsto H(x, y)$ $(y \in Y)$ to satisfy it, and in this way we get the decomposition (1.1).

Our results stated here together with Lemma 4.1 of [2] make it possible to characterize functions $H \in C^\infty(X_1 \times X_2 \times \ldots \times X_s)$ that admit a decomposition

$$H(x_1, \ldots, x_s) = \sum_{i_1=1}^{m_1} \ldots \sum_{i_s=1}^{m_s} c_{i_1 \ldots i_s} f^1_{i_1}(x_1) \ldots f^s_{i_s}(x_s)\,,$$

where $X_1, X_2, \ldots, X_s$ are smooth manifolds, $f^j_{i_j} \in C^\infty(X_j)$ and $c_{i_1 \ldots i_s} \in \boldsymbol{K}$.

2. NOTATION AND DEFINITIONS

For simplicity we shall suppose that functions and manifolds are of the class C^∞ (except Lemma 10.4) although in every statement lower diffetentiability is sufficient provided all assumptions have a good sense.

Manifolds will be denoted by X and Y (except Proposition 10.3 and Lemma 10.4). They can be real or complex (analytic) but always connected. The capital letter D (often with subscript) will stand for a linear differential operator from $C^\infty(X)$ to $C^\infty(X)$. The small d will be written for a linear differential operator on Y. The identity id: $C^\infty(X) \to C^\infty(X)$ is also considered to be a linear differential operator. The set of all linear differential operators on X will be denoted by $\mathscr{D}(X)$, while $\mathscr{D}^1(X)$ will stand for the set of all first order linear differential operators on X.

For a given k-tuple $D_1, D_2, \ldots, D_k$ of operators and for an m-tuple $f_1, f_2, \ldots, f_m$ of functions we define the matrix

$$W^{k,m}[D_1, D_2, \ldots, D_k; f_1, f_2, \ldots, f_m] = W^{k,m}[D_i; f_j] =$$
$$= (D_1, D_2, \ldots, D_k)^T (f_1, f_2, \ldots, f_m) = \begin{pmatrix} D_1 f_1 & D_1 f_2 & \ldots & D_1 f_m \\ D_2 f_1 & D_2 f_2 & \ldots & D_2 f_m \\ \ldots & \ldots & \ldots & \ldots \\ D_k f_1 & D_k f_2 & \ldots & D_k f_m \end{pmatrix}$$

where T denotes transposition. We shall write $W^m[D_i; f_j]$ if $m = k$. Analogously

$$W^{m,k}[g_1, g_2, \ldots, g_m;\ d_1, d_2, \ldots, d_k] = W^{m,k}[g_i; d_j] =$$
$$= (g_1, g_2, \ldots, g_m)^T (d_1, d_2, \ldots, d_k) = (W^{k,m}[d_j; g_i])^T .$$

For an m-tuple $D_1, D_2, \ldots, D_m$, a k-tuple $d_1, d_2, \ldots, d_k$ and a function $H \in C^\infty(X \times Y)$ we denote

$$W^{m,k}[D_i; d_j]\, H = W^{m,k}[D_i; d_j H] = W^{m,k}[D_i H; d_j] .$$

In a similar way we write

$$W^m[x_i, y_j]\, H = \begin{pmatrix} H(x_1, y_1) & H(x_1, y_2) & \ldots & H(x_1, y_m) \\ H(x_2, y_1) & H(x_2, y_2) & \ldots & H(x_2, y_m) \\ \ldots & \ldots & \ldots & \ldots \\ H(x_m, y_1) & H(x_m, y_2) & \ldots & H(x_m, y_m) \end{pmatrix}$$

for a function $H: X \times Y \to \boldsymbol{K}$, where $x_1, x_2, \ldots, x_m \in X$ and $y_1, y_2, \ldots, y_m \in Y$.

We shall write

$$f \equiv g \quad \text{on} \quad X \quad \text{if and only if} \quad f(x) = g(x) \quad \text{for each} \quad x \in X ,$$
$$f \not\equiv g \quad \text{on} \quad X \quad \text{if and only if} \quad f(x) \neq g(x) \quad \text{for some} \quad x \in X ,$$
$$f \neq g \quad \text{on} \quad X \quad \text{if and only if} \quad f(x) \neq g(x) \quad \text{for each} \quad x \in X ,$$

To abbreviate formulations we shall often use bold letters for m-tuples of functions $\boldsymbol{f} = (f_1, f_2, \ldots, f_m)^T$. Then (1.1) has the form

$$H = \boldsymbol{f}^T . \boldsymbol{g} \quad \text{on} \quad X \times Y .$$

The statement "components of $\boldsymbol{f}$ are linearly independent on X" means that functions $f_1, f_2, \ldots, f_m$ are linearly independent as elements of the linear space of all functions from X to $\boldsymbol{K}$.

If U is an open subset of X with coordinates $x_1, x_2, \ldots, x_n$, then for any two operators $D, D' \in \mathscr{D}(U)$ of the form

$$(2.1) \qquad D = \frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_n}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}} \quad \text{and} \quad D' = \frac{\partial^{\beta_1+\beta_2+\ldots+\beta_n}}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \ldots \partial x_n^{\beta_n}}$$

we will write $D \prec D'$ iff $\alpha_s \leqq \beta_s$ $(1 \leqq s \leqq n)$ but not $D = D'$, while $D < D'$ means that either $\alpha_1 + \alpha_2 + \ldots + \alpha_n < \beta_1 + \beta_2 + \ldots + \beta_n$, or $\alpha_1 + \alpha_2 + \ldots + \alpha_n = \beta_1 + \beta_2 + \ldots + \beta_n$, $\alpha_1 = \beta_1, \ldots, \alpha_{s-1} = \beta_{s-1}$ and $\alpha_s > \beta_s$ for some $s < n$. Notice that "$<$" is a linear ordering and

$$D < D' \quad \text{if and only if} \quad \frac{\partial}{\partial x_s} D < \frac{\partial}{\partial x_s} D'$$

holds for each $s = 1, 2, \ldots, n$.

An m-tuple of operators $D_1, D_2, \ldots, D_m \in \mathscr{D}(U)$ of the form (2.1) is said to be *complete* if $\mathrm{id} = D_1 < D_2 < \ldots < D_m$ and if for every $i = 2, 3, \ldots, m$, $D \prec D_i$ implies $D = D_k$ for some $k < i$. Obviously, if an operator D of the form (2.1) lies in a complete m-tuple, the numbers α_s satisfy $(\alpha_1 + 1)(\alpha_2 + 1)\ldots(\alpha_n + 1) \leqq m$. This inequality can be used when we need to go through the set of all complete m-tuples with a given m (see Remark 10.3).

3. LINEAR INDEPENDENCE IN $C^\infty(X)$

In this section we give necessary and sufficient conditions for linear independence of functions from $C^\infty(X)$ in terms of linear differential operators.

Lemma 3.1. *Let $D_1, D_2, \ldots, D_m \in \mathscr{D}(X)$ and let $f_1, f_2, \ldots, f_m \in C^\infty(X)$. If*

$$(3.1) \qquad \det W^m[D_i; f_j] \neq 0$$

on a dense subset of X, then $f_1, f_2, \ldots, f_m$ are linearly independent on every open subset of X.

Proof. Suppose that for an open subset $U \subset X$ we have

$$\sum_{j=1}^{m} c_j f_j = 0 \quad \text{on} \quad U \quad (c_j \in \boldsymbol{K}).$$

By differentiation we get

$$W^m[D_i; f_j]\,(c_1, c_2, \ldots, c_m)^T = 0 \quad \text{on} \quad U,$$

which gives $c_j = 0$ $(1 \leqq j \leqq m)$, because of (3.1).

In a certain sense the converse assertion is also true.

Theorem 3.2. *Let $f_1, f_2, \ldots, f_m \in C^\infty(X)$ be linearly independent on any open subset of X. Then the set $\tilde{X} = \{x \in X$, there is a neighbourhood U of x with co-*

ordinates and a complete m-tuple of differential operators $D_1, D_2, \ldots, D_m \in \mathscr{D}(U)$ *satisfying* (3.1) *on* $U\}$ *is open and dense in* X.

We postpone the proof of Theorem 3.2 to Section 6.

4. LEMMA ON LINEAR COMBINATIONS

To prove Theorem 3.2, we need to know a sufficient condition for a function f to be a linear combination of $f_1, f_2, \ldots, f_m$ stated in terms of linear differential operators. Here we establish such a condition. It will be also used in Section 7 on differential equations.

Lemma 4.1. *Let* $f_1, f_2, \ldots, f_m \in C^\infty(X)$ *and let operators* $D_1 = \mathrm{id}, D_2, \ldots, D_m$ *satisfy* (3.1) *on* X. *If* f *is such a function that for every* $D \in \mathscr{D}^1(X)$ *and every* $p = 1, 2, \ldots, m$

$$(4.1) \qquad \det W^{m+1}[D_i, DD_p; f_j, f] = 0 \quad on \quad X\,,$$

then f *is a linear combination of* $f_1, f_2, \ldots, f_m$ *on* X.

Proof. The relations (3.1) and (4.1) imply that the last column of the matrix from (4.1) is a linear combination of the previous ones:

$$(4.2) \qquad D_i f(x) = \sum_{j=1}^{m} b_j(x)\, D_i f_j(x) \quad (1 \leqq i \leqq m,\ x \in X)\,,$$

and

$$(4.3) \qquad DD_p f(x) = \sum_{j=1}^{m} b_j(x)\, DD_p f_j(x) \quad (x \in X)\,.$$

From (4.2) we get

$$b_j = \frac{\det W^m[D_i; f_1, \ldots, f_{j-1}, f, f_{j+1}, \ldots, f_m]}{\det W^m[D_i, f_s]} \quad \text{on} \quad X\,,$$

according to Cramer's rule. Consequently, $b_j \in C^\infty(X)$ are the same for all $D \in \mathscr{D}^1(X)$ and all p. Hence, carrying out differentiation D of (4.2) and substracting (4.3) with $p = i$, one obtains

$$0 = \sum_{j=1}^{m} D\, b_j(x)\, D_i f_j(x) \quad (1 \leqq i \leqq m,\ x \in X)\,,$$

which, along with (3.1), leads to $Db_j = 0$ on X, for any $D \in \mathscr{D}^1(X)$. This is why b_j's are constants. Then (4.2) with $D_1 = \mathrm{id}$ yields that f is a linear combination of $f_1, f_2, \ldots, f_m$ on X.

Remark 4.2. If X is such a manifold that every $D \in \mathscr{D}^1(X)$ can be written in the form $D = \sum_{s=1}^{n} c_s\, \partial/\partial x_s$, where $c_s \in C^\infty(X)$ and $\partial/\partial x_s \in \mathscr{D}^1(X)$, then in view of Lemma 4.1 it suffices to check (4.1) only for $D = \partial/\partial x_s$, $s = 1, 2, \ldots, n$. The same rule concerns also (7.1), (8.3), (9.2) and (10.2).

5. PROCEDURE OF CHOICE

Here we describe a procedure which enables us to find an appropriate maximal system of differential operators that are "linearly independent" as operators on a given set of functions.

Procedure 5.1. Let us consider a fixed m-tuple of functions $g_1, g_2, \ldots, g_m \in C^\infty(X)$. Let U be an open subset of X with coordinates $x_1, x_2, \ldots, x_n$ and let $g_\alpha \not\equiv 0$ on U for some $\alpha = 1, 2, \ldots, m$. We describe how to find an open subset $V \subset U$ and a complete q-tuple of operators $D_1, D_2, \ldots, D_q \in \mathscr{D}(U)$, $q \leqq m$, satisfying the following three conditions:

(i) rank $W^{q,m}[D_i; g_j] \equiv q$ on V,

(ii) rank $W^{q+1,m}[D_i, D; g_j] \equiv q$ on V (for each $D \in \mathscr{D}(U)$),

(iii) if $q > 1$ and $1 < k \leqq q$, then

$$\text{rank } W^{k,m}[D_1, \ldots, D_{k-1}, D; g_j] = k - 1 \text{ on } V, \text{ for any } D < D_k. \tag{5.1}$$

First, we find an $x_0 \in U$ such that $g_\alpha(x_0) \neq 0$. Then we have rank $W^{1,m}[\text{id}; g_j] = 1$ on V, where $V \subset U$ is an open neighbourhood of x_0. So we put $D_1 = \text{id}$. Suppose now that we have chosen a complete r-tuple $D_1, D_2, \ldots, D_r$ such that rank $W^{r,m}[D_i; g_j] = r$ on V, and that (5.1) holds for each $k = 2, 3, \ldots, r$. Suppose also that there exists an operator $D \in \mathscr{D}(U)$ satisfying

$$\text{rank } W^{r+1,m}[D_1, \ldots, D_r, D; g_j](x_0') = r + 1 \quad \text{for some} \quad x_0' \in V, \tag{5.2}$$

otherwise our procedure is finished with $q = r$. We put $D_{r+1} = D$, where D is the smallest (under the ordering $<$) operator of the form (2.1) that satisfies (5.2). Then rank $W^{r+1,m}[D_i; g_j] \equiv r + 1$ on V (V is restricted to a neighbourhood of x_0 if necessary). The choice of D_{r+1} ensures (5.1) with $k = r + 1$. So it remains to show that the $(r + 1)$-tuple $D_1, D_2, \ldots, D_{r+1}$ is complete. Notice first that $D_r < D_{r+1}$ follows from (5.1) with $k = r$. Since the r-tuple $D_1, D_2, \ldots, D_r$ is assumed to be complete, we need only to verify the following implication: if $D_{r+1} = \partial/\partial x_s \circ D'$, then $D' = D_j$ for some $j \leqq r$. Suppose on the contrary that $D_{r+1} = \partial/\partial x_s \circ D'$ and that $D' \neq D_j$ $(1 \leqq j \leqq r)$. The relation $D' < D_{r+1}$ implies that $D_{t-1} < D' < D_t$ for some $t \leqq r + 1$. From rank $W^{t-1,m}[D_i; g_j] = t - 1$ and (5.1) with $D = D'$ and $k = t$ we have

$$D'\boldsymbol{g} = \sum_{i=1}^{t-1} c_i D_i \boldsymbol{g} \text{ on } V \text{ (where } \boldsymbol{g} = (g_1, g_2, \ldots, g_m)^T),$$

with suitable functions $c_i \in C^\infty(V)$. Differentiating the last identity with respect to x_s we obtain

$$D_{r+1}\boldsymbol{g} = \frac{\partial}{\partial x_s} D'\boldsymbol{g} = \sum_{i=1}^{t-1} \frac{\partial c_i}{\partial x_s} D_i \boldsymbol{g} + \sum_{i=1}^{t-1} c_i \frac{\partial}{\partial x_s} D_i \boldsymbol{g} \quad \text{on} \quad V. \tag{5.3}$$

Since $D_1 < D_2 < \ldots < D_{t-1} < D'$, we also have

$$\frac{\partial}{\partial x_s} D_1 < \frac{\partial}{\partial x_s} D_2 < \ldots < \frac{\partial}{\partial x_s} D_{t-1} < \frac{\partial}{\partial x_s} D' = D_{r+1}\,.$$

Hence using (5.1) with $k = r + 1$, we conclude that there exist functions $c_{ij} \in C^\infty(V)$ satisfying

$$\frac{\partial}{\partial x_s} D_i g = \sum_{j=1}^{r} c_{ij} D_j g \quad \text{on} \quad V \quad (1 \leqq i \leqq k - 1)\,. \tag{5.4}$$

Finally, (5.3) and (5.4) imply that rank $W^{r+1,m}[D_i; g_j] < r + 1$ on V, which contradicts the choice of D_{r+1}.

The result of our considerations is the following. Starting with $D_1 = \text{id}$ and repeating the described choice, we find successively the desired operators $D_2, D_3, \ldots$ $\ldots, D_q$. (This process is finite because the condition rank $W^{q,m}[D_i; g_j] = q$ necessitates $q \leqq m$.)

Remark 5.2. It can be proved (see Section 6 for a special case $q = m$) that the number q of operators obtained by Procedure 5.1 is equal to the dimension of the linear space generated by the functions $g_1, g_2, \ldots, g_m \in C^\infty(U)$.

Example 5.3. Let us apply Procedure 5.1 to an m-tuple of functions $g_j(x, y) = (xy)^{j-1}$, $1 \leqq j \leqq n$, in $C^\infty(\boldsymbol{R} \times \boldsymbol{R})$. We explain why this application with $x_1 = x$, $x_2 = y$ and $U = \boldsymbol{R} \times \boldsymbol{R}^+$ yields the complete m-tuple id, $\partial/\partial x, \ldots, \partial^{m-1}/\partial x^{m-1}$. Denote $D_i = \partial^{i-1}/\partial x^{i-1}$ for $i = 1, 2, \ldots, m$. The functions g_j are defined so that

$$\det W^{k,k}[D_i; g_j]\,(x, y) = 1!\,2! \ldots (k-1)!\; y^{0+1+\ldots+(k-1)} \neq 0 \quad \text{on} \quad U\,,$$

for any $1 \leqq k \leqq m$. So it remains only check that rank $W^{k,m}[D_1, \ldots, D_{k-1}, D; g_j] < k$, where D is any partial derivative satisfying $D_{k-1} < D < D_k$. We verify that the last row of the mentioned matrix $W^{k,m}[D_1, \ldots, D_{k-1}, D; g_j]$ is a linear combination of the previous ones: there are functions $a_{D,i} \in C^\infty(U)$ such that the identities

$$D g_j = \sum_{i=1}^{k-1} a_{D,i} D_i g_j \quad (1 \leqq j \leqq m) \tag{5.5}$$

hold whenever $D_{k-1} < D < D_k$. The last fact follows from the equalities

$$\frac{\partial g_j}{\partial y}(x, y) = \frac{x}{y}\frac{\partial g_j}{\partial x}(x, y)\,, \quad 1 \leqq j \leqq m\,, \quad (x, y) \in U \tag{5.6}$$

by differentiating and using induction with respect to k. For example, the result of the application of $\partial/\partial x$ to the both sides of (5.6) is

$$\frac{\partial^2 g_j}{\partial x\,\partial y}(x, y) = \frac{1}{y}\frac{\partial g_j}{\partial x}(x, y) + \frac{x}{y}\frac{\partial^2 g_j}{\partial x^2}(x, y)\,,$$

which gives (5.5) with $D = \partial^2/\partial x\, \partial y$. The result of application of $\partial/\partial y$ to (5.6) is

$$\frac{\partial^2 g_j}{\partial y^2}(x, y) = -\frac{x}{y}\frac{\partial g_j}{\partial x}(x, y) + \frac{x}{y}\frac{\partial^2 g_j}{\partial y\, \partial x}(x, y) =$$

$$= -\frac{x}{y}\frac{\partial g_j}{\partial x}(x, y) + \frac{x}{y}\left(\frac{1}{y}\frac{\partial g_j}{\partial x}(x, y) + \frac{x}{y}\frac{\partial^2 g_j}{\partial x^2}(x, y)\right),$$

which gives (5.5) with $D = \partial^2/\partial y^2$. So (5.5) holds if $D_3 = \partial^2/\partial x^2 < D < D_4 = = \partial^3/\partial x^3$.

6. PROOF OF THEOREM 3.2

Let the function $f_1, f_2, \ldots, f_m$ be linearly independent on every open subset of X. The fact that the set $\tilde{X}$ is open being clear, we will show that X is dense. For any open subset $U \subset X$ we seek for an open subset $V \subset U$ and a complete m-tuple of operators $D_1, D_2, \ldots, D_m$ such that (3.1) holds on V. Let us apply Procedure 5.1 to the m-tuple $f_1, f_2, \ldots, f_m$. We find an open $V \subset U$, an integer $q \leqq m$ and a complete q-tuple of operators $D_1, D_2, \ldots, D_q$ such that rank $W^{q,m}[D_i; g_j] \equiv q \equiv \equiv$ rank $W^{q+1,m}[D_i, D; f_j]$ on V for each $D \in \mathscr{D}(V)$. We can change the ordering $f_1, f_2, \ldots, f_m$ so that $\det W^q[D_i; f_1, f_2, \ldots, f_q] \neq 0$ on some open $V' \subset V$. Since we have $\det W^{q+1}[D_i, D; f_1, f_2, \ldots, f_{q+1}] \equiv 0$ on V' for each $D \in \mathscr{D}(V')$, Lemma 4.1 implies that f_{q+1} is a linear combination of $f_1, f_2, \ldots, f_q$ on V', which is a contradiction.

7. DIFFERENTIAL EQUATIONS FOR FINITE-DIMENSIONAL SUBSPACES OF $C^\infty(X)$

Theorem 7.1. *Suppose that functions $f_1, f_2, \ldots, f_m \in C^\infty(X)$ and that operators $D_1 = \mathrm{id}$, $D_2, \ldots, D_m$ satisfy the condition $W^m[D_i; f_j] \neq 0$ on X. Then there is a unique system of equations*

$$DD_r f = \sum_{i=1}^{m} a_{Dri}(x)\, D_i f \tag{7.1}$$

in which D and r go through the sets $\mathscr{D}^1(X)$ and $\{1, 2, \ldots, m\}$, respectively, with the following property: A function $f \in C^\infty(X)$ is a solution of (7.1) if and only if f is a linear combination of $f_1, f_2, \ldots, f_m$.

Proof. For each $D \in \mathscr{D}^1(X)$ and each $r = 1, 2, \ldots, m$, the matrix $W^{m+1,m}[D_i, DD_r; f_j]$ has rank equal to m. Consequently, the last row of this matrix is a linear combination of the previous ones. So there are functions a_{Dri} such that

$$DD_r f_j \equiv \sum_{i=1}^{m} a_{Dri} \cdot D_i f_j \quad \text{on} \quad X \quad (1 \leqq j \leqq m)\,.$$

Cramer's rule implies that

$$(7.2) \qquad a_{Dri} \equiv \frac{\det W^m[D_1, \ldots, D_{i-1}, DD_r, D_{i+1}, \ldots, D_m; f_j]}{\det W^m[D_s; f_j]} \quad \text{on} \quad X\,.$$

Hence $a_{Dri} \in C^\infty(X)$ are determined uniquely. So there is a unique system of equations (7.1) with solutions $f_1, f_2, \ldots, f_m$. It is clear that every linear combination of $f_1, f_2, \ldots \ldots, f_m$ is a solution of this system. Conversely, if f is a solution of (7.1) with a_{Dri} as in (7.2), then $\det W^{m+1}[D_i, DD_r; f_j, f] \equiv 0$ on X for each $D \in \mathscr{D}^1(X)$. Now Lemma 4.1 implies that f is a linear combination of $f_1, f_2, \ldots, f_m$.

Remark 7.2. Some equations of the system (7.1) may be trivial: if the operator DD_r lies in $\{D_1, D_2, \ldots, D_m\}$, then (7.1) is reduced to

$$a_{Dri} = \begin{cases} 1 & \text{if} \quad DD_r = D_i\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Moreover, if the m-tuple $D_1, D_2, \ldots, D_m$ is chosen according to Procedure 5.1, then (5.1) and (7.2) lead to $a_{Dri} \equiv 0$ if $DD_r < D_i$. We hope to continue our discussion of the system (7.1) on another occasion.

Example 7.3. We find the system of differential equations for a 3-dimensional linear subspace of $C^\infty(\boldsymbol{R}^+ \times \boldsymbol{R}^+)$ generated by functions $f_1(x, y) = 1$, $f_2(x, y) = xy^2$ and $f_3(x, y) = x^2y$. A suitable (complete) 3-tuple of operators is $(D_1, D_2, D_3 = (\mathrm{id}, \partial/\partial x, \partial/\partial y)$, because

$$\det W^3[D_i; f_j] = \begin{vmatrix} 1 & xy^2 & x^2y \\ 0 & y^2 & 2xy \\ 0 & 2xy & x^2 \end{vmatrix} = -3x^2y^2 \neq 0 \quad \text{on} \quad \boldsymbol{R}^+ \times \boldsymbol{R}^+\,.$$

Applying $\partial/\partial x$ and $\partial/\partial y$ to each D_i, we obtain the family of operators $\{\partial/\partial x, \partial/\partial y, \partial^2/\partial x^2, \partial^2/\partial x\, \partial y, \partial^2/\partial y^2\}$, each of them will stand on the left-hand side of (7.1). However, only 3 of the 5 equations are nontrivial and we can easily find them:

$$(7.3) \qquad f_{xx} = \frac{4}{3x} f_x - \frac{2y}{3x^2} f_y\,,$$

$$f_{xy} = \frac{2}{3y} f_x + \frac{2}{3x} f_y\,,$$

$$f_{yy} = \frac{-2x}{3y^2} f_x + \frac{4}{3y} f_y\,.$$

For example, the first equation is computed from

$$\begin{vmatrix} 1 & xy^2 & x^2y & f \\ 0 & y^2 & 2xy & f_x \\ 0 & 2xy & x^2 & f_y \\ 0 & 0 & 2y & f_{xx} \end{vmatrix} = 0\,.$$

In the end let us emphasize once more that any solution of (7.3) is a linear combination of the starting functions f_1, f_2 and f_3.

8. NECESSARY CONDITIONS FOR DECOMPOSITION

The following theorem is a considerable extension of the previous results concerning the determinants (1.2) with one-dimensional x and y.

Theorem 8.1. *Suppose that a function $H \in C^\infty(X \times Y)$ can be written in the form (1.1) with $f_1, f_2, \ldots, f_m \in C^\infty(X)$ and $g_1, g_2, \ldots, g_m \in C^\infty(Y)$. If $U \subset X$ and $V \subset Y$ are any open subsets then we have*

$$(8.1) \qquad \det W^{m+1}[D_i; d_j]\, H \equiv 0 \quad on \quad U \times V$$

for any $D_1, D_2, \ldots, D_{m+1} \in \mathscr{D}(U)$ and for any $d_1, d_2, \ldots, d_{m+1} \in \mathscr{D}(V)$. Moreover, if $H \not\equiv 0$ on $U \times V$, then there are open subsets $\tilde{U} \subset U$ and $\tilde{V} \subset V$ and two complete $\tilde{m}$-tuples $\tilde{D}_1, \tilde{D}_2, \ldots, \tilde{D}_{\tilde{m}} \in \mathscr{D}(U)$ and $\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_{\tilde{m}} \in \mathscr{D}(V)$ satisfying

$$(8.2) \qquad \det W^{\tilde{m}}[\tilde{D}_i; \tilde{d}_j] \neq 0 \quad on \quad \tilde{U} \times \tilde{V}$$

and

$$(8.3) \qquad \det W^{\tilde{m}+1}[\tilde{D}_i, D\tilde{D}_p; \tilde{d}_j, d\tilde{d}_s]\, H \equiv 0 \quad on \quad \tilde{U} \times \tilde{V}$$

for any $p, s \in \{1, 2, \ldots, \tilde{m}\}$, any $D \in \mathscr{D}^1(\tilde{U})$ and any $d \in \mathscr{D}^1(\tilde{V})$. The number $\tilde{m} \leqq m$ may depend on the choice of $U \times V$ in $X \times Y$.

Proof. If H is of the form (1.1) on $X \times Y$, then $W^{m+1}[D_i; d_j]\, H$ is a product of the $(m+1) \times m$ matrix $W^{m+1,m}[D_i; f_j]$ and the $m \times (m \times 1)$ matrix $W^{m,m+1}[g_i; d_j]$. Consequently, rank $W^{m+1}[D_i; d_j]\, H \leqq m$, which yields (8.1).

Given $U \subset X$ and $V \subset Y$, we choose the smallest $\tilde{m} \leqq m$ such that

$$(8.4) \qquad \det W^{\tilde{m}+1}[D_i; d_j]\, H \equiv 0 \quad \text{on every} \quad \tilde{U} \times \tilde{V} \subset U \times V,$$

for any $D_1, D_2, \ldots, D_{\tilde{m}+1} \in D(\tilde{U})$ and any $d_1, d_2, \ldots, d_{\tilde{m}+1} \in \mathscr{D}(\tilde{V})$. Then there are $\tilde{m}$-tuples $\bar{D}_1, \bar{D}_2, \ldots, \bar{D}_{\tilde{m}} \in \mathscr{D}(\tilde{U})$ and $\bar{d}_1, \bar{d}_2, \ldots, \bar{d}_{\tilde{m}} \in \mathscr{D}(V)$ such that

$$(8.5) \qquad \det W^{\tilde{m}}[\bar{D}_i; \bar{d}_j H] \neq 0 \quad \text{on} \quad \tilde{U} \times \tilde{V},$$

where $\tilde{U} \subset U$ and $\tilde{V} \subset V$ are open subsets with coordinates. Now we use Procedure 5.1 for the $\tilde{m}$-tuple of functions $\bar{d}_1 H(x, y_0), \bar{d}_2 H(x, y_0), \ldots, \bar{d}_{\tilde{m}} H(x, y_0)$, where $y_0 \in V$ is fixed. We find a complete m'-tuple $\tilde{D}_1, \tilde{D}_2, \ldots, \tilde{D}_{m'} \in \mathscr{D}(\tilde{U})$, $m' \leqq \tilde{m}$, and an subset $U' \subset \tilde{U}$ such that

$$(8.6) \qquad \operatorname{rank} W^{m',\tilde{m}}[\tilde{D}_i; \bar{d}_j H](-, y_0) \equiv m' \quad \text{on} \quad U'$$

and

$$(8.7) \qquad \operatorname{rank} W^{m'+1,\tilde{m}}[\tilde{D}_i, D; \bar{d}_j H](-, y_0) \equiv m' \quad \text{on} \quad U'$$

for any $D \in \mathscr{D}(\tilde{U})$. Since (8.6) and (8.7) with $D = \bar{D}_1, \bar{D}_2, \ldots, \bar{D}_{\tilde{m}}$ yield

$$\operatorname{rank} W^{\tilde{m}}[\bar{D}_i; \bar{d}_j]\, H(x, y_0) \leqq m' \quad \text{for} \quad x \in U',$$

from (8.5) we conclude that $m' \geqq \tilde{m}$ and therefore $\tilde{m} = m'$. In view of (8.6), we may suppose that

$$\det W^{\tilde{m}}[\tilde{D}_i; \tilde{d}_j]\, H \neq 0 \quad \text{on} \quad \tilde{U} \times \tilde{V},$$

where the open subsets $\tilde{U} \subset U$ and $\tilde{V} \subset V$ are restricted if necessary. Using now Procedure 5.1 for the $\tilde{m}$-tuple of functions $\tilde{D}_1 H(x_0, y), \tilde{D}_2 H(x_0, y), \ldots, \tilde{D}_{\tilde{m}} H(x_0, y)$, where $x_0 \in \tilde{U}$ is fixed, and repeating the arguments as above, we find a complete $\tilde{m}$-tuple $\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_{\tilde{m}}$ such that (8.2) holds with suitable open subsets $\tilde{U} \subset U$ and $\tilde{V} \subset V$. Since (8.3) follows immediately from (8.4), the proof is complete.

9. SUFFICIENT CONDITIONS FOR DECOMPOSITION

Now we are in a position to prove our main result on the decomposition (1.1).

Theorem 9.1. *Let X and Y be smooth connected manifolds and let $H \in C^\infty(X \times Y)$. Suppose that there are two m-tuples of operators $D_1 = \mathrm{id}, D_2, \ldots, D_m \in \mathscr{D}(X)$ and $d_1 = \mathrm{id}, d_2, \ldots, d_m \in \mathscr{D}(Y)$ such that*

$$(9.1) \qquad \det W^m[D_i; d_j]\, H \neq 0 \quad \text{on} \quad X \times Y$$

and

$$(9.2) \qquad \det W^{m+1}[D_i, DD_r; d_j, dd_s]\, H \equiv 0 \quad \text{on} \quad X \times Y$$

for any $r, s \in \{1, 2, \ldots, m\}$, for any $D \in \mathscr{D}^1(X)$ and any $d \in \mathscr{D}^1(Y)$.

Then there exist functions $f_1, f_2, \ldots, f_m \in C^\infty(X)$ and $g_1, g_2, \ldots, g_m \in C^\infty(Y)$ such that H is of the form (1.1) on $X \times Y$.

Moreover, having fixed $x_0 \in X$ and $y_0 \in Y$, all such m-tuples $\boldsymbol{f} = (f_1, f_2, \ldots, f_m)$ and $\boldsymbol{g} = (g_1, g_2, \ldots, g_m)$ satisfying (1.1) have the form

$$(9.3) \qquad \begin{aligned} \boldsymbol{f}(x) &= C^T[d_1 H(x, y_0), d_2 H(x, y_0), \ldots, d_m H(x, y_0)]^T \quad (x \in X), \\ \boldsymbol{g}(y) &= C^{-1} W_0^{-1}[D_1 H(x_0, y), D_2 H(x_0, y), \ldots, D_m H(x_0, y)]^T \quad (y \in Y), \end{aligned}$$

where W_0 stands for $W^m[D_i, d_j H]\,(x_0, y_0)$ and C is any regular $m \times m$ constant matrix.

Proof. Let $D \in \mathscr{D}^1(X)$ and $d \in \mathscr{D}^1(Y)$. Due to (9.1) and (9.2), the last row of the matrix $W^{m+1}[D_i, DD_r; d_j, dd_s]\, H$ is a linear combination of the previous ones. It means that there are functions $a_{Dri} \in C^\infty(X \times Y)$ such that

$$(9.4) \qquad DD_r d_j H \equiv \sum_{i=1}^{m} a_{Dri} \,.\, D_i d_j H$$

for every $j = 1, 2, \ldots, m$, and

$$(9.5) \qquad DD_r dd_s H \equiv \sum_{i=1}^{m} a_{Dri} \,.\, D_i dd_s H \,.$$

From (9.4) we can compute a_{Dri} using Cramer's rule:

$$a_{Dri} \equiv \frac{\det W^m[D_1, \dots, D_{i-1}, DD_r, D_{i+1}, \dots, D_m; d_j]\, H}{\det W^m[D_q; d_j]\, H}.$$

Hence a_{Dri} is independent of $d \in \mathscr{D}^1(Y)$ and $s = 1, 2, \dots, m$. If we now carry out differentiation d in the equation (9.4) and substract the equation (9.5) with $s = j$, we get

$$0 \equiv \sum_{i=1}^{m} da_{Dri} \, . \, D_i d_j H$$

for every $j = 1, 2, \dots, m$. That is why $da_{Dri}(x, y) = 0$ for every $(x, y) \in X \times Y$ and every $d \in \mathscr{D}^1(Y)$, which means that a_{Dri} is independent of $y \in Y$. Consequently, for every $y \in Y$, the function $H(-, y)$ is a solution of the family of equations

$$DD_r f = \sum_{i=1}^{m} a_{Dri}(x) \, . \, D_i f, \tag{9.6}$$

where r goes through the set $\{1, 2, \dots, m\}$ and D goes through the set $\mathscr{D}^1(X)$. Let $y_0 \in Y$ be fixed. From (9.4) we see that $d_1 H(-, y_0), d_2 H(-, y_0), \dots, d_m H(-, y_0)$ and $H(-, y) = d_1 H(-, y)$ with arbitrary fixed $y \in Y$ are solutions of the system (9.6). According to Theorem 7.1, every solution of (9.6) is a linear combination of $d_1 H(-, y_0), d_2 H(-, y_0), \dots, d_m H(-, y_0)$. So we have

$$H(x, y) = \sum_{j=1}^{m} d_j H(x, y_0)\, g_j(y) \tag{9.7}$$

for every $(x, y) \in X \times Y$. Consequently, putting $f_j(x) = d_j H(x, y_0)$ we get the decomposition (1.1) for H. To compute $g_j(y)$, differentiate (9.7) and put $x = x_0$:

$$D_i H(x_0, y) = \sum_{j=1}^{m} D_i d_j H(x_0, y_0)\, g_j(y)$$

for $j = 1, 2, \dots, m$. This can be rewritten in the matrix form as

$$(D_1 H(x_0, y), D_2 H(x_0, y), \dots, D_m H(x_0, y))^T = W_0 \boldsymbol{g}(y), \tag{9.8}$$

where $W_0 = W^m[D_i; d_j H](x_0, y_0)$ and $\boldsymbol{g} = (g_1, g_2, \dots, g_m)^T$. Multiplying (9.8) by W_0^{-1}, we get (9.3) with C equal to the unit matrix. It remains to show that if

$$H(x, y) = \boldsymbol{f}^T(x) \, . \, \boldsymbol{g}(y) = \boldsymbol{f}^{-T}(x) \, . \, \boldsymbol{g}^{-}(y), \tag{9.9}$$

where $\boldsymbol{f}, \boldsymbol{f}^{-} \in C^\infty(X, \boldsymbol{K}^m)$ and $\boldsymbol{g}, \boldsymbol{g}^{-} \in C^\infty(Y, \boldsymbol{K}^m)$, then

$$\boldsymbol{f}^{-} = C^T \boldsymbol{f} \text{ on } X \text{ and } \boldsymbol{g}^{-} = C^{-1} \boldsymbol{g} \text{ on } Y, \tag{9.10}$$

where C is a constant regular matrix. (9.9) leads to the equality

$$W^m[D_i; d_j]\, H = W^m[D_i; f_s]\, W^m[g_s; d_j] = W^m[D_i; \bar{f}_s]\, W^m[\bar{g}_s; d_j],$$

where $f_s, \bar{f}_s, g_s, \bar{g}_s$ are components of $\boldsymbol{f}, \boldsymbol{f}^{-}, \boldsymbol{g}, \boldsymbol{g}^{-}$, respectively. Since the matrix

$W^m[D_i; d_j]\, H$ is regular (see (9.1)), the matrix

$$C = W^m[g_s; d_j]\,(W^m[\bar{g}_s; d_j])^{-1} = (W^m[D_i; f_s])^{-1}\, W^m[D_i; \bar{f}_s]$$

is also regular and depends neither on $x \in X$ nor on $y \in Y$. Using the first row and column of $W^m[D_i; \bar{f}_s] = W^m[D_i; f_s]\, C$ and $W^m[\bar{g}_s; d_j] = C^{-1} W^m[g_s; d_j]$, respectively, we obtain (9.10) because D_1 and d_1 are identical operators.

10. LOCAL SUFFICIENT CONDITIONS FOR GLOBAL DECOMPOSITION

Here we weaken sufficient conditions for global decomposition found above to local ones.

Theorem 10.1. *Let X and Y be smooth connected manifolds, let $H \in C^\infty(X \times Y)$ and let $m \geqq 1$ be an integer. Suppose that for every couple $(x, y) \in X \times Y$ there is a neighbourhood $\tilde{U} \times \tilde{V}$ of (x, y) and two m-tuples $D_1 = \mathrm{id}, D_2, \ldots, D_m \in \mathscr{D}(\tilde{U})$, $d_1 = \mathrm{id}, d_2, \ldots, d_m \in \mathscr{D}(\tilde{V})$ such that*

$$(10.1) \qquad \det W^m[D_i; d_j]\, H \neq 0 \quad \text{on} \quad \tilde{U} \times \tilde{V}$$

and

$$(10.2) \qquad \det W^{m+1}[D_i, DD_r; d_j, dd_s]\, H \equiv 0 \quad \text{on} \quad \tilde{U} \times \tilde{V}$$

for any $r, s \in \{1, 2, \ldots, m\}$ and for any $D \in \mathscr{D}^1(\tilde{U})$, $d \in \mathscr{D}^1(\tilde{V})$.

Then there exist functions $f_1, f_2, \ldots, f_m \in C^\infty(X)$, $g_1, g_2, \ldots, g_m \in C^\infty(Y)$ such that H is of the form (1.1) on $X \times Y$.

Remark 10.2. Under the assumptions of Theorem 10.1, Theorem 9.1 ensures only a local decomposition (1.1) of H in the neighbourhood $\tilde{U} \times \tilde{V}$ of each point $(x, y) \in X \times Y$. The goal of Theorem 10.1 is to show that such local decompositions can be "glued together". We emphasize that such a conclusion is not correct unless the domain of definition of H is a Cartesian product (see the example of a "stapler" in [4, pp. 43–44]).

Remark 10.3. If we look for m-tuples of operators satisfying the assumptions of Theorem 10.1, we can restrict ourselves to the class of complete m-tuples. Indeed, if (10.1) and (10.2) hold for some pair of m-tuples $D_1, D_2, \ldots, D_m$ and $d_1, d_2, \ldots, d_m$ that may depend on the couple (x, y), then there are complete m-tuples satisfying (10.1) and (10.2) on some neighbourhood of (x, y). This fact can be proved in the same way as the second part of Theorem 8.1.

To prove Theorem 10.1 we need the following general result on the decomposition (1.1) without any regularity properties of the function H.

Proposition 10.4. (Neuman [5] and [6]). *Let X and Y be two nonempty sets. If a function $H: X \times Y \to \boldsymbol{K}$ can be written in the form (1.1) on $X \times Y$, then*

$$(10.3) \qquad \det W^{m+1}[x_i; y_j]\, H = 0$$

for every $(m+1)$*-tuple* $x_1, x_2, \ldots, x_{m+1} \in X$ *and every* $(m+1)$*-tuple* $y_1, y_2, \ldots$ $\ldots, y_{m+1} \in Y$.

Conversely, if there are two m-tuples $x_1, x_2, \ldots, x_m \in X$ *and* $y_1, y_2, \ldots, y_m \in Y$ *such that*

$$(10.4) \qquad \det W^m[x_i; y_j]\, H \neq 0$$

and (10.3) *holds for every* $x_{m+1} \in X$, $y_{m+1} \in Y$, *then H is of the form* $H(x, y) = f^T(x) \,.\, g(y)$, *where the components of* $f: X \to K^m$ *and* $g: Y \to K^m$ *are linearly independent on their domains of definition. If H admits another decomposition* $H(x, y) = f^{\sim T}(x) \,.\, g^{\sim}(y)$ *with some mappings* $f^{\sim}: X \to K^m$ *and* $g^{\sim}: Y \to K^m$, *then there is a unique regular matrix C such that* $f = C^T f^{\sim}$ *on X and* $g = C^{-1} g^{\sim}$ *on Y.*

Moreover, if X and Y are topological spaces and H is continuous on $X \times Y$, *then* $f \in C(X, K^m)$ *and* $g \in C(Y, K^m)$. *If X and Y are smooth manifolds and* $H \in C^\infty(X \times Y)$, *then* $f \in C^\infty(X, K^m)$ *and* $g \in C^\infty(Y, K^m)$.

The following lemma forms the main part of the proof of Theorem 10.1 and may be of interest in itself.

Lemma 10.5. *Let X and Y be arcwise connected topological spaces. Let* $H \in C(X \times Y)$ *and let* $\gamma_1 \in C([0, 1], X)$, $\gamma_2 \in C([0, 1], Y)$ *be continuous curves. Suppose that for every couple* $(x, y) \in X \times Y$ *there is a neighbourhood* $\tilde{U} \times \tilde{V}$ *of* (x, y) *and mappings* $f^{\sim} \in C(\tilde{U}, K^m)$ *and* $g^{\sim} \in C(\tilde{V}, K^m)$ *with linearly independent components on any open subset of their domain of definition such that*

$$(10.5) \qquad H = f^{\sim T} \,.\, g^{\sim} \quad \text{on} \quad \tilde{U} \times \tilde{V}.$$

Then there is a neighbourhood U of $\gamma_1([0, 1])$ *in X and a neighbourhood V of* $\gamma_2([0, 1])$ *in Y and mappings* $f \in C(U, K^m)$, $g \in C(V, K^m)$ *such that H is of the form* (1.1) *on* $U \times V$.

The proof of Lemma 10.5 is rather technical and we postpone it to Section 11. Due to this lemma and Proposition 10.4 we can proceed to

Proof of Theorem 10.1. In view of Theorem 9.1 the assumptions of Theorem 10.1 ensure that the assumptions of Lemma 10.5 are also satisfied. Fix $(x_0, y_0) \in X \times Y$. There is a neighbourhood $\tilde{U} \times \tilde{V}$ of (x_0, y_0) such that H is of the form (1.1) on $\tilde{U} \times \tilde{V}$ with $\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_m \in C(\tilde{U})$ and $\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_m \in C(\tilde{V})$ linearly independent on their domains of definition. According to [1, Sec. 4.2.5] there are $x_1, x_2, \ldots, x_m \in \tilde{U}$ and $y_1, y_2, \ldots, y_m \in \tilde{V}$ such that (10.4) holds. Given arbitrary $(x, y) \in X \times Y$, we find curves $\gamma_1 \in C([0, 1]), X)$, $\gamma_2 \in C([0, 1]), Y)$ such that $x_1, x_2, \ldots, x_m, x \in \gamma_1([0, 1])$ and $y_1, y_2, \ldots, y_m, y \in \gamma_2([0, 1])$. Due to Lemma 10.5 there are neighbourhoods U of $\gamma_1([0, 1])$ in X and V of $\gamma_2([0, 1])$ in Y such that H is of the form (1.1) on $U \times V$. In view of the first part of Proposition 10.4

$$\det W^{m+1}[x_i, x; y_j, y]\, H = 0\,.$$

Using the second part we conclude that H is of the form (1.1) on $X \times Y$. Because of

$H \in C^\infty(X \times Y)$ we obtain $\boldsymbol{f} \in C^\infty(X, \boldsymbol{K}^m)$, $\boldsymbol{g} \in C^\infty(Y, \boldsymbol{K}^m)$. The linear independence of their components on every open subset follows from (10.1) and Lemma 3.1.

11. PROOF OF LEMMA 10.5

For the reader's convenience we first outline the main idea of the proof. To avoid unnecessary repetition, we agree that the letters U, V will always stand for nonempty open sets, while the letters $\boldsymbol{f}$, $\boldsymbol{g}$ will denote continuous mappings into $\boldsymbol{K}^m$ whose m components are linearly independent on any open subset of their domain of definition.

Suppose that the hypotheses of Lemma 10.5 are fulfilled. The first step of our proof is to show:

(i) There are sets U_0, V_0 and mappings $\boldsymbol{f}$, $\boldsymbol{g}_0$ such that $\gamma_1([0, 1]) \subset U_0 \subset X$, $\gamma_2(0) \in V_0 \subset Y$ and $H \equiv \boldsymbol{f}^T \cdot \boldsymbol{g}_0$ on $U_0 \times V_0$.

Having fixed U_0 and $\boldsymbol{f} \in C(U_0, \boldsymbol{K}^m)$ from (i), we will consider the set

$$P = \{t \in [0, 1]\text{: there are sets } U_t, V_t \text{ and a mapping } \boldsymbol{g}_t \text{ such that } \gamma_1([0, 1]) \subset U_t \subset U_0,\ \gamma_2([0, t]) \subset V_t \subset Y \text{ and } H = \boldsymbol{f}^T \cdot \boldsymbol{g}_t \text{ on } U_t \times V_t\}.$$

Then (i) implies that $0 \in P$. Obviously, P is open in $[0, 1]$ and $t \in P$ implies that $[0, t] \subset P$. Since the conclusion of Lemma 10.5 can be stated as $1 \in P$, we need only to verify that $t \in P$ whenever $[0, t) \subset P$. This can be done as follows. Suppose that $[0, t) \subset P$, where t is fixed, and consider the set

$$R = \{\tau \in [0, 1]\text{: there are sets } \hat{U}_\tau, \hat{V}_\tau \text{ and a mapping } \boldsymbol{g}_\tau^\wedge \text{ such that } \gamma_1([0, \tau]) \subset \hat{U}_\tau \subset U_0,\ \gamma_2([0, t]) \subset \hat{V}_\tau \subset Y \text{ and } H = \boldsymbol{f}^T \cdot \boldsymbol{g}_\tau^\wedge \text{ on } \hat{U}_\tau \times \hat{V}_\tau\}.$$

We need only to show that $1 \in R$ because then we can put $U_t = \hat{U}_1$, $V_t = \hat{V}_1$, $\boldsymbol{g}_t = \boldsymbol{g}_1^\wedge$ and conclude that $t \in P$. Since R is obviously open in $[0, 1]$ and $[0, \tau] \subset R$ whenever $\tau \in R$, it remains to prove the following two assertions:

(ii) $0 \in R$,

(iii) if $[0, \tau) \subset R$ for some $\tau \leqq 1$, then $\tau \in R$.

Now we give the proofs of (i)–(iii).

ad (i). In view of (10.5) we have $H = \boldsymbol{f}^{\sim T} \boldsymbol{g}^\sim$ on $\tilde{U}_0 \times \tilde{V}_0$, a neighbourhood of the point $(\gamma_1(0), \gamma_2(0))$. Put $\boldsymbol{g}_0 = \boldsymbol{g}^\sim$, $V_0' = \tilde{V}_0$ and consider the set $Q = \{\tau \in [0, 1]$: there are sets U_τ', V_τ' and a mapping $\boldsymbol{f}_\tau$ such that $\gamma_1([0, \tau]) \subset U_\tau' \subset X$, $\gamma_2(0) \subset V_\tau' \subset$ $\subset V_0'$ and $H \equiv \boldsymbol{f}_\tau^T \cdot \boldsymbol{g}_0$ on $U_\tau' \times V_\tau'\}$. Obviously $0 \in Q$, Q is open in $[0, 1]$ and $[0, \tau] \subset Q$ if $\tau \in Q$. Our goal is to show that $1 \in Q$. So we need only to prove that $\tau \in Q$ if $[0, \tau) \subset Q$. Suppose that $[0, \tau) \subset Q$, with a fixed $\tau \leqq 1$. From the assumptions of Lemma 10.5 we have $H \equiv \boldsymbol{f}^{\sim T} \boldsymbol{g}^\sim$ on $\tilde{U} \times \tilde{V}$, a neighbourhood of $(\gamma_1(\tau), \gamma_2(0))$. Since $\gamma_1(\sigma) \to \gamma_1(\tau)$ as $\sigma \to \tau$, we have $\gamma_1([\sigma, \tau]) \subset \tilde{U}$ for some $\sigma < \tau$. Now $\sigma \in Q$ implies that $H \equiv \boldsymbol{f}_\sigma^T \cdot \boldsymbol{g}_0$ on $U_\sigma' \times V_\sigma'$, where U_σ' and V_σ' are as in the definition of Q. Then $H \equiv f^{\sim T} . g^{\sim} \equiv f_\sigma^T . g_0$ on $(\tilde{U} \cap U'_\sigma) \times (\tilde{V} \cap V'_\sigma)$, which is a nonempty open set containing $(\gamma_1(\sigma), \gamma_2(0))$. Proposition 10.4 yields $f_\sigma \equiv C^T f^{\sim}$ on $\tilde{U} \cap U_\sigma$ and $g_0 \equiv C^{-1} g^{\sim}$ on $\tilde{V} \cap V'_\sigma$ with a constant regular matrix C. Consequently, we can put $U'_\tau = \tilde{U} \cup U'_\sigma$, $V'_\tau = \tilde{V} \cap V'_\sigma$ and define $f_\tau \in C(U'_\tau)$ by

$$f_\tau(x) = \begin{cases} f_\sigma(x) & \text{if } x \in U'_\sigma, \\ C^T f^{\sim}(x) & \text{if } x \in \tilde{U}. \end{cases}$$

Then $\gamma_1([0, \tau]) \subset U'_\tau \subset X$, $\gamma_2(0) \in V'_\tau \subset V'_0$ and $U'_\tau \times V'_\tau \subset (U'_\sigma \times V'_\sigma) \cup \cup (\tilde{U} \times (\tilde{V} \cap V'_\sigma))$. Since $H \equiv f_\sigma^T . g_0 \equiv f_\tau^T . g_0$ on $U'_\sigma \times V'_\sigma$ and $f_\tau^T . g_0 \equiv (C^T f^{\sim})^T . . (C^{-1} g^{\sim}) \equiv f^{\sim T} . g^{\sim} \equiv H$ on $\tilde{U} \times (\tilde{V} \cap V'_\sigma)$, we conclude that $H \equiv f_\tau^T . g_0$ on $U'_\tau \times V'_\tau$. So $\tau \in Q$.

ad (ii). Starting with $H \equiv f^T . g_0$ on $U_0 \times V_0$ and repeating the arguments given above in the proof of (i) with the curve γ_1 and the point $\gamma_2(0)$ replaced by $\gamma_2|_{[0,t]}$ and $\gamma_1(0)$, respectively, we find $\hat{U}_0$, $\hat{V}_0$ and $g_0^{\wedge}$ as required for 0 to be in R.

ad (iii). Suppose that $[0, t) \subset P$, $[0, \tau) \subset R$ and that $H \equiv f^{\sim T} g^{\sim}$ on $\tilde{U} \times \tilde{V}$, a neighbourhood of the point $(\gamma_1(\tau), \gamma_2(t))$. We find $\sigma < \tau$ and $s < t$ such that $\gamma_1([\sigma, \tau]) \subset \tilde{U}$ and $\gamma_2([s, t]) \subset \tilde{V}$. Since $\sigma \in R$ and $s \in P$, we have $H \equiv f^T . g_\sigma^{\wedge}$ on $\hat{U}_\sigma \times \hat{V}_\sigma$ and $H \equiv f^T . g_s$ on $U_s \times V_s$, where $\hat{U}_\sigma$, $\hat{V}_\sigma$ and U_s, V_s are as in the definitions of R and P, respectively. The open sets $\tilde{U} \cap \hat{U}_\sigma \cap U_s$ and $\tilde{V} \cap \hat{V}_\sigma \cap V_s$ are nonempty, because $(\gamma_1(\sigma), \gamma_2(s))$ lies in their product. Consequently, Proposition 10.4 yields $f \equiv C_1^T f^{\sim}$ on $\tilde{U} \cap \hat{U}_\sigma$, $g_\sigma^{\wedge} \equiv C_1^{-1} g^{\sim}$ on $\tilde{V} \cap \hat{V}_\sigma$, $f \equiv C_2^T f^{\sim}$ on $\tilde{U} \cap U_s$ and $g_s \equiv \equiv C_2^{-1} g^{\sim}$ on $\tilde{V} \cap V_s$, with constant regular matrices C_1 and C_2. However, $C_2^T f^{\sim} \equiv \equiv C_1^T f^{\sim}$ on $\tilde{U} \cap \hat{U}_\sigma \cap U_s$ implies that $C_1 = C_2$. So we will write $C_1 = C_2 = C$. Putting $\hat{U}_\tau = (\hat{U}_\sigma \cup \tilde{U}) \cap U_s$ and $\hat{V}_\tau = (\hat{V}_\sigma \cap \tilde{V}) \cup V_s$, we easily observe that $\gamma_1([0, \tau]) \subset \hat{U}_\tau \subset U_0$ and $\gamma_2([0, t]) \subset \hat{V}_\tau \subset Y$. Since $g_\sigma^{\wedge} \equiv g_s$ $(\equiv C^{-1} g^{\sim})$ on $\tilde{V} \cap \hat{V}_\sigma \cap V_s$, the following definition of a continuous mapping $g_\tau^{\wedge} : V \to K^m$ is correct:

$$g_\tau^{\wedge}(y) = \begin{cases} g_s(y) & \text{if } y \in V_s, \\ g_\sigma^{\wedge}(y) & \text{if } y \in \hat{V}_\sigma \cap \tilde{V}. \end{cases}$$

This definition implies that

$$f^T . g_\tau^{\wedge} \equiv f^T . g_s \equiv H \quad \text{on} \quad U_s \times V_s,$$

$$f^T . g_\tau^{\wedge} \equiv f^T . g_\sigma^{\wedge} \equiv H \quad \text{on} \quad \hat{U}_\sigma \times (\hat{V}_\sigma \cap \tilde{V})$$

and

$$f^T . g_\tau^{\wedge} \equiv (C^T f^{\sim})^T . g_\sigma^{\wedge} \equiv (f^{\sim T} C) . (C^{-1} g^{\sim}) \equiv f^{\sim T} . g^{\sim} \equiv H \quad \text{on} \quad (\tilde{U} \cap U_s) \times (\hat{V}_\sigma \cap \tilde{V}).$$

These identities yield $H \equiv f^T . g_\tau^{\wedge}$ on $\hat{U}_\tau \times \hat{V}_\tau$, because $\hat{U}_\tau \times \hat{V}_\tau$ is a subset of

$$(U_s \times V_s) \cup [\hat{U}_\sigma \times (\hat{V}_\sigma \cap \tilde{V})] \cup [(\tilde{U} \cap U_s) \times (\hat{V}_\sigma \cap \tilde{V})].$$

So we conclude that $\tau \in R$, which completes the proof of (iii).

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