Ondrej Kováčik; Jiří Rákosník

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ON SPACES $L^{p(x)}$ AND $W^{k,p(x)}$

ONDREJ KOVÁČIK, Žilina and JIŘÍ RÁKOSNÍK, Praha

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1. INTRODUCTION AND PRELIMINARIES

Consider the nonlinear Dirichlet b.v.p.

$$\sum_{|\alpha|\leqq k} (-1)^{|\alpha|} D^\alpha a_\alpha(x, \delta_k u) = f \quad \text{on} \quad \Omega\,, \tag{1.1}$$

$$u = 0 \quad \text{on} \quad \partial\Omega\,, \tag{1.2}$$

where $\delta_k u = \{D^\alpha u\colon |\alpha| \leqq k\}$. One of the common approaches to the weak solvability of the problem (1.1), (1.2) is based on the Browder theorem and assumes that the coefficients satisfy both the growth conditions

$$|a_\alpha(x, \xi)| \leqq g(x) + c \sum_{|\alpha|\leqq k} |\xi_\alpha|^{p-1} \tag{1.3}$$

with $g \in L^{p'}(\Omega)$ and the coercivity condition

$$\sum_{|\alpha|\leqq k} a_\alpha(x, \xi)\, \xi_\alpha \geqq c_1 \sum_{|\alpha|\leqq k} |\xi_\alpha|^p - c_2 \tag{1.4}$$

with some $p \in (1, \infty)$. It is then natural to look for a weak solution in the Sobolev space $W_0^{k,p}(\Omega)$.

Consider a more general situation, when $\Omega = \Omega_1 \cup \Omega_2$, $1 < p_1 < p_2 < \infty$, and the conditions (1.3), (1.4) are satisfied with p_i on Ω_i. If we simply use the above scheme to find the weak solution of (1.1), (1.2) in $W^{k,p}(\Omega)$, we see that the validity of conditions (1.3) and (1.4) requires $p = \max\{p_1, p_2\}$ and $p = \min\{p_1, p_2\}$, respectively. Even more difficult situation occurs when p is a function of $x \in \Omega$.

The aim of this paper is to suggest appropriate analogues of the Lebesgue spaces L^p and of the Sobolev spaces $W^{k,p}$. It is clear that we cannot simply replace p by $p(x)$ in the usual definition of the norm in L^p. However, the Lebesgue spaces can be considered as particular cases of the Orlicz spaces belonging to a larger family of so called modular spaces. This approach enables to define corresponding counterparts of the Luxemburg and Orlicz norms in $L^{p(x)}$. If the function p is finite a.e. in Ω, then $L^{p(x)}$ is a particular case of the so called Orlicz-Musielak spaces treated by J. Musielak in [6] where some details for the spaces $L^{p(x)}$ and further references can be found. We extend the definition of $L^{p(x)}$ for functions p taking the values from $[1, \infty]$.

Our paper is organized in the following way. In Section 2 we define the spaces $L^{p(x)}$ and investigate their properties interesting from the point of view of the above b.v.p. It appears that spaces $L^{p(x)}$ and L^p have many common properties except a very important one: the p-mean continuity. In Section 3 we introduce the generalized Sobolev spaces $W^{k,p(x)}$ and prove some theorems on continuous and compact embeddings and on equivalent norms. In the last section we deal with the Nemyckii operators in $L^{p(x)}$ and $W^{k,p(x)}$ and use the results of the previous sections to establish an existence theorem for a weak solution to the b.v.p. (1.1), (1.2) with coefficients of variable growth.

Throughout the paper the terms measure, measurable etc. will mean the Lebesgue measure, Lebesgue measurable etc. All sets and functions are supposed measurable. The Lebesgue measure and the characteristic function of a set $A \subset \mathbb{R}^N$ will be denoted by $|A|$ and χ_A, respectively. The symbol Ω will stand for a set in $\mathbb{R}^N$ with $|\Omega| > 0$.

By $\mathscr{P}(\Omega)$ we denote the family of all (measurable) functions $p: \Omega \to [1, \infty]$.

Notation 1.1. For $p \in \mathscr{P}(\Omega)$ we put $\Omega_1^p = \Omega_1 = \{x \in \Omega: p(x) = 1\}$, $\Omega_\infty^p = \Omega_\infty = \{x \in \Omega: p(x) = \infty\}$, $\Omega_0^p = \Omega_0 = \Omega \setminus (\Omega_1 \cup \Omega_\infty)$, $p_* = \operatorname{ess\,inf}_{\Omega_0} p(x)$ and $p^* = \operatorname{ess\,sup}_{\Omega_0} p(x)$ if $|\Omega_0| > 0$, $p_* = p^* = 1$ if $|\Omega_0| = 0$, $c_p = \|\chi_{\Omega_1}\|_\infty + \|\chi_{\Omega_0}\|_\infty + \|\chi_{\Omega_\infty}\|_\infty$, and $r_p = c_p + 1/p_* - 1/p^*$. We use the convention $1/\infty = 0$.

2. GENERALIZED LEBESGUE SPACES

Let $p \in \mathscr{P}(\Omega)$. On the set of all functions on Ω we define the functionals ϱ_p and $\|\cdot\|_p$ by

$$\varrho_p(f) = \int_{\Omega \setminus \Omega_\infty} |f(x)|^{p(x)}\,dx + \operatorname{ess\,sup}_{\Omega_\infty} |f(x)|, \tag{2.1}$$

$$\|f\|_p = \inf\{\lambda > 0: \varrho_p(f/\lambda) \leqq 1\}. \tag{2.2}$$

It is easy to see that ϱ_p has the following properties:

(2.3) $\varrho_p(f) \geqq 0$ *for every function* f.

(2.4) $\varrho_p(f) = 0$ *if and only if* $f = 0$.

(2.5) $\varrho_p(-f) = \varrho_p(f)$ *for every* f.

(2.6) ϱ_p *is convex*.

(2.7) *If* $|f(x)| \geqq |g(x)|$ *for a.e.* $x \in \Omega$ *and if* $\varrho_p(f) < \infty$, *then* $\varrho_p(f) \geqq \varrho_p(g)$; *the last inequality is strict if* $|f| \neq |g|$.

(2.8) *If* $0 < \varrho_p(f) < \infty$, *then the function* $\lambda \mapsto \varrho_p(f/\lambda)$ *is continuous and decreasing on the interval* $[1, \infty)$.

The properties (2.3)–(2.6) characterize ϱ_p as the convex modular in the sense of [6].

(2.9) $\varrho_p(f/\|f\|_p) \leqq 1$ *for every f with* $0 < \|f\|_p < \infty$.

Indeed, taking $\gamma_n \downarrow \|f\|_p$ we use the Fatou lemma, (2.8) and (2.2) to obtain $\varrho_p(f/\|f\|_p) \leqq \liminf_{n\to\infty} \varrho_p(f/\gamma_n) \leqq 1$. ■

(2.10) *If* $p^* < \infty$, *then* $\varrho_p(f/\|f\|_p) = 1$ *for every f with* $0 < \|f\|_p < \infty$.

For $0 < \lambda \leqq \|f\|_p$ we have $\varrho_p(f/\lambda) \leqq (\|f\|_p/\lambda)^{p^*} \varrho_p(f/\|f\|_p)$. Hence, if $\varrho_p(f/\|f\|_p) < 1$, we can find $\lambda < \|f\|_p$ such that $\varrho_p(f/\lambda) \leqq 1$, which contradicts (2.2). ■

As a consequence of (2.6), (2.4) and (2.9) we have:

(2.11) *If* $\|f\|_p \leqq 1$, *then* $\varrho_p(f) \leqq \|f\|_p$.

The *generalized Lebesgue space* $L^{p(x)}(\Omega)$ is the class of all functions f such that $\varrho_p(\lambda f) < \infty$ for some $\lambda = \lambda(f) > 0$. The properties (2.3)–(2.6) and (2.9) yield that $L^{p(x)}(\Omega)$ is a normed linear space if endowed with the norm (2.2) which corresponds to the well-known Luxemburg norm in Orlicz spaces. If $p(x) \equiv p$ is a constant function, then the norm (2.2) coincides with the usual L^p-norm and so the notation is not confusional.

Let $M: \Omega \times R \to [0, \infty]$ be a non-negative measurable function such that for a.e. $x \in \Omega$ the function $M(x, \cdot)$ is lower semicontinuous, convex, even and satisfies $\lim_{x\to 0} M(x, u) = M(x, 0) = 0$. The so called Orlicz-Musielak space $L^M(\Omega)$ consists of all functions f on Ω such that $\int_\Omega M(x, \lambda f(x))\, dx < \infty$ for some $\lambda > 0$ (cf. [6]). If the function p is finite a.e. in Ω then $L^{p(x)}(\Omega) = L^M(\Omega)$, where

(2.12) $M(x, u) = |u|^{p(x)}$.

Given $p \in \mathscr{P}(\Omega)$ we define the conjugate function $p' \in \mathscr{P}(\Omega)$,

$$p'(x) = \begin{cases} \infty & \text{for } x \in \Omega_1^p , \\ 1 & \text{for } x \in \Omega_\infty^p , \\ p(x)/(p(x) - 1) & \text{for other } x \in \Omega . \end{cases}$$

Theorem 2.1 (generalized Hölder inequality). *Let* $p \in \mathscr{P}(\Omega)$. *Then the inequality*

$$\int_\Omega |f(x)\, g(x)|\, dx \leqq r_p \|f\|_p \|g\|_{p'}$$

holds for every $f \in L^{p(x)}(\Omega)$ *and* $g \in L^{p'(x)}(\Omega)$ *with the constant* r_p *defined in* 1.1.

Proof. Obviously, we can suppose that $\|f\|_p \neq 0$, $\|g\|_{p'} \neq 0$ and $|\Omega_0| > 0$. For a.e. $x \in \Omega_0$ we have $1 < p(x) < \infty$, $|f(x)| < \infty$ and $|g(x)| < \infty$. Putting $a = f(x)/\|f\|_p$, $b = g(x)/\|g\|_{p'}$, $p = p(x)$, $p' = p'(x)$ in the well-known inequality

$$ab \leqq \frac{a^p}{p} + \frac{b^{p'}}{p'} ,$$

integrating over Ω_0 and using (2.9) we obtain

$$\int_{\Omega_0} \frac{|f(x)\,g(x)|}{\|f\|_p\,\|g\|_{p'}}\,\mathrm{d}x \leqq \operatorname*{ess\,sup}_{\Omega_0} \frac{1}{p(x)}\,\varrho_p(f/\|f\|_p) + \operatorname*{ess\,sup}_{\Omega_0} \frac{1}{p'(x)}\,\varrho_p(g/\|g\|_{p'}) \leqq$$
$$\leqq 1 + 1/p_* - 1/p^* .$$

Thus,

$$\int_\Omega |f(x)\,g(x)|\,\mathrm{d}x \leqq (1 + 1/p_* - 1/p^*)\,\|f\|_p\,\|g\|_{p'}\,\|\chi_{\Omega_0}\|_\infty +$$
$$+ \|f\chi_{\Omega_1}\|_1\,\|g\chi_{\Omega_1}\|_\infty + \|f\chi_{\Omega_\infty}\|\,\|g\chi_{\Omega_\infty}\|_1 \leqq r_p\|f\|_p\,\|g\|_{p'} . \quad \blacksquare$$

Corollary 2.2. *Let $p, r, q \in \mathscr{P}(\Omega)$ be such that $p(x) \leqq r(x) \leqq q(x) < \infty$ for a.e. $x \in \Omega$ and $p \neq q$. Then there exists a constant $c > 0$ such that for every $f \in L^{p(x)}(\Omega) \cap$ $\cap\, L^{q(x)}(\Omega)$ the inequality*

$$\text{(2.13)} \qquad \|f\|_r \leqq c\|f\|_p^\mu\,\|f\|_q^\nu$$

holds, where

$$\text{(2.14)} \qquad \mu = \begin{cases} \operatorname*{ess\,sup}_{\Omega} \dfrac{p(x)}{r(x)}\,\dfrac{q(x) - r(x)}{q(x) - p(x)} & \text{if} \quad \|f\|_p > 1 , \\[2ex] \operatorname*{ess\,inf}_{\Omega} \dfrac{p(x)}{r(x)}\,\dfrac{q(x) - r(x)}{q(x) - p(x)} & \text{if} \quad \|f\|_p \leqq 1 , \end{cases}$$

$$\text{(2.15)} \qquad \nu = \begin{cases} \operatorname*{ess\,sup}_{\Omega} \dfrac{q(x)}{r(x)}\,\dfrac{r(x) - p(x)}{q(x) - p(x)} & \text{if} \quad \|f\|_q > 1 , \\[2ex] \operatorname*{ess\,inf}_{\Omega} \dfrac{q(x)}{r(x)}\,\dfrac{r(x) - p(x)}{q(x) - p(x)} & \text{if} \quad \|f\|_q \leqq 1 \end{cases}$$

(*here we consider* $0/0 = 1$).

Proof. It suffices to consider $f \neq 0$. At first, assume that $r(x) < q(x)$ for a.e. $x \in \Omega$. Define functions $s, t \in \mathscr{P}(\Omega)$,

$$s(x) = \frac{q(x) - p(x)}{q(x) - r(x)}, \quad t(x) = \frac{q(x) - p(x)}{r(x) - p(x)} .$$

Then $1 < s(x), t(x) < \infty$ and $1/s(x) + 1/t(x) = 1$ for a.e. $x \in \Omega$ and so, by Theorem 2.1,

$$\text{(2.16)} \qquad \varrho_r\left(\frac{f}{\|f\|_p^\mu\,\|f\|_q^\nu}\right) \leqq r_s \left\|\frac{|f|^{p/s}}{\|f\|_p^{\mu r}}\right\|_s \left\|\frac{|f|^{r-p/s}}{\|f\|_q^{\nu r}}\right\|_t .$$

According to (2.14) and (2.9) we have

$$\text{(2.17)} \qquad \varrho_s\left(\frac{|f|^{p/s}}{\|f\|_p^{\mu r}}\right) \leqq \varrho_p\left(\frac{f}{\|f\|_p}\right) \leqq 1 .$$

Similarly,

$$\text{(2.18)} \qquad \varrho_t\left(\frac{|f|^{r-p/s}}{\|f\|_q^{\nu r}}\right) \leqq \varrho_q\left(\frac{f}{\|f\|_q}\right) \leqq 1 .$$

Since $r_s \geqq 1$, we can use the convexity of ϱ_r and estimates (2.16)–(2.18) to obtain

$$(2.19) \qquad \varrho_r\left(\frac{f}{r_s\|f\|_p^\mu \|f\|_q^\nu}\right) \leqq \frac{1}{r_s}\varrho_r\left(\frac{f}{\|f\|_p^\mu \|f\|_q^\nu}\right) \leqq 1\,,$$

i.e. the inequality (2.13) holds with $c = r_s$.

Now, assume that $|G| > 0$ where $G = \{x \in \Omega\colon r(x) = q(x)\}$. Then

$$\operatorname*{ess\,inf}_{\Omega} \frac{p(x)}{r(x)}\frac{q(x) - r(x)}{q(x) - p(x)} = 0\,, \quad \operatorname*{ess\,sup}_{\Omega} \frac{q(x)}{r(x)}\frac{r(x) - p(x)}{q(x) - p(x)} = 1$$

and so $\|f\|_p^\mu \geqq 1$, $\|f\|_q^\nu \geqq \|f\|_q$. Hence,

$$\varrho_r\left(\frac{f\chi_G}{\|f\|_p^\mu \|f\|_q^\nu}\right) = \varrho_q\left(\frac{f\chi_G}{\|f\|_p^\mu \|f\|_q^\nu}\right) \leqq \varrho_q\left(\frac{f\chi_G}{\|f\|_q}\right) \leqq \varrho_q\left(\frac{f}{\|f\|_q}\right) \leqq 1\,.$$

Since $r(x) < q(x)$ for a.e. $x \in \Omega \setminus G$, by the first part of the proof (cf. (2.19)) we have

$$\varrho_r\left(\frac{f\chi_{\Omega\setminus G}}{\|f\|_p^\mu \|f\|_q^\nu}\right) \geqq r_s\,.$$

Thus,

$$\varrho_r\left(\frac{f}{\|f\|_p^\mu \|f\|_q^\nu}\right) \leqq \varrho_r\left(\frac{f\chi_G}{\|f\|_p^\mu \|f\|_q^\nu}\right) + \varrho_r\left(\frac{f\chi_{\Omega\setminus G}}{\|f\|_p^\mu \|f\|_q^\nu}\right) \leqq r_s + 1$$

and, similarly as in the first part of the proof, we conclude the inequality (2.13) with $c = r_s + 1$. ∎

For functions f on Ω we define

$$(2.20) \qquad |||f|||_p = \sup_{\varrho_{p'}(g) \leqq 1} \int_\Omega f(x)\,g(x)\,\mathrm{d}x\,.$$

This is an analogue of the Orlicz norm in Orlicz spaces (cf. [5], chap. 9) and it is easy to see that it is a norm on the class of functions f with $|||f|||_p < \infty$.

(2.21) *Let* $|||f|||_p < \infty$ *and* $\varrho_{p'}(g) < \infty$. *Then*

$$\left|\int_\Omega f(x)\,g(x)\,\mathrm{d}x\right| \leqq \begin{cases} |||f|||_p & \text{if } \varrho_{p'}(g) \leqq 1\,,\\ |||f|||_p\,\varrho_{p'}(g) & \text{if } \varrho_{p'}(g) > 1\,. \end{cases}$$

The first case follows from (2.20). Assume $\varrho_{p'}(g) > 1$. The convexity of ϱ_p yields $\varrho_{p'}(\varrho_{p'}(g)^{-1}\,g) \leqq \varrho_{p'}(g)^{-1}\,\varrho_{p'}(g) = 1$ and so

$$\left|\int_\Omega f(x)\,g(x)\,\mathrm{d}x\right| = \varrho_{p'}(g)\left|\int_\Omega f(x)\,\varrho_{p'}(g)^{-1}\,g(x)\,\mathrm{d}x\right| \leqq \varrho_{p'}(g)\,|||f|||_p\,. \quad ∎$$

(2.22) *If* $|\Omega_1| = |\Omega_\infty| = 0$ *and if* $\varrho_p(f) < \infty$, $|||f|||_p \leqq 1$, *then* $\varrho_p(f) \leqq 1$.

Suppose, to the contrary, that $\varrho_p(f) > 1$. According to (2.8) there exists $\lambda > 1$ such that $\varrho_p(f/\lambda) = 1$. Putting

$$g(x) = |f(x)/\lambda|^{p(x)-1}\operatorname{sign} f(x)\,, \quad x \in \Omega\,,$$

we have $\varrho_{p'}(g) = \varrho_p(f/\lambda) = 1$ and so

$$|||f|||_p \geqq \int_\Omega f(x)\, g(x)\, dx = \lambda \varrho_p(f/\lambda) = \lambda > 1 ,$$

which is a contradiction. ∎

(2.23) *If* $|||f|||_p \leqq 1$, *then* $\varrho_p(f) \leqq c_p |||f|||_p$.

First, suppose that $\varrho_p(f) < \infty$. We have

$$(2.24) \qquad \varrho_p(f) = \|\chi_{\Omega_1}\|_\infty\, \varrho_p(f_1) + \|\chi_{\Omega_0}\|_\infty\, \varrho_p(f_0) + \|\chi_{\Omega_\infty}\, \varrho_p(f_\infty) ,$$

where $f_j = f\chi_{\Omega_j}$, $j = 1, 0, \infty$. Put

$$g_1(x) = \operatorname{sign} f_1(x) , \quad g_0(x) = |f_0(x)|^{p(x)-1} \operatorname{sign} f_0(x) , \quad x \in \Omega .$$

Then $\varrho_{p'}(g_1) = \operatorname*{ess\,sup}_{x\in\Omega_1} |g_1(x)| = 1$ and, according to (2.22),

$$\varrho_{p'}(g_0) = \int_{\Omega_0} |f(x)|^{p(x)}\, dx \leqq 1 .$$

Hence, (2.21) yields

$$(2.25) \qquad \varrho_p(f_j) = \int_\Omega f(x)\, g_j(x)\, dx \leqq |||f|||_p , \quad j = 1, 0 .$$

If $|\Omega_\infty| > 0$, then for every $\delta \in (0, 1)$ there exists a set $A \subset \Omega_\infty$ such that $0 < |A| < \infty$ and $|f(x)| \geqq \delta \operatorname*{ess\,sup}_{y\in\Omega_\infty} |f(y)|$ for $x \in A$. Then for

$$g_\infty = |A|^{-1}\, \chi_A \operatorname{sign} f \quad \text{we have} \quad \varrho_{p'}(g_\infty) = \int_A |A|^{-1}\, |\operatorname{sign} f(x)|\, dx \leqq 1$$

and so

$$|||f||| \geqq \int_\Omega f(x)\, g_\infty(x)\, dx =$$
$$= |A|^{-1} \int_A |f(x)|\, dx \geqq \delta \operatorname*{ess\,sup}_{y\in\Omega_\infty} |f(x)| = \delta\, \varrho_p(f_\infty) .$$

Letting $\delta \to 1-$ we obtain

$$(2.26) \qquad \varrho_p(f_\infty) \leqq |||f|||_p .$$

Relations (2.24)–(2.26) yield the desired inequality (2.23).

To avoid the assumption $\varrho_p(f) < \infty$ we use the truncations

$$f_n(x) = \min\{n, |f(x)|\}\, \chi_{G_n}(x) , \quad n \in \mathbb{N} ,$$

where $\{G_n\}$ is a sequence of sets such that $G_n \subset G_{n+1} \subset \Omega$, $|G_n| < \infty$ for $n \in \mathbb{N}$ and $\Omega = \bigcup_{n=1}^{\infty} G_n$. Then $\varrho_p(f_n) < \infty$, $|||f_n|||_p \leqq |||f|||_p \leqq 1$ and, according to the first part of the proof, $\varrho_p(f_n) \leqq c_p |||f|||_p$. It suffices to let $n \to \infty$. ∎

Theorem 2.3 (on equivalent norms). $L^{p(x)}(\Omega) = \{f\colon |||f|||_p < \infty\}$ *and for every* $f \in L^{p(x)}(\Omega)$ *the inequalities*

$$(2.27) \qquad c_p^{-1} \|f\|_p \leqq |||f|||_p \leqq r_p \|f\|_p$$

hold, where c_p *and* r_p *are constants defined in* 1.1.

Proof. Let $f \in L^{p(x)}(\Omega)$. If $\varrho_{p'}(g) \leqq 1$, then $\|g\|_{p'} \leqq 1$ and the Hölder inequality yields $\int_\Omega f(x)\, g(x)\, dx \leqq r_p\|f\|_p \|g\|_{p'} \leqq r_p\|f\|_p$. This gives the second inequality (2.27) and, consequently, $|||f|||_p < \infty$.

On the contrary, let $0 < |||f|||_p < \infty$. Since $|||f/(c_p|||f|||_p)|||_p = c_p^{-1} \leqq 1$, we use (2.23) to get $\varrho_p(f/(c_p|||f|||_p)) \leqq c_p c_p^{-1} = 1$. The first inequality (2.27) follows and yields $f \in L^{p(x)}(\Omega)$. ■

We shall say that functions $f_n \in L^{p(x)}(\Omega)$ *converge modularly* to a function $f \in L^{p(x)}(\Omega)$, if $\lim_{n\to\infty} \varrho_p(f - f_n) = 0$.

In [5] it is shown that in Orlicz spaces there is a substantial difference between the norm convergence and the modular convergence. We shall show that a similar difference is in the space $L^{p(x)}(\Omega)$. According to (2.11) the norm convergence is stronger that the modular one.

(2.28) *If* $p^* < \infty$, *then* $\varrho_p(f_n) \to 0$ *if and only if* $\|f_n\|_p \to 0$.

Suppose, that $\varrho_p(f_n) \to 0$, and take $\varepsilon \in (0, 1]$. For sufficiently large n we have $\varrho_p(f_n) < \varepsilon \leqq 1$ and so

$$\varrho_p(f_n\, \varrho_p(f_n)^{-1/p^*}) \leqq \varrho_p(f_n)^{-1} \int_{\Omega\setminus\Omega_\infty} |f_n(x)|^{p(x)}\, dx + \\ + \varrho_p(f_n)^{-1/p^*} \operatorname*{ess\,sup}_{\Omega_\infty} |f_n(x)| \to \varrho_p(f_n)^{-1} \varrho_p(f_n) = 1 ,$$

i.e.,

$$\|f_n\|_p \leqq \varrho_p(f_n)^{1/p^*} < \varepsilon^{1/p^*} .$$

Hence, $\|f_n\|_p \to 0$. ■

Theorem 2.4. *The topology of the normed linear space $L^{p(x)}(\Omega)$ given by the norm (2.2) or (2.20) coincides with the topology of modular convergence if and only if $p^* < \infty$.*

Proof. Suppose that $p^* = \infty$. Then there exist sets $G_{n+1} \subset G_n \subset \Omega \setminus \Omega_\infty$ such that $|G_n| < \infty$ and

$$(2.29) \qquad |G_n| \to 0 ,$$

$$(2.30) \qquad p(x) > n \quad \text{on} \quad G_n , \quad n \in \mathbb{N} ,$$

$$(2.31) \qquad \sup \{n: |G_n \setminus G_{n+1}| > 0\} = \infty .$$

Fix $\lambda \in (0, 1)$, put $\omega_n = |G_n \setminus G_{n+1}|$ and $a_n = \lambda^n \omega_n^{-1}$ if $\omega_n > 0$, $a_n = 0$ otherwise. Consider the functions $f(x) = (\sum_{n=1}^{\infty} a_n \chi_{G_n \setminus G_{n+1}}(x))^{1/p(x)}$, $x \in \Omega$, and $f_n = f\chi_{G_n}$. Then

$$(2.32) \qquad \varrho_p(f) = \int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)}\, dx = \sum_{n=1}^{\infty} a_n \omega_n \leqq \sum_{n=1}^{\infty} \lambda^n < \infty .$$

On the other hand, (2.30) yields

$$(2.33) \qquad \varrho_p(f_n/\lambda) = \int_{G_n} |f(x)/\lambda|^{p(x)}\, dx = \\ = \sum_{k=n}^{\infty} \int_{G_k\setminus G_{k+1}} |f(x)/\lambda|^{p(x)}\, dx \geqq \sum_{k=n}^{\infty} a_k \omega_k \lambda^{-k} = \infty ,$$

because, according to (2.31), the last series of non-negative numbers contains an infinite number of members $a_k \omega_k \lambda^{-k} = 1$. Now, (2.32) and (2.29) yield $\varrho_p(f_n) \to 0$, but (2.33) gives $\|f_n\|_p \geqq \lambda$.

The sufficiency of condition $p^* < \infty$ is proved in (2.28). ∎

(2.34) *If* $p^* < \infty$ *and if* $f_n \to 0$ *in* $L^{p(x)}(\Omega)$, *then* $f_n \to 0$ *in measure.*

If, to the contrary, there are $\varepsilon, \delta \in (0, 1]$ and a subsequence $\{n_k\}$ such that $\inf_k |\{x \in \Omega: |f_{n_k}(x)| > \varepsilon\}| \geqq \delta$, then $\varrho_p(f_{n_k}) \geqq \delta \varepsilon^{p^*}$. This, by (2.28), contradicts the assumption $f_n \to 0$. ∎

Theorem 2.5. *The space* $L^{p(x)}(\Omega)$ *is complete.*

Proof. Let $\{f_n\}$ be a Cauchy sequence of functions from $L^{p(x)}(\Omega)$ and let $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that

(2.35) $$\int_\Omega |f_m(x) - f_n(x)| \, |g(x)| \, \mathrm{d}x < \varepsilon$$

for every $m, n \geqq n_0$ and for every function g such that $\varrho_{p'}(g) \leqq 1$. We decompose Ω into pairwise disjoint subsets G_k of finite measure and define functions $g_k = (1 + |G_k|)^{-1} \chi_{G_k}$, $k \in \mathbb{N}$. Then

$$\varrho_{p'}(g_k) \leqq \int_{G_k} (1 + |G_k|)^{-p(x)} \, \mathrm{d}x + (1 + |G_k|)^{-1} \leqq 1 ,$$

and inserting g_k for g in (2.35) we get

$$\int_{G_k} |f_m(x) - f_n(x)| \, \mathrm{d}x \leqq \varepsilon(1 + |G_k|) , \quad m, n \geqq n_0 , \quad k \in \mathbb{N} .$$

This means that the sequence $\{f_n\}$ is Cauchy — and so convergent — in each $L^1(G_k)$. By induction we find subsequences $\{f_n^{(k)}\}_n$ and functions $f^{(k)} \in L^1(G_k)$ such that $f_n^{(k)}(x) \to f^{(k)}(x)$ for a.e. $x \in G_k$, $k \in \mathbb{N}$. Thus, $f_m^{(m)}(x) \to \sum_{k=1}^{\infty} f^{(k)}(x) \chi_{G_k}(x) = f(x)$ for a.e. $x \in \Omega$, and replacing f_m by $f_m^{(m)}$ in (2.35) and using the Fatou lemma we obtain

$$\int_\Omega |f(x) - f_n(x)| \, |g(x)| \, \mathrm{d}x \leqq \sup_m \int_\Omega |f_m^{(m)}(x) - f_n(x)| \, |g(x)| \, \mathrm{d}x \leqq \varepsilon$$

for every $n \geqq n_0$ and every g with $\varrho_{p'}(g) \leqq 1$. Hence, $|||f - f_n|||_p \leqq \varepsilon$. ∎

According to (2.2) and (2.11) f satisfies $\varrho_p(f) \leqq 1$ if and only if $\|f\|_p \leqq 1$. Hence, Theorem 2.3 yields: *If* $g \in L^{p'(x)}(\Omega)$, *then* G *given by*

(2.26) $$G(f) = \int_\Omega f(x) \, g(x) \, \mathrm{d}x , \quad f \in L^{p(x)}(\Omega) ,$$

is a linear continuous functional on $L^{p(x)}(\Omega)$ *with the norm satisfying* $c_p^{-1} \|g\|_{p'} \leqq \|G\| \leqq r_p \|g\|_{p'}$.

Theorem 2.6. *The following conditions are equivalent:*

(i) $p \in L^\infty(\Omega)$.

(ii) *For every linear continuous functional* G *on* $L^{p(x)}(\Omega)$ *there exists a unique function* $g \in L^{p'(x)}(\Omega)$ *such that* (2.36) *holds.*

Proof. Assume that (ii) holds. Then, obviously, $|\Omega_\infty| = 0$ and $L^{p(x)}(\Omega)$ is the Orlicz-

Musielak space $L^M(\Omega)$ with M satisfying (2.12). A. Kozek [3] proved that if (ii) holds then the function M satisfies the $\tilde{\Delta}_2$-condition: there exists $K \geqq 1$ and a function $h \in L^1(\Omega)$ such that for every $u \in \mathbb{R}$ and a.e. $x \in \Omega$ the inequality $M(2u, x) \leqq \leqq KM(u, x) + h(x)$, i.e.

$$(2.37) \qquad (2^{p(x)} - K)\,|u|^{p(x)} \leqq h(x)$$

holds.

Suppose that $p^* = \infty$. Then the set $E = \{x \in \Omega\colon p(x) \geqq 1 + \log_2 K\}$ has a positive measure and for $x \in E$ we have

$$(2.38) \qquad 2^{p(x)} \geqq 2K\,.$$

From (2.37) and (2.38) we obtain the estimate

$$h(x) \geqq \sup_u\,(2^{p(x)} - K)\,|u|^{p(x)} \geqq K \sup_{u \geqq 1} u^{1+\log_2 K} = \infty\,, \quad x \in E\,,$$

which contradicts the integrability of h. Thus, $p^* < \infty$ and (i) holds.

If $p \in L^\infty(\Omega)$, then the function M from (2.12) satisfies the $\tilde{\Delta}_2$-condition and so, according to H. Hudzik [2], the condition (ii) holds. Let us note that this part of the proof can be made also directly by the usual method for the classical Lebesgue spaces based on the approximation by step function and on the use of the Radon-Nikodym theorem. ∎

Corollary 2.7. *The dual space to* $L^{p(x)}(\Omega)$ *is* $L^{p'(x)}(\Omega)$ *if and only if* $p \in L^\infty(\Omega)$. *The space* $L^{p(x)}(\Omega)$ *is reflexive if and only if*

$$(2.39) \qquad 1 < \operatorname*{ess\,inf}_\Omega p(x) \leqq \operatorname*{ess\,sup}_\Omega p(x) < \infty\,.$$

Given two Banach spaces X and Y the symbol $X \hookrightarrow Y$ means that X *is (continuously) embedded in* Y.

Theorem 2.8. *Let* $0 < |\Omega| < \infty$ *and* $p, q \in \mathscr{P}(\Omega)$. *Then*

$$(2.40) \qquad L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$$

if and only if

$$(2.41) \qquad p(x) \leqq q(x) \quad \textit{for a.e.} \quad x \in \Omega\,.$$

The norm of the embedding operator (2.40) *does not exceed* $|\Omega| + 1$.

Proof. First, assume (2.41). Then

$$\Omega^p_\infty \subset \Omega^q_\infty$$

(cf. Notation 1.1). It suffices to prove that

$$(2.42) \qquad \|f\|_p \leqq |\Omega| + 1$$

for every $f \in L^{q(x)}(\Omega)$ with $\|f\|_q \leqq 1$. By (2.11) we have

$$\varrho_q(f) = \int_{\Omega \setminus \Omega_\infty{}^q} |f(x)|^{q(x)}\,\mathrm{d}x + \operatorname*{ess\,sup}_{\Omega_\infty{}^q} |f(x)| \leqq 1\,,$$

in particular, $|f(x)| \leqq 1$ for a.e. $x \in \Omega_\infty^q$. So, we can write

$$\varrho_p(f) \leqq |\{x \in \Omega \setminus \Omega_\infty^q : |f(x)| \leqq 1\}| + \int_{\Omega\setminus\Omega_{\infty}{}^q} |f(x)|^{q(x)} \, dx + $$
$$+ |\Omega_\infty^q \setminus \Omega_\infty^p| + \operatorname*{ess\,sup}_{\Omega_\infty{}^q} |f(x)| \leqq |\Omega| + \varrho_q(f) \leqq |\Omega| + 1 .$$

We use the convexity of ϱ_p to obtain

$$\varrho_p(f/(|\Omega| + 1)) \leqq (|\Omega| + 1)^{-1} \varrho_p(f) \leqq 1 .$$

The inequality (2.42) follows.

Suppose, on the contrary, that (2.41) does not hold, i.e. there exists a subset Ω^* of Ω such that $|\Omega^*| > 0$ and

$$p(x) > q(x) , \quad x \in \Omega^* .$$

In contradistinction with (2.40), we shall construct a function $f \in L^{q(x)}(\Omega) \setminus L^{p(x)}(\Omega)$.

If

$$(2.43) \qquad |\Omega_\infty^p \cap \Omega^*| > 0 ,$$

then there exists a set $A \subset \Omega_\infty^p \cap \Omega^*$, $0 < |A| < \infty$, and a number $r \in (1, \infty)$ such that $1 \leqq q(x) \leqq r < \infty = p(x)$ for all $x \in A$. We find sets A_k such that

$$(2.44) \qquad A = \bigcup_{k=1}^{\infty} A_k , \quad A_k \cap A_j = \emptyset \quad \text{for} \quad k \neq j , \quad |A_k| = 2^{-k}|A| \quad \text{for} \quad k \in \mathbb{N} ,$$

and define the function $f = \sum_{k=1}^{\infty} (3/2)^{k/r} \chi_{A_k}$ on Ω. Then

$$\|f\|_p \geqq \|f\chi_A\|_\infty = \infty ,$$

but

$$\varrho_q(f) = \int_A |f(x)|^{q(x)} \, dx = \sum_{k=1}^{\infty} \int_{A_k} (3/2)^{kq(x)/r} \, dx \leqq$$
$$\leqq \sum_{k=1}^{\infty} (3/2)^k |A_k| = |A| \sum_{k=1}^{\infty} (3/4)^k = 3|A| < \infty ,$$

i.e., $f \in L^{q(x)}(\Omega)$.

If (2.43) does not hold, then $1 \leqq q(x) < p(x) < \infty$ for a.e. $x \in \Omega^*$ and there exists a set $A \subset \Omega^*$, $0 < |A| < \infty$, and numbers $a > 0$, $r \in (1, \infty)$ such that $q(x) + a \leqq p(x) \leqq r$ for $x \in A$. We find sets A_k satisfying (2.44) and define the function $f(x) = \sum_{k=1}^{\infty} (2^k k^{-2})^{1/q(x)} \chi_{A_k}(x)$, $x \in \Omega$. Then

$$\varrho_p(f) = \sum_{k=1}^{\infty} 2^k k^{-2} |A_k| = |A| \sum_{k=1}^{\infty} k^{-2} < \infty ,$$

i.e. $f \in L^{q(x)}(\Omega)$. On the other hand, for every $\lambda \in (0, 1]$ we have

$$\varrho_p(\lambda f) \geqq \lambda^r \sum_{k=1}^{\infty} \int_{A_k} (2^k k^{-2})^{p(x)/q(x)} \, dx \geqq \lambda^r \sum_{k=1}^{\infty} (2^k k^{-2})^{1 + a/r} |A_k| =$$
$$= \lambda^r |A| \sum_{k=1}^{\infty} 2^{ak/r} k^{-2(1 + a/r)} = \infty ,$$

and so, $f \notin L^{p(x)}(\Omega)$. ∎

One very important property of the Lebesgue and Orlicz spaces is the mean continuity of their elements. We shall show that this is the point in which the spaces $L^{p(x)}(\Omega)$ differ from the classical Lebesgue spaces. Further on, we shall assume the functions to be extended by zero outside Ω.

We shall say that a function $f \in L^{p(x)}(\Omega)$ is *$p(x)$-mean continuous* if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\varrho_p(f_h - f) < \varepsilon$ for $h \in \mathbb{R}^N$, $|h| < \delta$, where $f_h(x) = f(x + h)$, $x \in \mathbb{R}^N$.

Example 2.9. Let $N = 1$, $\Omega = (-1, 1)$ and let $1 \leqq r < s < \infty$. Put

$$p(x) = \begin{cases} r & \text{for} \quad x \in [0, 1)\,, \\ s & \text{for} \quad x \in (-1, 0) \end{cases} \quad \text{and} \quad f(x) = \begin{cases} x^{-1/s} & \text{for} \quad x \in [0,1)\,, \\ 0 & \text{for} \quad x \in (-1, 0)\,. \end{cases}$$

Then $p \in \mathscr{P}(\Omega)$ and, obviously, $f \in L^{p(x)}(\Omega)$. However, given $h \in (0, 1)$, $\varrho_p(f_h/\lambda) \geqq \geqq \lambda^{-1} \int_{-h}^{0} (x + h)^{-1}\, dx = \infty$ for every $\lambda > 0$, and so $f_h \notin L^{p(x)}(\Omega)$. We shall work on the principle of the previous example to show that for a rather wide class of functions $p \in \mathscr{P}(\Omega)$ we cannot expect the $p(x)$-mean continuity for all functions from $L^{p(x)}(\Omega)$.

Theorem 2.10. *Let Ω contain a ball $B(x_0, r) = \{x \in \mathbb{R}^N : |x - x_0| < r\}$ on which the function p is continuous and non-constant. Then there exists a function $f \in \in L^{p(x)}(\Omega)$ which is not $p(x)$-mean continuous.*

Proof. According to the assumptions, there exists a point $z \in B(x_0, r)$ in which p does not attend its local extremum. Then there exist sequences of points $x_n, y_n \in \in B(x_0, r)$ such that $\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = z$ and $p(x_n) < p(z) < p(y_n)$ for $n \in \mathbb{N}$. The continuity of p yields the existence of such numbers $r_n > 0$ that

$$p(x) < \tfrac{1}{2}(p(z) + p(x_n)) < p(z) \quad \text{for} \quad x \in B(x_n, r_n)\,, \tag{2.45}$$

$$p(x) > p(z) \quad \text{for} \quad x \in B(y_n, r_n)\,. \tag{2.46}$$

Put $q_n = \frac{1}{2}(p(z) + p(x_n))$ and let f_n be functions on Ω such that

$$\operatorname{supp} f_n \subset B(x_n, r_n),\ f_n \in L^{q_n}(B(x_n, r_n)) \setminus L^{p(z)}(B(x_n, r_n)) \ \text{and} \ \|f_n\|_{q_n} = 1\,,$$

and define the function f by

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} f_n(x)\,.$$

By the use of (2.45) and of Theorem 2.8 we obtain

$$\|f\|_p \leqq \sum_{n=1}^{\infty} 2^{-n} \|f_n\|_p \leqq \sum_{n=1}^{\infty} 2^{-n} \|f_n\|_{q_n} (|B(x_n, r_n)| + 1) \leqq$$
$$\leqq 1 + \sup_n |B(x_n, r_n)| < \infty\,.$$

On the other hand, we put $h_n = y_n - x_n$ and, according to (2.46) and to Theorem

2.8, we have

$$\|f_{h_n}\|_p \geqq \|f_{h_n}\chi_{B(y_n,r_n)}\|_p \geqq (1 + |B(y_n, r_n)|)^{-1} \|f_{h_n}\chi_{B(y_n,r_n)}\|_{p(z)} =$$
$$= (1 + |B(y_n, r_n)|)^{-1} \|f\chi_{B(x_n,r_n)}\|_{p(z)} = \infty .$$

Thus, $f_{h_n} - f \notin L^{p(x)}(\Omega)$ and since $h_n \to 0$, the function f is not $p(x)$-mean continuous. ■

(2.47) *If $p^* < \infty$, then the set of all bounded functions on Ω is dense in $L^{p(x)}(\Omega)$.*

Indeed, if $G_n = \{x \in \Omega \setminus \Omega_\infty: |x| < n\}$, then the functions f_n,

$$f_n(x) = \begin{cases} f(x), & \text{if } |f(x)| \leqq n \text{ and } x \in G_n \cup \Omega_\infty, \\ n \operatorname{sign} f(x), & \text{if } |f(x)| > n \text{ and } x \in G_n \cup \Omega_\infty, \\ 0 \text{ in other points of } \Omega, \end{cases}$$

are bounded on Ω and the Lebesgue Dominated Convergence Theorem yields $\varrho_p(f - f_n) \to 0$ as $n \to \infty$. Hence, by Theorem 2.4, $f_n \to f$. ■

Theorem 2.11. *Let $p \in \mathscr{P}(\Omega) \cap L^\infty(\Omega)$. Then the set $C(\Omega) \cap L^{p(x)}(\Omega)$ is dense in $L^{p(x)}(\Omega)$. If, moreover, Ω is open, then the set $C_0^\infty(\Omega)$ is dense in $L^{p(x)}(\Omega)$.*

Proof. Let $f \in L^{p(x)}(\Omega)$ and $\varepsilon > 0$. By (2.47), there exists a bounded function $g \in L^{p(x)}(\Omega)$ such that

(2.48) $$\|f - g\|_p < \varepsilon .$$

By the Luzin theorem there exists a function $h \in C(\Omega)$ and an open set U such that

$$|U| < \min \left\{1, \left(\frac{\varepsilon}{2\|g\|_\infty}\right)^{p^*}\right\},$$

$g(x) = h(x)$ for all $x \in \Omega \setminus U$ and $\sup |h(x)| = \sup_{\Omega \setminus U} |g(x)| \leqq \|g\|_\infty$. Hence,

$$\varrho_p((g - h)/\varepsilon) \leqq \max \{1, (2\|g\|_\infty/\varepsilon)^{p^*}\} |U| \leqq 1 ,$$

i.e. $\|g - h\|_p \leqq \varepsilon$, which together with (2.48) gives

(2.49) $$\|f - h\|_p \leqq 2\varepsilon .$$

Assume, moreover, that Ω is open. Since $p \in L^\infty(\Omega)$, we have $C_0^\infty(\Omega) \subset L^{p(x)}(\Omega)$ and $\varrho_p(h/\varepsilon) < \infty$, and so there exists a bounded open set $G \subset \Omega$ such that $\varrho_p(h\chi_{\Omega \setminus G}/\varepsilon) \leqq 1$, i.e.

(2.50) $$\|h - h\chi_G\|_p \leqq \varepsilon .$$

Let m be a polynomial satisfying $\sup_G |h(x) - m(x)| < \varepsilon \min \{1, |G|^{-1}\}$. Then $\varrho_p((h\chi_G - m\chi_G)/\varepsilon) \leqq \min \{1, |G|^{-1}\} |G| \leqq 1$, i.e.

(2.51) $$\|h\chi_G - m\chi_G\|_p \leqq \varepsilon .$$

Finally, considerations similar to those leading to (2.50) yield that for a sufficiently small positive number a the compact set $K_a = \{x \in G: \operatorname{dist}(x, \partial G) \geqq a\}$ satisfies

$\|m\chi_G - m\chi_{K_a}\|_p \leqq \varepsilon$. Taking $\varphi \in C_0^\infty(G)$ such that $0 \leqq \varphi(x) \leqq 1$ for $x \in G$ and $\varphi(x) = 1$ for $x \in K_a$ we obtain the estimate

$$\|m\chi_G - m\varphi\|_p \leqq \|m\chi_G - m\chi_{K_a}\|_p \leqq \varepsilon ,$$

which together with (2.49)–(2.51) gives

$$\|f - m\varphi\|_p < 4\varepsilon .$$

Obviously, $m\varphi \in C_0^\infty(\Omega)$. ∎

Corollary 2.12. *If $p \in \mathscr{P}(\Omega) \cap L^\infty(\Omega)$, then $L^{p(x)}(\Omega)$ is separable.*

Proof. Let G_n be bounded sets such that $G_n \subset G_{n+1} \subset \Omega$ for $n \in \mathbb{N}$ and $\Omega = \bigcup_n G_n$. Using the same considerations as in the proof of Theorem 2.11 we obtain that the set of all functions $m\chi_{G_n}$, where $n \in \mathbb{N}$ and m is a polynomial on $\mathbb{R}^N$ with rational coefficients, is dense in $L^{p(x)}(\Omega)$. ∎

3. GENERALIZED SOBOLEV SPACES

In this section we shall always assume that $\Omega \subset \mathbb{R}^N$ is a non-empty open set, $p \in \mathscr{P}(\Omega)$ and k is a given natural number. To avoid anyway rather complicated assumptions we shall consider only *bounded* domains Ω.

Given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}_0^N$, we set $|\alpha| = \alpha_1 + \ldots + \alpha_N$ and $D^\alpha = D_1^{\alpha_1} \ldots D_N^{\alpha_N}$, where $D_i = \partial/\partial x_i$ is the generalized derivative operator.

The *generalized Sobolev space* $W^{k,p(x)}(\Omega)$ is the class of all functions f on Ω such that $D^\alpha f \in L^{p(x)}(\Omega)$ for every multi-index α with $|\alpha| \leqq k$, endowed with the norm

$$\|f\|_{k,p} = \sum_{|\alpha| \leqq k} \|D^\alpha f\|_{k,p} . \tag{3.1}$$

By $W_0^{k,p(x)}(\Omega)$ we denote the subspace of $W^{k,p(x)}(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ with respect to the norm (3.1).

We can use the standard arguments to derive the following statement from Theorem 2.5 and Corollaries 2.7 and 2.12.

Theorem 3.1. *The spaces $W^{k,(p(x)}(\Omega)$ and $W_0^{k,p(x)}(\Omega)$ are Banach spaces, which are separable if $p \in L^\infty(\Omega)$ and reflexive if p satisfies (2.39).*

As a consequence of Theorem 2.8 we have:

(3.2) *If $q(x) \leqq p(x)$ for a.e. $x \in \Omega$, then $W^{k,p(x)}(\Omega) \hookrightarrow W^{k,q(x)}(\Omega)$.*

Besides this trivial embedding, it would be useful to know finer estimates of the type of Sobolev inequality. We may ask whether there exists the embedding

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega) ,$$

with

$$1/q(x) = 1/p(x) - 1/N , \quad x \in \Omega .$$

The following example shows that, in general, this we cannot expect.

Example 3.2. Let $N = 2$, $\Omega = \{x \in \mathbb{R}^2 : |x| < 1\}$, $1 < r < s < 2$ and $\sigma = 2(s - r)/r$. Denote by A the set of those points $x \in \Omega$, whose polar coordinates $t = |x|$, $\varphi = \arccos x_1$, satisfy the inequalities $0 < \varphi < t^\sigma$ and define

$$p(x) = \begin{cases} r & \text{if} \quad x \in \Omega \setminus A , \\ s & \text{if} \quad x \in A . \end{cases}$$

The Sobolev conjugate is

$$q(x) = \begin{cases} 2r/(2 - r) & \text{if} \quad x \in \Omega \setminus A , \\ 2s/(2 - s) & \text{if} \quad x \in A . \end{cases}$$

The function $f(x) = |x|^\mu$ with $\mu = (s - 2)/r$ belongs to $W^{1,p(x)}(\Omega)$, because

$$\varrho_p(f) = \int_A |x|^{\mu s} \, dx + \int_{\Omega\setminus A} |x|^{\mu r} \, dx < \int_0^1 t^{\mu s + \sigma + 1} \, dt + 2\pi \int_0^1 t^{\mu r + 1} \, dt ,$$

$$\varrho_p(\operatorname{grad} f) = |\mu|^s \int_A |x|^{(\mu - 1)s} \, dx + |\mu|^r \int_{\Omega\setminus A} |x|^{(\mu - 1)r} \, dx <$$

$$< \int_0^1 t^{(\mu - 1)s + \sigma + 1} \, dt + 2\pi \int_0^1 t^{(\mu - 1)r + 1} \, dt$$

and $\mu s + \sigma + 1 > 0$, $\mu r + 1 > 0$, $(\mu - 1) s + \sigma + 1 > -1$, $(\mu - 1) r + 1 > -1$. However,

$$\varrho_q(f) > 2\pi \int_0^1 t^{2s\mu/(2 - s) + \sigma + 1} \, dt = \infty ,$$

because $2s\mu/(2 - s) + \sigma + 1 = -1$, and so, $f \notin W^{1,q(x)}(\Omega)$.

The idea of the example lies in combination of two unfavourable properties of the function p: Its discontinuity and the non-regularity of the set of points of discontinuity. We shall show that for some classes of functions p it is possible to get the embedding $W^{k,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ with functions q approximating the Sobolev conjugate.

We shall say that the function $p \in \mathscr{P}(\Omega)$ is *$*$-continuous* on Ω if $\lim_{\substack{y \to x \\ y \in \Omega}} p(y) = p(x)$ for every $x \in \Omega$ (i.e. even if $p(x) = \infty$).

Given $k \in \mathbb{N}$, $k < N$, we define the function S,

$$S(t) = \frac{Nt}{N - kt} , \quad 1 \leqq t < N/k , \tag{3.3}$$

which associates the Sobolev conjugate $q = Np/(N - kp)$ with a number $p \in [1, N/k)$.

Theorem 3.3. *Let $k < N$ and let the function $p \in \mathscr{P}(\bar{\Omega})$ be continuous on $\bar{\Omega}$ and such that $p(x) < N/k$ for every $x \in \bar{\Omega}$. Then for every number $\varepsilon \in (0, k/(N - k))$ there exists a constant $c > 0$ such that*

$$\|f\|_q \leqq c\|f\|_{k,p} , \quad f \in W_0^{k,p(x)}(\Omega) , \tag{3.4}$$

where

$$1 \leqq q(x) \leqq S(p(x)) - \varepsilon , \quad x \in \Omega . \tag{3.5}$$

Proof. Without loss of generality we can suppose that p is continuous on $\mathbb{R}^N$ and that $\sup p(x) = \sup_\Omega p(x) = p^* < N/k$.

Let $0 < \varepsilon < k/(N-k)$. The function S defined in (3.3) is continuous, increasing and $S([1, N/k)) = [N/(N-k), \infty)$. Hence, for every $t \in [1, N/k)$ there exists a unique $s \in (t, \infty)$ such that $S(s) = S(t) + \varepsilon$ and we can find numbers

$$p_1 = 1 < p_2 < r_1 < p_3 < r_2 < \ldots < p_m < r_{m-1} < r_m = p^* \tag{3.6}$$

such that

$$S(p_i) \geqq S(r_i) - \varepsilon\,, \quad i = 1, \ldots, m\,. \tag{3.7}$$

Put $G_1 = p^{-1}([1, r_1))$, $G_i = p^{-1}((p_i, r_i))$ for $i = 2, \ldots, m-1$ and $G_m = p^{-1}((p_m, p^*])$. The sets G_i are open and $\bigcup_{i=1}^{m} G_i = \mathbb{R}^N$. Let $\varphi_i \in C_0^\infty(G_i)$, $i = 1, \ldots, m$, be such that $0 \leqq \varphi_i(x) \leqq 1$ for $x \in \mathbb{R}^N$ and $\sum_{i=1}^{m} \varphi_i(x) = 1$ for $x \in \bar{\Omega}$.

Let $f \in C_0^\infty(\Omega)$ and set $f_i = f\varphi_i$, $i = 1, \ldots, m$. Then $f_i \in C_0^\infty(G_i \cap \Omega)$ and, according to (3.5)–(3.7), for $x \in G_i$ we have

$$p_i \leqq p(x) \leqq r_i\,, \quad q(x) \leqq S(p_i)\,. \tag{3.8}$$

The usual Sobolev embedding theorem yields

$$W_0^{k,p_i}(G_i \cap \Omega) \hookrightarrow L^{S(p_i)}(G_i \cap \Omega)\,, \quad i = 1, \ldots, m\,. \tag{3.9}$$

Using Theorem 2.8 and inequalities (3.8) we successively obtain

$$\|f\|_q \leqq \sum_{i=1}^{m} \|f_i\|_q \leqq (1 + |\Omega|) \sum_{i=1}^{m} \|f_i\|_{S(p_i)} \leqq c_1(1 + |\Omega|) \sum_{i=1}^{m} \|f_i\|_{k,p_i} \leqq$$
$$\leqq c_1 c_2 m \varkappa (1 + |\Omega|)^2 \sum_{i=1}^{m} \|f_i\|_{k,p}\,,$$

where c_1 is the greatest of the constants of embeddings (3.9), $c_2 = \sup\{|D^\alpha \varphi_i(x)|: \alpha \in \mathbb{N}_0^N, |\alpha| \leqq k, i = 1, \ldots, m, x \in \Omega\}$ and $\varkappa = \#\{\alpha \in \mathbb{N}_0^N: |\alpha| \leqq k\}$.

Similar considerations lead to the following extended assertion.

Theorem 3.4. *Let $k < N$ and let function $p \in \mathscr{P}(\bar{\Omega})$ be $*$-continuous on $\bar{\Omega}$. Then for every $\varepsilon \in (0, k/(N-k))$ and $\eta \in (0, (N-k)/k)$ there exists a constant $c > 0$ such that* (3.4) *holds with q satisfying*

$$1 \leqq q(x) \leqq \begin{cases} \min\left\{S(p(x)) - \varepsilon,\ S\left(\dfrac{N}{k} - \eta\right)\right\}, & \text{if } p(x) < \dfrac{N}{k} + \eta\,, \\ \infty \quad \text{for other } x \in \Omega\,. \end{cases} \tag{3.10}$$

Moreover, every function $f \in W_0^{k,p(x)}(\Omega)$ is after a possible change on a set of zero measure continuous on $\{x \in \Omega: p(x) > N/k\}$.

Theorem 3.5. *Let the functions $p, q \in \mathscr{P}(\Omega)$ be $*$-continuous on Ω. If*

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)\,, \tag{3.11}$$

then

$$(3.12) \qquad \frac{1}{q(x)} \geqq \frac{1}{p(x)} - \frac{k}{N} \quad \textit{for a.e.} \quad x \in \Omega .$$

(Recall that we set $1/\infty = 0$.)

Proof. Note that $1/p$ and $1/q$ are continuous functions on Ω with values in $[0, 1]$. Suppose, that

$$\frac{1}{q(z)} < \frac{1}{p(z)} - \frac{k}{N}$$

for some $z \in \Omega$. Then $p(z) < N/k$ and there exist numbers $s \in (1, \infty)$, $r \in (1, N/k)$, and a ball $B = \{y: |y - z| < t\}$ such that

$$(3.13) \qquad \frac{1}{q(y)} < \frac{1}{s} < \frac{1}{r} - \frac{k}{N} < \frac{1}{p(y)} - \frac{k}{N}, \quad y \in B .$$

Then, by Theorem 2.8, $W_0^{k,r}(B) \hookrightarrow W_0^{k,p(x)}(B)$, $L^{q(x)}(B) \hookrightarrow L^s(B)$, and since the second inequality (3.13) yields $W_0^{k,r}(B) \setminus L^s(B) \neq \emptyset$, we have $W_0^{k,p(x)}(B) \setminus L^{q(x)}(B) \neq \emptyset$, which contradicts (3.11). ■

There is a gap between the necessary condition (3.12) and the sufficient condition (3.5) for the embedding $W_0^{k,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$. We could not fill it up and estimate the behaviour of the constant c from (3.4) (i.e. of the norm of the embedding operator) when $\varepsilon \to 0$ or $\eta \to 0$. Of course, the constants obtained in the proofs of Theorems 3.3 and 3.4 tend to infinity when $\varepsilon \to 0$ or $\eta \to 0$ because, in general, the number of intervals (p_i, r_i) and so the number of members of the partition of unity increases to infinity.

The idea of the proof of Theorem 3.3 indicates that "reasonable" functions p need not necessarily be continuous.

Theorem 3.6. (i) *If* $k > N$, *then* $W_0^{k,p(x)}(\Omega) \hookrightarrow C(\bar{\Omega})$.

(ii) *If* $k = N$ *and if there exists a number* $p_1 \in (1, \infty)$ *and open sets* $G_1, G_2 \subset \mathbb{R}^N$ *such that* $\bar{\Omega} \subset G_1 \cup G_2$ *and* $p(x) \geqq p_1$ *for a.e.* $x \in G_2 \cap \Omega$, *then the embedding* (3.4) *holds where* q *is an arbitrary function from* $\mathcal{P}(\Omega)$ *such that* $q_0 = \operatorname{ess\,sup}_{G_1 \cap \Omega} q(x) < \infty$.

(iii) *If* $k < N$ *and if there exists* $\varepsilon > 0$, *open sets* $G_i \subset \mathbb{R}^N$ *and numbers* p_i, r_i, $i = 1, \ldots, m$, *such that* $\bar{\Omega} \subset \bigcup_{i=1}^{m} G_i$,

$$(3.14) \qquad 1 = p_1 < p_2 < r_1 < p_3 < r_2 < \ldots$$
$$\ldots < p_{m-1} < r_{m-2} < N/k < p_m < r_{m-1} < r_m = \infty ,$$

$$(3.15) \qquad S(p_i) = S(r_i) - \varepsilon , \quad i = 1, \ldots, m - 1$$

and

$$(3.16) \qquad p_i \leqq p(x) \leqq r_i \quad \textit{for} \quad i = 1, \ldots, m \quad \textit{and for a.e.} \quad x \in G_i \cap \Omega ,$$

then there exists the embedding (3.4), *where*

$$(3.17)\qquad q(x) = \begin{cases} \min\{S(p(x)) - \varepsilon,\ S(p_{m-1})\} & \text{for } x \in \Omega \cap \bigcup_{i=1}^{m-1} G_i,\\ \infty \quad \text{for other } x \in \Omega. \end{cases}$$

Moreover, every function $f \in W_0^{k,p(x)}(\Omega)$ *is after a possible change on a set of zero measure continuous on the interior of the set* $\{x \in \Omega: p(x) > N/k\}$.

Proof. (i) We use Theorem 2.8 and the Sobolev embedding theorem to obtain $W_0^{k,p(x)}(\Omega) \hookrightarrow W_0^{k,1}(\Omega) \hookrightarrow C(\bar{\Omega})$.

(ii) We use a partition of unity on $\bar{\Omega}$ subordinated to the covering G_1, G_2, Theorem 2.8 and the embeddings $W_0^{k,1}(\Omega \cap G_1) \hookrightarrow L^{q_0}(\Omega \cap G_1)$, $W_0^{k,p_1}(\Omega \cap G_2) \hookrightarrow L^{\infty}(\Omega \cap G_2)$ which follow from the Sobolev embedding theorem.

(iii) The proof of this assertion is similar to that of Theorems 3.3 and 3.4. ∎

Theorem 3.7. (i) *If the sets* Ω *and* $\Omega \cap G_1$, $\Omega \cap G_2$, *and* $\Omega \cap G_i$, $i = 1, \ldots, m$, *consist of a finite number of components with Lipschitz boundary, then the assertions* (i), (ii) *and* (iii), *respectively, in Theorem* 3.6 *hold with* $W^{k,p(x)}(\Omega)$ *in place of* $W_0^{k,p(x)}(\Omega)$.

(ii) *If* $k = N$ *and if there exists a number* $p_1 \in (1, \infty)$ *and open sets* $G_1, G_2 \subset \Omega$ *consisting of a finite number of components with Lipschitz boundaries and such that* $|\Omega \setminus (G_1 \cup G_2)| = 0$ *and* $p(x) \geqq p_1$ *for a.e.* $x \in G_2 \cap \Omega$, *then* $W^{k,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, *where* $q \in \mathscr{P}(\Omega)$ *is such that* $\operatorname{ess\,sup} q(x) < \infty$.

(iii) *If* $k < N$ *and if there exist numbers* $\varepsilon > 0$, p_i, r_i *and open sets* $G_i \subset \Omega$, $i = 1, \ldots, m$, *consisting of a finite number of components with Lipschitz boundaries and such that* $|\Omega \setminus \bigcup_{i=1}^{m} G_i| = 0$ *and the relations* (3.14)–(3.16) *hold, then* $W^{k,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, *where* q *is defined in* (3.17).

Theorem 3.7 extends the class of functions $p \in \mathscr{P}(\Omega)$ for which a Sobolev type embedding theorem holds and allows to consider even the spaces $W^{k,p(x)}(\Omega)$. We shall omit the proof because it is analogous to the previous one. The difference consists in that we need not use a partition of unity. The assumption that the corresponding domains have Lipschitz boundaries enables us to use the Sobolev embedding theorem for $W^{k,p}$.

Given two Banach spaces X and Y the symbol $X \hookrightarrow\hookrightarrow Y$ means that there is a *compact embedding* of X in Y.

Theorem 3.8. (i) *If* $k > N$, *then* $W_0^{k,p(x)}(\Omega) \hookrightarrow\hookrightarrow C(\bar{\Omega})$.

(ii) *If the assumptions of Theorem* 3.6(ii) *are fulfilled, then* $W_0^{k,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$ *where* q *satisfies* $q_0 = \operatorname{ess\,sup}_{G_1 \cap \Omega} q(x) < \infty$.

(iii) *If the assumptions of Theorem* 3.6(iii) *are fulfilled, then* $W_0^{k,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{r(x)}(\Omega)$ *for every function* $r \in \mathscr{P}(\Omega)$ *such that* $r(x) \leqq q(x) - \eta$ *for a.e.* $x \in \Omega$, *where* $\eta > 0$ *and* q *satisfies* (3.17).

(iv) *If, moreover, the assumptions of Theorem* 3.7 *concerning the components* *with Lipschitz boundaries are satisfied, the assertions* (i)–(iii) *hold with* $W^{k,p(x)}(\Omega)$ *in place of* $W_0^{k,p(x)}(\Omega)$.

Proof. (i) Using Theorem 2.8 and the compact embedding theorem for usual Sobolev spaces we obtain

$$W_0^{k,p(x)}(\Omega) \hookrightarrow W_0^{k,1}(\Omega) \hookrightarrow\hookrightarrow C(\bar{\Omega}) .$$

(ii) Let $\varphi_i \in C_0^\infty(G_i)$, $i = 1, 2$, be such that $0 \leqq \varphi_i(x) \leqq 1$ for $x \in \mathbb{R}^N$ and $\varphi_1(x) + \varphi_2(x) = 1$ for $x \in \bar{\Omega}$. Let $\{f_n\}$ be a bounded sequence in $W_0^{k,p(x)}(\Omega)$. Then the sequences of functions $f_n^{(1)} = f_n\varphi_1$ and $f_n^{(2)} = f_n\varphi_2$ are bounded in $W_0^{k,p(x)}(\Omega \cap G_1)$ and $W_0^{k,p(x)}(\Omega \cap G_2)$, respectively. Since there is a compact embedding $W_0^{k,p(x)}(\Omega \cap G_1) \hookrightarrow\hookrightarrow L^{q_0}(\Omega \cap G_1)$, we can find a subsequence $\{f_{n_i}^{(1)}\}$ such that

$$(3.18) \qquad f_{n_i}^{(1)} \to f^{(1)} \quad \text{in} \quad L^{q_0}(\Omega \cap G_1) .$$

According to the compact embedding $W_0^{k,p(x)}(\Omega \cap G_2) \hookrightarrow\hookrightarrow L^\infty(\Omega \cap G_2)$ we can find a subsequence $\{f_{n_{i_j}}^{(2)}\}$ of $\{f_{n_i}^{(2)}\}$ such that

$$(3.19) \qquad f_{n_{i_j}}^{(2)} \to f^{(2)} \quad \text{in} \quad L^\infty(\Omega \cap G_2) .$$

We extend the functions $f^{(1)}$ and $f^{(2)}$ by zero into $\mathbb{R}^N$ and put $f = f^{(1)} + f^{(2)}$. Then, by Theorem 2.8,

$$\|f - f_{n_{i_j}}\|_q \leqq (|\Omega| + 1)\left(\|f^{(1)} - f_{n_{i_j}}^{(1)}\|_{q_0} + \|f^{(2)} - f_{n_{i_j}}^{(2)}\|_\infty\right),$$

which together with (3.18) and (3.19) yields $f_{n_{i_j}} \to f$ in $L^{q(x)}(\Omega)$.

(iii) Set $G = \Omega \cap \bigcup_{i=1}^{m-1} G_i$. It suffices to prove that

$$W_0^{k,p(x)}(G) \hookrightarrow\hookrightarrow L^{r(x)}(G) .$$

Then the proof can be finished in a similar way as in the case (ii), because $p(x) \geqq p_m > N/k$ for a.e. $x \in G_m$ and so $W_0^{k,p(x)}(G_m \cap \Omega) \hookrightarrow\hookrightarrow L^\infty(G_m \cap \Omega)$.

Let $\{f_n\}$ be a bounded sequence in $W_0^{k,p(x)}(\Omega)$. By Theorem 3.6(iii), there is the embedding $W_0^{k,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, and so

$$(3.20) \qquad \|f_n\|_q \leqq K$$

for some $K > 0$. Since $W_0^{k,1}(G) \hookrightarrow\hookrightarrow L^1(G)$ and $W_0^{k,p(x)}(G) \hookrightarrow W_0^{k,1}(G)$, we have $W_0^{k,p(x)}(G) \hookrightarrow\hookrightarrow L^1(G)$. Hence, the sequence $\{f_n\}$ contains a subsequence which is Cauchy in $L^1(G)$. We shall denote it again with $\{f_n\}$. By Corollary 2.2,

$$(3.21) \qquad \|f_m - f_n\|_r \leqq c\|f_m - f_n\|_1^\mu \|f_m - f_n\|_q^\nu ,$$

where the numbers μ, ν satisfy the estimates

$$\mu \geqq \eta\, S(p_{m-1})^{-2} > 0 , \quad 0 \leqq \nu \leqq 1 .$$

Hence, from (3.20), (3.21) we obtain

$$\|f_m - f_n\|_r \leqq c \max\{1, 2K\} \|f_m - f_n\|_1^\mu ,$$

and since $\{f_n\}$ is Cauchy in $L^1(G)$, it is also Cauchy in $L^{r(x)}(G)$. ■

The proof of the following theorem is similar.

Theorem 3.9. *If the assumptions of Theorem* 3.3 *are satisfied, then for every* $\varepsilon \in (0, k/(N-k))$ *the compact embedding* $W_0^{k,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$ *holds with* q *satisfying* (3.5). *If the assumptions of Thoerem* 3.4 *are satisfied, then for every* $\varepsilon \in (0, k/(N-k))$ *and* $\eta \in (0, (N-k)/k)$ *the compact embedding* $W_0^{k,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$ *holds with* q *satisfying* (3.10).

Theorem 3.10. *Let* Ω, p *and* k *satisfy some of the following assumptions*:

(i) $k > N$;

(ii) $k = N$ *and there exists a number* $p_1 \in (1, \infty)$ *and open sets* $G_1, G_2 \subset \mathbb{R}^N$ *such that* $\bar{\Omega} \subset G_1 \cup G_2$ *and* $p(x) \geqq p_1$ *for a.e.* $x \in G_1 \cap \Omega$;

(iii) $k = N$ *and there exists* $p_1 \in (1, \infty)$ *and open sets* $G_1, G_2 \subset \Omega$ *consisting of finite number of components with Lipschitz boundaries and such that* $|\Omega \setminus (G_1 \cup G_2)| = 0$ *and* $p(x) \geqq p_1$ *for a.e.* $x \in G_2 \cap \Omega$;

(iv) $k < N$ *and there exist numbers* p_i, q_i *and open sets* G_i, $i = 1, \ldots, m$, *satisfying* (3.14), (3.16) *and such that* $\bar{\Omega} \subset \bigcup_{i=1}^{m} G_i$ *and*

$$r_i < S(p_i)\,, \quad i = 1, \ldots, m-1\,; \tag{3.22}$$

(v) $k < N$ *and there exist numbers* p_i, q_i *and open sets* $G_i \subset \Omega$, $i = 1, \ldots, m$, *consisting of finite number of components with Lipschitz boundaries and such that* $|\Omega \setminus \bigcup_{i=1}^{m} G_i| = 0$ *and the inequalities* (3.14), (3.16) *and* (3.22) *hold*;

(vi) p *is* $*$*-continuous on* $\bar{\Omega}$.

Then $W_0^{k,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{p(x)}(\Omega)$ *and*

$$]\!]f[\![_{k,p} = \sum_{|\alpha|=k} \|D^\alpha f\|_p \tag{3.23}$$

is an equivalent norm in $W_0^{k,p(x)}(\Omega)$.

Proof. The compact embedding $W_0^{k,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{p(x)}(\Omega)$ can be proved in the same way as Theorem 3.8. Note that in cases (iv) and (v) we have the inequalities

$$p_i \leqq p(x) \leqq r_i < S(p_i) \quad \text{for a.e.} \quad x \in G_i\,, \tag{3.24}$$

which play the role of (3.8) in the proof of Theorem 3.3. Since the last inequality (3.24) is strict, we obtain the compact embedding (cf. the proof of Theorem 3.8(ii)). If $p \in \mathscr{P}(\Omega) \cap C(\bar{\Omega})$, then some of the assumptions (i), (ii), (iv) is satisfied.

Obviously, (3.23) is a seminorm in $W_0^{k,p(x)}(\Omega)$ satisfying $]\!]f[\![_{k,p} \leqq \|f\|_{k,p}$. The converse inequality $\|f\|_{k,p} \leqq c]\!]f[\![_{k,p}$ can be proved in a standard way with use of the compact embedding $W_0^{k,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{p(x)}(\Omega)$. ∎

Since the equivalent norm (3.23) plays an important role in applications, we shall prove yet another assertion of that type which extends the class of admissible functions $p \in \mathscr{P}(\Omega)$.

Theorem 3.11. *Let Ω and G be domains in $\mathbb{R}^N$, $|\Omega| < \infty$ and let $p \in \mathscr{P}(\Omega) \cap L^\infty(\Omega)$. Put $G' = \{t' \in \mathbb{R}^{N-1}: (t', \tau) \in G$ for some $\tau \in \mathbb{R}\}$ and $G(t') = \{\tau \in \mathbb{R}: (t', \tau) \in G\}$ for $t' \in \mathbb{R}^{N-1}$. Suppose that $|G(t')| \leqq A < \infty$ for $t' \in G'$ and that there exists a one-to-one mapping $\Phi: G \to \Omega$ satisfying the conditions for the change of variables in Lebesgue integral and such that*

$$(3.25) \qquad 0 < \inf_G |J_\Phi(t)| \leqq \sup_G |J_\Phi(t)| < \infty ,$$

where J_Φ is the Jacobian of Φ,

$$(3.26) \qquad D_N\Phi \in L^\infty(G) ,$$

$$(3.27) \qquad p(\Phi(t', \tau)) = q(t') \quad \text{for a.e.} \quad (t', \tau) \in G .$$

Then (3.23) is an equivalent norm in $W_0^{k,p(x)}(\Omega)$.

Proof. We shall consider $k = 1$. For $k > 1$ the proof can be accomplished by induction. Let $f \in C_0^\infty(\Omega)$, $f \neq 0$, and put $\lambda = \|f\|_p$. For $t' \in G'$ satisfying (3.27) and for a.e. $\tau \in \mathbb{R}$ we have

$$f(\Phi(t', \tau)) = \int_{-\infty}^{\tau} \frac{\partial}{\partial \xi} f(\Phi(t', \xi))\, d\xi = \int_{-\infty}^{\tau} \operatorname{grad} f(\Phi(t', \xi))\, D_N\Phi(t', \xi)\, d\xi ,$$

and so, using the Hölder inequality, (3.26) and (3.27), we obtain

$$|f(\Phi(t))|^{q(t')} \leqq \max\{\| |D_N\Phi| \|_\infty^{p^*}, 1\} \max\{A^{p^*-1}, 1\} \, . \\ \cdot \int_{G(t')} |\operatorname{grad} f(\Phi(t', \tau))|^{q(t')}\, d\tau .$$

Hence, according to (2.10) and (3.25),

$$\begin{aligned} 1 &= \int_\Phi |f(x)/\lambda|^{p(x)}\, dx = \int_G |f(\Phi(t))/\lambda|^{p(\Phi(t))}\, |J_\Phi(t)|\, dt \leqq \\ &\leqq \sup_G |J_\Phi(t)| \int_G |f(\Phi(t', \tau))/\lambda|^{q(t')}\, d\tau\, dt' \leqq \\ &\leqq c_1 \int_{G'} \int_{G(t')} \int_{G(t')} \left| \frac{|\operatorname{grad} f(\Phi(t', \xi))|}{\lambda} \right|^{q(t')} d\xi\, d\tau\, dt' \leqq \\ &\leqq c_1 A (\inf_G |J_\Phi(t)|)^{-1} \int_G \left| \frac{|\operatorname{grad} f(\Phi(t))|}{\lambda} \right|^{p(\Phi(t))} |J_\Phi(t)|\, dt = \\ &= c_2 \int_\Omega \left| \frac{|\operatorname{grad} f(x)|}{\lambda} \right|^{p(x)} dx , \end{aligned}$$

where $c_1 = \sup_G |J_\Phi(t)| \max\{\| |D_N\Phi| \|_\infty^{p^*}, 1\} \max\{A^{p^*-1}, 1\}$. Putting $c = \max\{c_2, 1\}$, we obtain

$$1 \leqq \int_\Omega \left| \frac{c|\operatorname{grad} f(x)|}{\lambda} \right|^{p(x)} dx ,$$

i.e.,

$$]\!]f[\![_{k,p} \geqq \| |\operatorname{grad} f| \|_p \geqq c^{-1}\lambda = c^{-1}\|f\|_p . \quad \blacksquare$$

Example 3.12. Let Ω, G_1, G_2 be domains in $\mathbb{R}^N$ such that $G_1 \cup G_2 \subset \Omega$, $|\Omega \setminus (G_1 \cup G_2)| = 0$ and G_1, G_2 have Lipschitz boundaries. Let $1 \leqq p_1 < p_2 \leqq \infty$ and set

$$p(x) = \begin{cases} p_1 & \text{for} \quad x \in G_1 , \\ p_2 & \text{for} \quad x \in G_2 , \end{cases} \qquad q(x) = \begin{cases} q_1 & \text{for} \quad x \in G_1 , \\ q_2 & \text{for} \quad x \in G_2 , \end{cases}$$

where $q_i = S(p_i) = Np_i/(N - kp_i)$ if $p_i < N/k$, $q_i \in [1, \infty)$ arbitrary if $p_i = N/k$, and $q_i = \infty$ if $p_i > N/k$, $i = 1, 2$. Then

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$$

by Theorem 3.7,

$$W^{k,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{r(x)}(\Omega)$$

for every $r \in \mathscr{P}(\Omega)$ with $\operatorname{ess\,inf}_{\Omega}(q(x) - r(x)) > 0$ by Theorem 3.8, and $]\!]\cdot[\![_{k,p}$ is an equivalent norm in $W_0^{k,p(x)}(\Omega)$ by Theorem 3.10.

Example 3.13. Let $\Omega = (0, 1) \times (0, 1)$, let $p \in \mathscr{P}(\Omega)$ and $q \in L^\infty(0, 1)$ satisfy $1 \leqq \leqq p(x_1, x_2) = q(x_1) < \infty$ for a.e. $(x_1, x_2) \in \Omega$. Set $G = \Omega$ and $\Phi(t) = t$, $t \in G$. Then the assumptions of Theorem 3.11 are satisfied and so $]\!]\cdot[\![_{k,p}$ is an equivalent norm in $W_0^{k,p(x)}(\Omega)$.

Example 3.14. Let $\Omega = \{x \in \mathbb{R}^2 \colon 1/2 < |x| < 1\}$ and let $p \in \mathscr{P}(\Omega) \cap L^\infty(\Omega)$ be 0-homogeneous, i.e. $p(\lambda x) = p(x)$ for every $\lambda > 0$. Then $\Omega_1 = \{x \in \mathbb{R}^2 \colon \frac{1}{4} < |x| < 2, x_1 < |x_2|\}$, $\Phi(t_1, t_2) = t_2(\cos t_1, \sin t_1)$, $G_1 = (\frac{1}{4}\pi, \frac{7}{4}\pi) \times (\frac{1}{4}, 2)$, and $\Omega_2 = = \{x \in \mathbb{R}^2 \colon \frac{1}{4} < |x| < 2, x_1 > -|x_2|\}$, Φ, $G_2 = (-\frac{3}{4}\pi, \frac{3}{4}\pi) \times (\frac{1}{4}, 2)$ satisfy the assumptions of Theorem 3.11 with $q(t_1) = p(\cos t_1, \sin t_1)$. Hence, using the partition of unity we easily obtain that $W_0^{k,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{p(x)}(\Omega)$ and $]\!]\cdot[\![_{k,p}$ is an equivalent norm in $W_0^{k,p(x)}(\Omega)$.

Example 3.15. Let $\Omega = \{x \in \mathbb{R}^2 \colon 1/2 < |x| < 1\}$ and let $p \in \mathscr{P}(\Omega) \cap L^\infty(\Omega)$ be such that $p(x) = \tilde{p}(|x|)$ for $x \in \Omega$. Let Ω_1 and Ω_2 be as in Example 3.14 and set $\Phi_i(t) = = t_1(\cos t_2, \sin t_2)$, $i = 1, 2$, $G_1 = (\frac{1}{4}, 2) \times (\frac{1}{4}\pi, \frac{7}{4}\pi)$, $G_2 = (\frac{1}{4}, 2) \times (-\frac{3}{4}\pi, \frac{3}{4}\pi)$. Then the assumptions of Theorem 3.11 are satisfied with $q = \tilde{p}$ and, consequently, $W_0^{k,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{p(x)}(\Omega)$ and $]\!]\cdot[\![_{k,p}$ is an equivalent norm in $W_0^{k,p(x)}(\Omega)$.

We can use Theorems 2.1, 2.6 and the standard reasoning to obtain the following characterization of the dual space $(W_0^{k,p(x)}(\Omega))^*$:

Theorem 3.16. *Let $p \in \mathscr{P}(\Omega) \cap L^\infty(\Omega)$. Then for every $G \in (W_0^{k,p(x)}(\Omega))^*$ there exists a unique system of functions $\{g_\alpha \in L^{p'(x)}(\Omega) \colon |\alpha| \leqq k\}$ such that*

$$G(f) = \sum_{|\alpha| \leqq k} \int_\Omega D^\alpha f(x)\, g_\alpha(x)\, \mathrm{d}x\,, \quad f \in W_0^{k,p(x)}(\Omega)\,.$$

In this section we shall show a general scheme of application of spaces $W^{k,p(x)}(\Omega)$ to a Dirichlet boundary value problems for nonlinear partial differential equations with coefficients of a variable growth. We shall assume that $\Omega \subset \mathbb{R}^N$ is a *non-empty open set*, $p \in \mathscr{P}(\Omega)$ and k is a given natural number. Given $u \in W^{k,p(x)}(\Omega)$ we shall write $\delta_k u = \{D^\alpha u : \alpha \in \mathbb{N}_0^N, |\alpha| \leqq k\}$.

First, we shall investigate the Nemyckii operators in $W^{k,p(x)}(\Omega)$. We shall say that a function $h: \Omega \times \mathbb{R}^m \to \mathbb{R}$, $m \in \mathbb{N}$, *satisfies the Carathéodory conditions*, $h \in \mathrm{CAR}(\Omega, m)$, if for every $\xi \in \mathbb{R}^m$ the function $h(\cdot, \xi)$ is measurable on Ω and if for a.e. $x \in \Omega$ the function $h(x, \cdot)$ is continuous on $\mathbb{R}^m$. Every function $h \in \mathrm{CAR}(\Omega, m)$, generates a *Nemyckii operator H* which maps an m-tuple $(u_1, \ldots, u_m)$ of functions on Ω onto

$$H(u_1, \ldots, u_m)(x) = h(x, u_1(x), \ldots, u_m(x)), \quad x \in \Omega .$$

We shall use the convention $1/\infty = 0$ and $0 \,.\, \infty = 1$.

Theorem 4.1. *Let $h \in \mathrm{CAR}(\Omega, m)$ and let functions $p_i, r \in \mathscr{P}(\Omega)$ be finite a.e. in Ω and such that* $\operatorname{ess\,sup}_\Omega (p_i(x) - r(x)) = \beta_i < \infty$, *$i = 1, \ldots, m$. If there exists a non-negative function $g \in L^{r(x)}(\Omega)$ and a constant $c > 0$ such that*

$$|h(x, \xi)| \leqq g(x) + c \sum_{i=1}^{m} |\xi_i|^{p_i(x)/r(x)}$$

for every $\xi \in \mathbb{R}^m$ and a.e. $x \in \Omega$, then the Nemyckii operator H maps the space $\Lambda_m(\Omega) = L^{p_1(x)}(\Omega) \times \ldots \times L^{p_m(x)}(\Omega)$ in $L^{r(x)}(\Omega)$.

Proof. Let $u = (u_1(x), \ldots, u_m(x)) \in \Lambda_m(\Omega)$ and let $\lambda_i \in (0, 1]$ be such that $\varrho_{p_i}(\lambda_i u_i) < \infty$, $i = 1, \ldots, m$, and $\varrho_r(\lambda_{m+1} g/c) < \infty$. Set $\lambda = \min\{\lambda_i : i = 1, \ldots, m + 1\}$. Then

$$\varrho_r\left(\frac{\lambda u}{c(m+1)}\right) = \int_\Omega (m+1)^{-r(x)} \left(\frac{\lambda}{c} g(x) + \sum_{i=1}^{m} \lambda |u_i(x)|^{p_i(x)/r(x)}\right)^{r(x)} \mathrm{d}x \leqq$$

$$\leqq \int_\Omega \left(\left|\frac{\lambda}{c} g(x)\right|^{r(x)} + \sum_{i=1}^{m} \lambda^{r(x)} |u_i(x)|^{p_i(x)}\right) \mathrm{d}x \leqq$$

$$\leqq \int_\Omega |\lambda_{m+1}\, g(x)/c|^{r(x)} \,\mathrm{d}x + \sum_{i=1}^{m} \int_\Omega \lambda_i^{r(x) - p_i(x)} |\lambda_i\, u_i(x)|^{p_i(x)} \,\mathrm{d}x \leqq$$

$$\leqq \varrho_r(\lambda_{m+1} g/c) + \sum_{i=1}^{m} \lambda_i^{-\beta_i} \varrho_{p_i}(\lambda_i u_i) < \infty . \quad \blacksquare$$

Theorem 4.2. *Let $p_i, r \in \mathscr{P}(\Omega) \cap L^\infty(\Omega)$, $i = 1, \ldots, m$. If the Nemyckii operator H maps $\Lambda_m(\Omega)$ in $L^{r(x)}(\Omega)$, then it is continuous and bounded.*

Proof. It is sufficient to assume that $H(0) = 0$ and to prove the continuity and boundedness of H in $0 \in \Lambda_m(\Omega)$.

Suppose that H is not continuous in 0. Then, according to Theorem 2.4, there exists $\alpha > 0$ and functions $\varphi_n \in \Lambda_m(\Omega)$, $n \in \mathbb{N}$, such that

$$(4.1) \qquad \sum_{n=1}^{\infty} \varrho_{p_i}(\varphi_{n,i}) < \infty\,, \quad i = 1, \ldots, m\,,$$

$$(4.2) \qquad \varrho_r(H(\varphi_n)) > \alpha\,, \quad n \in \mathbb{N}\,.$$

First, suppose that $|\Omega| < \infty$. Using the induction we shall construct sequences of numbers $\varepsilon_k > 0$, $n_k \in \mathbb{N}$, and of sets $G_k \subset \Omega$ such that for every $k \in \mathbb{N}$ the following conditions hold:

$$(4.3) \qquad \varepsilon_{k+1} < \tfrac{1}{2}\varepsilon_k\,,$$

$$(4.4) \qquad |G_k| \leqq \varepsilon_k\,,$$

$$(4.5) \qquad \int_{G_k} |H(\varphi_{n_k})(x)|^{r(x)}\,\mathrm{d}x > \tfrac{2}{3}\alpha$$

and

$$(4.6) \qquad \int_D |H(\varphi_{n_k})(x)|^{r(x)}\,\mathrm{d}x < \tfrac{1}{3}\alpha \quad \text{for every} \quad D \subset \Omega\,, \quad |D| \leqq 2\varepsilon_{k+1}\,.$$

Put $\varepsilon_1 = |\Omega|$, $n_1 = 1$ and $G_1 = \Omega$. Suppose we have already found ε_k, n_k and G_k. Since the function $|H(\varphi_{n_k})|^r$ is integrable on Ω, there exists $\varepsilon_{k+1} > 0$ such that (4.6) holds. If $\varepsilon_k \leqq 2\varepsilon_{k+1}$, then $|G_k| \leqq 2\varepsilon_{k+1}$ and (4.6) with $D = G_k$ contradicts (4.5). Thus, (4.3) holds.

According to (4.1), $\|\varphi_{n,i}\|_{p_i} \to 0$ for $i = 1, \ldots, m$ and so, by (2.34), $\varphi_{n,i} \to 0$ in measure. Since the Nemyckii operator is continuous with respect to the convergence in measure (see [4]), we have $H(\varphi_n) \to 0$ in measure and there exists $n_{k+1} > n_k$ such that

$$|G_{k+1}| \leqq \varepsilon_{k+1}\,,$$

where

$$G_{k+1} = \left\{x \in \Omega \colon |H(\varphi_{n_{k+1}})(x)|^{r(x)} \geqq \frac{\alpha}{3|\Omega|}\right\}.$$

Hence, (4.4) holds as well as (4.5), because

$$\int_{G_{k+1}} |H(\varphi_{n_{k+1}})(x)|^{r(x)}\,\mathrm{d}x =$$
$$= \int_\Omega |H(\varphi_{n_{k+1}})(x)|^{r(x)}\,\mathrm{d}x - \int_{\Omega\setminus G_{k+1}} |H(\varphi_{n_{k+1}})(x)|^{r(x)}\,\mathrm{d}x > \alpha - \alpha/3 = \tfrac{2}{3}\alpha\,.$$

Now, the sets $D_k = G_k \setminus \bigcup_{i=k+1}^{\infty} G_i$ are pairwise disjoint and from (4.3), (4.4) and (4.6) we have

$$|G_k \setminus D_k| \leqq \Big|\bigcup_{i=k+1}^{\infty} G_i\Big| \leqq \sum_{i=k+1}^{\infty} \varepsilon_i < 2\varepsilon_{k+1}$$

and

$$(4.7) \qquad \int_{G_k\setminus D_k} |H(\varphi_{n_k})(x)|^{r(x)}\,\mathrm{d}x < \alpha/3\,.$$

The function $\psi = \sum_{k=1}^{\infty} \varphi_{n_k}\chi_{D_k}$ belongs to $\Lambda_m(\Omega)$ by (4.1). On the other hand, by (4.5) and (4.7), for $\lambda > 0$ we have

$$\int_\Omega |\lambda\, H(\psi)(x)|^{r(x)}\, dx \geqq \min\{1, \lambda^{r^*}\} \sum_{k=1}^{\infty} \int_{D_k} |H(\varphi_{n_k})(x)|^{r(x)}\, dx \geqq$$

$$\geqq \min\{1, \lambda^{r^*}\} \times$$

$$\times \sum_{k=1}^{\infty} [\int_{G_k} |H(\varphi_{n_k})(x)|^{r(x)}\, dx - \int_{G_k \setminus D_k} |H(\varphi_{n_k})(x)|^{r(x)}\, dx] = \infty\,,$$

i.e. $H(\psi) \notin L^{r(x)}(\Omega)$, which contradicts the assumption of the theorem. Thus, the continuity of H is proved in the case $|\Omega| < \infty$.

Now, let $|\Omega| = \infty$. By induction we construct an increasing sequence $\{n_k\}$ and pairwise disjoint sets $D_k \subset \Omega$ such that

$$(4.8) \qquad |D_k| < \infty\,, \quad \int_{D_k} |H(\varphi_{n_k})(x)|^{r(x)}\, dx > \alpha/2$$

for $k \in \mathbb{N}$. We set $n_1 = 1$ and, according to (4.2) find a set D_1 satisfying (4.8). Suppose we have already found n_k and D_k. Then $|\bigcup_{i=1}^{k} D_i| < \infty$ and by the first part of the proof there exists $n_{k+1} > n_k$ such that

$$\int_{\bigcup_{i=1}^{k} D_i} |H(\varphi_{n_{k+1}})(x)|^{r(x)}\, dx < \alpha/2\,.$$

By (4.2) there exists $G_{k+1} \subset \Omega$ with a finite measure and such that

$$\int_{G_{k+1}} |H(\varphi_{n_{k+1}})(x)|^{r(x)}\, dx > \alpha\,.$$

Hence, we put $D_{k+1} = G_{k+1} \setminus \bigcup_{i=1}^{k} D_i$. The function $\psi = \sum_{k=1}^{\infty} \varphi_{n_k} \chi_{D_k}$ satisfies $\psi \in \Lambda_m(\Omega)$ and $H(\psi) \notin L^{r(x)}(\Omega)$ which contradicts the assumption of the theorem. Thus H is continuous again.

Since H is continuous, by (2.11) there exists a number $R > 0$ such that

$$(4.9) \qquad \varrho_r(H(\varphi)) \leqq 1 \quad \text{if} \quad \|\varphi\|_{\Lambda_m} \leqq R\,.$$

Let $u = (u_1, \ldots, u_m) \in \Lambda_m(\Omega)$, $u \neq 0$, and let $a > 0$ be such that $aR < \|u\|_{\Lambda_m} \leqq \leqq (a + 1) R$. Then for every $i = 1, \ldots, m$,

$$\varrho_{p_i}(R^{-1} m u_i) \leqq [m(a + 1)]^{p_i^*}\, \varrho_{p_i}(R^{-1}(a + 1)^{-1} u_i) \leqq$$

$$\leqq [m(R^{-1} \|u\|_{\Lambda_m} + 1)]^{p_i^*}$$

and there exists $k_i \in \mathbb{N}$ satisfying

$$[m(R^{-1} \|u\|_{\Lambda_m} + 1)]^{p_i^*} \leqq k_i < [m(R^{-1} \|u\|_{\Lambda_m} + 1)]^{p_i^*} + 1$$

and such that

$$\varrho_p(R^{-1} m u_i) \leqq k_i\,.$$

Thus, there exist sets G_j^i, $j = 1, \ldots, k_i$, such that $\Omega = \bigcup_{j=1}^{k_i} G_j^i$ and

$$(4.10) \qquad \int_{G^i_j} |m\, u_i(x)\, R^{-1}|^{p_i(x)}\, dx \leqq 1\,, \quad j = 1, \ldots, k_i\,, \quad i = 1, \ldots, m\,.$$

For every m-tuple $l = (l_1, \ldots, l_m) \in \prod_{i=1}^{m} \{1, \ldots, k_i\}$ we put $G_l = \bigcap_{i=1}^{m} G^i_{l_i}$ and $u_l = u \chi_{G_l}$.

From (4.10) we have $\varrho_{p_i}(mu_{l,i}/R) \leqq 1$, i.e. $\|u_l\|_{\Lambda_m} \leqq R$. This and (4.9) yield

$$\varrho_r(H(u)) \leqq \sum_l \int_{G_l} |H(u)(x)|^{r(x)}\,dx \leqq \sum_l \varrho_r(H(u_l)) \leqq \prod_{i=1}^m k_i \leqq$$

$$\leqq \prod_{i=1}^m \{[m(R^{-1}\|u\|_{\Lambda_m} + 1)]^{p_i^*} + 1\} := K(\|u\|_{\Lambda_m}).$$

Thus, $\|H(u)\|_r \leqq K(1)$ for $\|u\|_{\Lambda_m} \leqq 1$, and the operator H is bounded. ∎

As an immediate consequence we obtain:

Theorem 4.3. *Let the functions $p, r \in \mathscr{P}(\Omega)$ be finite a.e. in Ω and such that $\operatorname{ess\,sup}_\Omega (p(x) - r(x)) < \infty$. Let $m = \#\{\alpha \in \mathbb{N}_0^N: |\alpha| \leqq k\}$ and let $h \in \mathrm{CAR}(\Omega, m)$ be such that the inequality*

$$|h(x, \xi)| \leqq g(x) + c \sum_{|\alpha| \leqq k} |\xi_\alpha|^{p(x)/r(x)}$$

with some $c > 0$ and $g \in L^{r(x)}(\Omega)$ holds for every $\xi \in \mathbb{R}^m$ and a.e. $x \in \Omega$. Then the operator $H: u \mapsto h(x, \delta_k u(x))$ maps the space $W^{k,p(x)}(\Omega)$ in $L^{r(x)}(\Omega)$. If, moreover, $p, r \in L^\infty(\Omega)$, then H is continuous and bounded.

Corollary 4.4. *Let $p \in \mathscr{P}(\Omega)$ satisfy (2.39). Let $h \in \mathrm{CAR}(\Omega, m)$, $m = \#\{\alpha \in \mathbb{N}_0^N: |\alpha| \leqq k\}$, and let $g \in L^{p'(x)}(\Omega)$ and $c > 0$ be such that*

$$|h(x, \xi)| \leqq g(x) + c \sum_{|\alpha| \leqq k} |\xi_\alpha|^{p(x)-1} \tag{4.11}$$

holds for every $\xi \in \mathbb{R}^m$ and a.e. $x \in \Omega$. Let $\alpha \in \mathbb{N}_0^N$, $|\alpha| \leqq k$. Then the operator $T_\alpha: W^{k,p(x)}(\Omega) \to (W^{k,p(x)}(\Omega))^*$ defined by $T_\alpha u(v) = \int_\Omega h(x, \delta_k u(x))\, D^\alpha v(x)\,dx$, $u, v \in W^{k,p(x)}(\Omega)$, is continuous and bounded.

Proof. We use Theorem 4.3 with $r = p'$ to obtain $H(u) \in L^{p'(x)}(\Omega)$. Theorem 2.1 yields

$$|T_\alpha u(v)| \leqq r_p \|H(u)\|_{p'} \|D^\alpha v\|_p \leqq r_p \|H(u)\|_{p'} \|v\|_{k,p}$$

and so $T_\alpha u \in (W^{k,p(x)}(\Omega))^*$. Since $p, p' \in L^\infty(\Omega)$, the continuity and boundedness of the operator T_α follows from Theorem 4.3. ∎

We are now ready to show an application of generalized Sobolev spaces $W^{k,p(x)}(\Omega)$ to Dirichlet boundary value problems for partial differential equations.

Boundary value problem. Consider a differential operator A of order $2k$ in the divergence form,

$$A\,u(x) = \sum_{|\alpha| \leqq k} (-1)^{|\alpha|} D^\alpha a_\alpha(x, \delta_k u(x)),$$

where the functions $a_\alpha \in \mathrm{CAR}(\Omega, m)$, $m = \#\{\alpha \in \mathbb{N}_0^N: |\alpha| \leqq k\}$, fulfill the growth condition (4.11) with $g \in L^{ps(x)}(\Omega)$ and $c > 0$. Let Q be a Banach space of functions on Ω equipped with a norm $\|\cdot\|_Q$ and such that $C_0^\infty(\Omega)$ is dense in Q and, moreover, $W_0^{k,p(x)}(\Omega) \hookrightarrow Q$. By Theorem 2.11, for Q we can take e.g. $L^{p(x)}(\Omega)$. Let $f \in Q^*$ and $u_0 \in$

$\in W^{k,p(x)}(\Omega)$ and denote by $\langle\cdot,\cdot\rangle_Q$ the duality on Q. A function $u \in W^{k,p(x)}(\Omega)$ is a *weak solution to the Dirichlet boundary value problem* (A, u_0, f) *for the equation*

$$Au = f$$

with the boundary condition given by u_0, if

$$u - u_0 \in W_0^{k,p(x)}(\Omega)$$

and if the identity

$$\sum_{|\alpha| \leqq k} \int_\Omega a_\alpha(x, \delta_k u(x))\, D^\alpha v(x)\,\mathrm{d}x = \langle f, v\rangle$$

holds for every $v \in W_0^{k,p(x)}(\Omega)$.

From Corollary 4.4 we obtain that the operator $T\colon W_0^{k,p(x)}(\Omega) \to (W_0^{k,p(x)}(\Omega))^*$ defined by

$$\langle\!\langle Tw, v\rangle\!\rangle = \sum_{|\alpha| \leqq k} \int_\Omega a_\alpha(x, \delta_k[w(x) + u_0(x)])\, D^\alpha v(x)\,\mathrm{d}x\,, \tag{4.12}$$

where $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ stands for the duality in $W_0^{k,p(x)}(\Omega)$, is continuous and bounded. It is clear that $u \in (A, u_0, f)$ if and only if $u = u_0 + w$ and $Tw = f$.

Theorem 4.5. *Let* $p \in \mathscr{P}(\Omega)$ *satisfy* (2.39). *Let the functions* a_α *satisfy* (4.11) *and let for every* $\xi, \eta \in \mathbb{R}^m$ *and for a.e.* $x \in \Omega$ *the conditions*

$$\sum_{|\alpha| \leqq k} [a_\alpha(x, \xi) - a_\alpha(x, \eta)]\,(\xi_\alpha - \eta_\alpha) \geqq 0\,, \tag{4.13}$$

$$\sum_{|\alpha| \leqq k} a_\alpha(x, \xi)\,\xi_\alpha \geqq c_1 \sum_{|\alpha| \leqq k} |\xi_\alpha|^{p(x)} - c_2 \tag{4.14}$$

hold with some constants $c_1, c_2 > 0$. *Then the boundary value problem* (A, u_0, f) *has at least one weak solution* $u \in W^{k,p(x)}(\Omega)$. *If, moreover, the inequality* (4.13) *is strict for* $\xi \neq \eta$, *then the solution is unique.*

Proof. It suffices to verify that the operator $T\colon W_0^{k,p(x)}(\Omega) \to (W_0^{k,p(x)}(\Omega))^*$ from (4.12) satisfies the assumptions of the well-known Browder theorem (see e.g. [1]).

The space $W_0^{k,p(x)}(\Omega)$ is reflexive by Theorem 3.1.

The operator T is continuous and bounded by Corollary 4.4.

Inserting $\xi = \delta_k u(x)$, $\eta = \delta_k v(x)$ into (4.13) and integrating over Ω we obtain the monotonicity of T.

Similarly, from (4.14) we obtain

$$\lim_{\|w\|_{k,p}\to\infty} \frac{\langle Tw, w\rangle}{\|w\|_{k,p}} \geqq c_1 \lim_{\|w\|_{k,p}\to\infty} \|w\|_{k,p}^{-1} \sum_{|\alpha| \leqq k} \varrho_p(D^\alpha w)\,.$$

Since $\|w\|_{k,p} \leqq m \max\{\|D^\alpha w\|_p\colon |\alpha| \leqq k\}$ and since we can consider $\|w\|_{k,p} \geqq 1$, we have

$$\sum_{|\alpha| \leqq k} \varrho_p(D^\alpha w) \geqq \|w\|_{k,p}^{p_*} \sum_{|\alpha| \leqq k} \varrho_p(|D^\alpha w|/\|w\|_{k,p}) \geqq$$
$$\geqq \|w\|_{k,p}^{p_*-1}\, m^{-p^*} \varrho_p(|D^\beta w|/\|D^\beta w\|_{k,p}) = \|w\|_{k,p}^{p_*-1}\, m^{-p^*}\,,$$

where β is the multi-index for which the maximum of $\|D^\alpha w\|_p$ is attained. Thus,

$$\lim_{\|w\|_{k,p}\to\infty} \frac{\langle Tw, w\rangle}{\|w\|_{k,p}} \geqq c_1 m^{-p^*} \lim_{\|w\|_{k,p}\to\infty} \|w\|_{k,p}^{p_*-1} = \infty$$

and the operator T is coercive.

If the inequality (4.13) is strict, then T is strictly monotone, which yields the unicity of the solution. ∎

Remark 4.6. If Ω and p satisfy the assumptions of Theorem 3.10 or 3.11 then the condition (4.14) in Theorem 4.5 can be weakened in the following way:

$$\sum_{|\alpha|\leqq k} \alpha_\alpha(x, \xi)\, \xi_\alpha \geqq c_1 \sum_{|\alpha|=k} |\xi|^{p(x)} - c_2 .$$

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Authors' addresses: O. Kováčik, 010 08 Žilina, Zvolenská 15; J. Rákosník, 115 67 Praha, Žitná 25, Czechoslovakia (Matematický ústav ČSAV).