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AN APPLICATION OF POPOV'S METHOD IN THE THEORY OF ELECTRICAL NETWORKS

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0. We shall investigate the stability of an equilibrium state of such electrical systems which may be considered as a connection of a passive linear n-port containing constant lumped elements, with a nonlinear purely resistive n-port. For this purpose we shall first establish certain relations between the properties of the admittance (impedance) matrix of a linear passive n-port and the behavior of its free transients. On applying the extended Popov's criteria (see [1]), there result certain assertions on the dependence of regimes accross common terminals on the initial state of the linear system. These results yield the desired conditions for stability of an equilibrium of the entire connection.

1. Let us begin with some basic definitions.

If $\tilde{G}$ is an oriented graph with branches $h_1, \ldots, h_r$ which contains at least one loop, and $\tilde{L}$, $\tilde{R}$, $\tilde{S}$ are constant symmetric positive semidefinite $r \times r$ matrices such that each element $\tilde{L}_{ik}$, $\tilde{R}_{ik}$, $\tilde{S}_{ik}$ is assigned to a pair (h_i, h_k), $i, k = 1, \ldots, r$ of branches, then the quadruple $\tilde{N}_J = (\tilde{G}, \tilde{L}, \tilde{R}, \tilde{S})$ will be called a passive network on a current basis, or simply J-network.

Let X be an $r \times m$ matrix, whose columns constitute a complete set of linearly independent real solutions of the equation $\tilde{a}'x = 0$, where $\tilde{a}$ is the branch-node incidence matrix of $\tilde{G}$ (see [2]); if $\tilde{X}'(\tilde{L} + \tilde{R} + \tilde{S})\tilde{X}$ is a positive definite matrix, $\tilde{N}_J$ will be called J-regular.

Let $\tilde{N}_J$ be J-regular; if $\tilde{E}(t)$ is a real r-vector function locally integrable on $\langle 0, \infty)$ (or, more generally, a distribution vanishing on $(-\infty, 0)$), and $\tilde{J}_0$, $\tilde{q}_0$ are real constant r-vectors such that the component $\tilde{E}_i(t)$, $\tilde{J}_{0i}$, $\tilde{q}_{0i}$ of $\tilde{E}(t)$, $\tilde{J}_0$, $\tilde{q}_0$, respectively, is assigned to the branch h_i, $i = 1, \ldots, r$, then the r-vector $\tilde{J}(t)$ given by

$$\tilde{J}(t) = \tilde{X}(\tilde{X}'\tilde{Z}(D)\,\tilde{X})^{-1}\,\tilde{X}(\tilde{E}(t) + \tilde{L}\tilde{J}_0\delta_0 - \tilde{S}\tilde{q}_0 H_0)\,, \tag{1.1}$$

where $H_0 = 1$ for $t \geqq 0$, $H_0 = 0$ for $t < 0$, $\delta_0 = H_0'$ and

$$\tilde{Z}(D) = \tilde{L}D + \tilde{R} + \tilde{S}D^{-1}\,, \tag{1.2}$$

will be called the solution of $\tilde{N}_J$ on a current basis corresponding to $\tilde{E}(t)$, $\tilde{J}_0$, $\tilde{q}_0$.

Note 1. Due to the assumption on the definiteness of $\tilde{X}'(\tilde{L} + \tilde{R} + \tilde{S})\tilde{X}$, the matrix $(\tilde{X}' Z(p)\tilde{X})^{-1}$, which has rational functions of p as its elements, always exists (see [2]); moreover, as there is an isomorphism between the system of all Heaviside operators and the system of all rational functions of p, the operator matrix $(\tilde{X}'\tilde{Z}(D)\tilde{X})^{-1}$ also exists. Consequently, in the sequel we shall not distingnisch in notation between operator matrices and the corresponding matrices having rational functions as their elements.

Analogously, if $\tilde{G}$ is an oriented graph with branches $h_1, \ldots, h_r$ which contains at least one loop, and $\tilde{C}$, $\tilde{A}$, $\tilde{M}$ are constant symmetric positive semidefinite $r \times r$ matrices whose elements are assigned to pairs of branches, then the quadruple $N_V = (\tilde{G}, \tilde{C}, \tilde{A}, \tilde{M})$ will be called a passive network on a voltage basis, or briefly a V-network.

Let d be an $r \times l$ matrix whose columns constitute a complete set of linearly independent columns of the branch-node incidence matrix $\tilde{a}$; if $\tilde{d}'(\tilde{C} + \tilde{A} + \tilde{M})\tilde{d}$ is positive definite, then $\tilde{N}_V$ will be called V-regular.

Let $\tilde{N}_V$ be V-regular; if $I(t)$ is a real r-vector function locally integrable on $\langle 0, \infty)$, and $\tilde{V}_0$, $\tilde{F}_0$ re real constant r-vectors whose elements are assigned to branches of $\tilde{G}$, then the r-vector $\tilde{V}(t)$ given by

$$\tilde{V}(t) = \tilde{d}(\tilde{d}'\tilde{Y}(D)\tilde{d})^{-1}\tilde{d}'(\tilde{I}(t) + \tilde{C}\tilde{V}_0\delta_0 - \tilde{M}\tilde{F}_0 H_0), \tag{1.3}$$

where

$$\tilde{Y}(D) = \tilde{C}D + \tilde{A} + \tilde{M}D^{-1}, \tag{1.4}$$

will be called the solution of $\tilde{N}_V$ on a voltage basis corresponding to $\tilde{I}(t)$, $\tilde{V}_0$, $\tilde{F}_0$.

The physical meaning of the concepts related to a J-network certainly needs no comment (also see [2]); as for the meaning of those related to a V-network, see the Appendix 1.

Next, let us introduce the concept of a linear passive n-port.

Let G be an oriented graph in which n distinct pairs of nodes $(u_{i_s}^*, u_i)$, $i = 1, \ldots, n$ are distinguished; if either

a) $N = (G, L, R, S)$ is a J-network, or

b) $N = (G, C, A, M)$ is a V-network,

then N will be called a linear passive n-port. The nodes u_i^*, u_i will be called the initial and the end terminal, respectively, of the i-th port (u_i^*, u_i), $i = 1, \ldots, n$.

Given an n-port N, let us construct a network $\tilde{N}$ as follows: Let $\tilde{G}$ be the graph obtained by completing G by branches $h_1, h_2, \ldots, h_n$ such that u_i^* is the initial, u_i the end node of h_i, $i = 1, 2, \ldots, n$.

In case a), define $r \times r$ matrices

$$\tilde{L} = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & L \end{array}\right], \quad \tilde{R} = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & R \end{array}\right], \quad \tilde{S} = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & S \end{array}\right], \tag{1.5}$$

where the zero matrices standing in the upper left corner have type $n \times n$, and put $\tilde{N} = (\tilde{G}, \tilde{L}, \tilde{R}, \tilde{S})$.

In case b) let

$$(1.6) \qquad \tilde{C} = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & C \end{array}\right], \quad \tilde{A} = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & A \end{array}\right], \quad \tilde{M} = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & M \end{array}\right],$$

and put $\tilde{N} = (\tilde{G}, \tilde{C}, \tilde{A}, \tilde{M})$.

It is obvious that in case a) $\tilde{N}$ indeed is a J-network, and in case b) a V-network.

Moreover, we shall always assume that the n-port N is such that, in case a), for every branch h_i with $1 \leqq i \leqq n$, there is a loop of $\tilde{G}$ containing h_i and no other h_k with $1 \leqq k \leqq n$, $k \neq i$; and in case b) that all nodes u_i^*, u_i are distinct for $i, k = 1, \ldots, n$.

The n-port N will be called J-, (V-) regular, if $\tilde{N}$ is J-, (V-) regular.

Next, introduce the notation

$$(1.7) \qquad \tilde{J} = \left[\begin{array}{c} J \\ \hline J^* \end{array}\right], \quad \tilde{E} = \left[\begin{array}{c} E \\ \hline 0 \end{array}\right], \quad \tilde{J}_0 = \left[\begin{array}{c} 0 \\ \hline J_0 \end{array}\right], \quad \tilde{q}_0 = \left[\begin{array}{c} 0 \\ \hline q_0 \end{array}\right],$$

$$(1.8) \qquad \tilde{V} = \left[\begin{array}{c} V \\ \hline V^* \end{array}\right], \quad \tilde{I} = \left[\begin{array}{c} I \\ \hline 0 \end{array}\right], \quad \tilde{V}_0 = \left[\begin{array}{c} 0 \\ \hline V_0 \end{array}\right], \quad \tilde{F}_0 = \left[\begin{array}{c} 0 \\ \hline F_0 \end{array}\right],$$

where J, E, V, I are n-vectors, $J^*, J_0, q_0, V^*, V_0, F_0$ $(r - n)$-vectors.

Let the n-port N be J-regular; then the pair (J_0, q_0) will be called the initial state of N; and any pair (J, E), where J, E are given by (1.7) with $\tilde{J}, \tilde{E}, \tilde{J}_0, \tilde{q}_0$ satisfying (1.1), will be called a regime on N corresponding to the initial state (J_0, q_0). Furthermore, a matrix $A(p)$ fulfilling the equation $J = A(D)E$ for any regime (J, E) corresponding to the initial state $(0, 0)$ will be called the admittance matrix of N.

From (1.1) it is apparent that for a J-regular n-port the admittance matrix always exists.

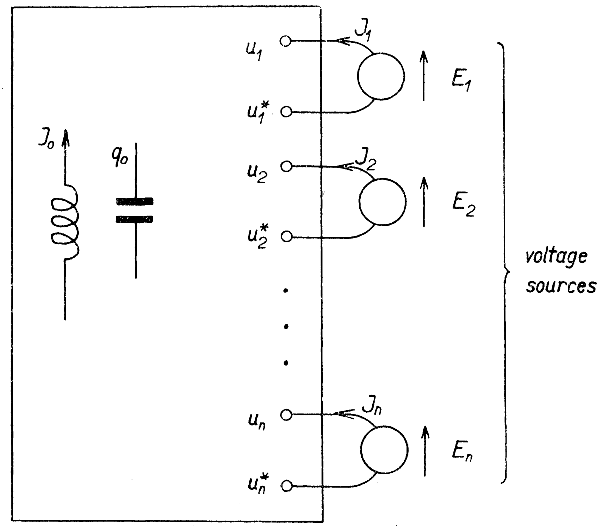
Analogously, let the n-port N be V-regular; then the pair (V_0, F_0) will be called the initial state of N; and any pair (V, I), where V, I are given by (1.8) with $\tilde{V}, \tilde{I}, \tilde{V}_0, \tilde{F}_0$ satisfying (1.3), will be called a regime on N corresponding to (V_0, F_0). Furthermore, a matrix $Z(p)$ fulfilling the equation $V = Z(D)I$ for any regime (V, I) corresponding to the zero initial state will be called the impedance matrix of N.

Evidently, any V-regular n-port possesses an impedance matrix.

The physical meaning of the concepts already defined is straightforward. Obviously, if N is J-regular, then (J, E) may be interpreted as a pair of currents and voltages acting simultaneously on individual ports of N, provided the initial currents and charges "inside" N are represented by vectors J_0, q_0, respectively. (See Fig. 1a.)

Similarly, if N is V-regular, then (V, I) describes voltages and currents appearing simultaneously on the ports of N, provided the initial voltages across the involved capacities and initial magnetic fluxes in coils are given by V_0, F_0, respectively. (See Fig. 1b.)

Fig. 1a.



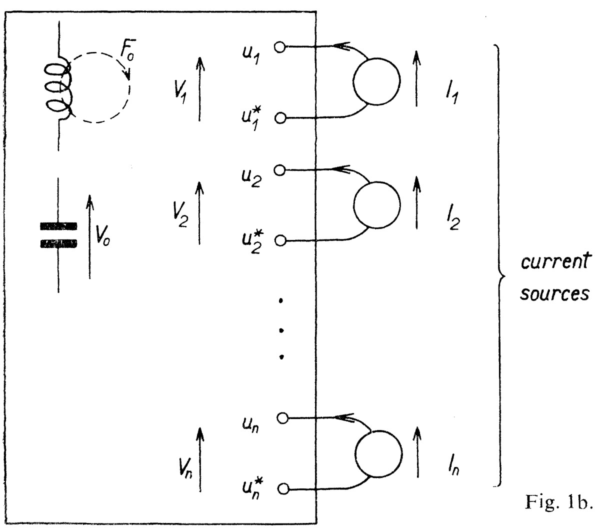


Fig. 1b.

The concepts of the admittance and impedance matrix then have the usual meaning. Note also that in case of a J-regular n-port, J is uniquely determined by E (with fixed (J_0, q_0)); the converse, however, need not be true. A similar statement is true for V-regular n-ports.

Let the system $\mathfrak{S}_n$ of matrices have the same meaning as in [2], or [3] p. 29; let $\mathfrak{R}_n$ be a subsystem of $\mathfrak{S}_n$ such that if $A(p) \in \mathfrak{R}_n$, then

1. $A(p)$ has no poles on the imaginary axis nor at infinity,

2. $A(\infty) = 0$.

Then we have the following assertion:

Lemma 1.1. *Let N be a J-regular n-port such that its admittance matrix $A(p)$ belongs to $\mathfrak{R}_n$. Then, for any regime (J, E) on N corresponding to the initial state (J_0, q_0), we have*

$$J(t) = \int_0^t a(t - \tau)\, E(\tau)\, d\tau + i_0(t)\,, \tag{1.9}$$

where $a = A(D)\,\delta_0$ and $i_0(t)$ is independent of $E(t)$. Moreover, there are positive constants K, K_1, K_2, λ independent of E, J_0, q_0 such that

$$\|a(t)\|,\ \|a'(t)\| \leqq K \exp(-\lambda t)\,, \tag{1.10}$$

$$\|i_0(t)\|,\ \|i_0'(t)\| \leqq (K_1\|J_0\| + K_2\|q_0\|) \exp(-\lambda t) \tag{1.11}$$

for every $t \geqq 0$.

Furthermore, if $\tilde{N}$ is a J-network, then the positive definiteness of matrices $\tilde{X}'\tilde{L}\tilde{X}$, $\tilde{X}'\tilde{R}\tilde{X}$ implies that N is J-regular and $A(p) \in \mathfrak{R}_n$.

Note 2. It is obvious that $A(p) \in \mathfrak{R}_n$ need not imply that the matrices $\tilde{X}'\tilde{L}\tilde{X}$, $\tilde{X}'\tilde{R}\tilde{X}$ are definite. On the other hand, making use of results obtained in [2], we have the following corollary: *If $\tilde{N}$ is a J-network containing no mutual couplings and if every loop of $\tilde{G}$ contains both an inductance and a resistance, then $\tilde{X}'\tilde{L}\tilde{X}$, $\tilde{X}'\tilde{R}\tilde{X}$ are definite.*

Proof of Lemma 1.1. Recalling the assumption on loops of $\tilde{G}$ passing through a branch h_i with $1 \leqq i \leqq n$ and the fact that for $\tilde{X}$ we can take any matrix whose columns correspond to a complete set of linearly independent loops of $\tilde{G}$, it is readily seen that, with a proper numeration of loops, we may set

$$\tilde{X} = \left[\begin{array}{c|c} I & 0 \\ \hline X_1 & X_2 \end{array}\right], \tag{1.12}$$

where I is an $n \times n$ unit matrix. Denoting

$$A^* = \left[\begin{array}{c|c} A_{11}^* & A_{12}^* \\ \hline A_{12}^{*\prime} & A_{22}^* \end{array}\right] = (\tilde{X}'\tilde{Z}(p)\,\tilde{X})^{-1} \tag{1.13}$$

(A^* exists due to the J-regularity of N), where A_{11}^* is an $n \times n$ matrix, we have $A^* \in \mathfrak{S}_m$. (See [2], p. 455.) Furthermore, by (1.12),

$$\tilde{A} = \tilde{X}A^*\tilde{X}' = \tag{1.14}$$

$$= \left[\begin{array}{c|c} A_{11}^* & A_{11}^*X_1' + A_{12}^*X_2' \\ \hline X_1A_{11}^* + X_2A_{12}^{*\prime} & X_1A_{11}^*X_1' + X_1A_{12}^*X_2' + X_2A_{12}^{*\prime}X_1' + X_2A_{22}^*X_2' \end{array}\right].$$

Obviously $\tilde{A} \in \mathfrak{S}_r$, and consequently $A^*_{11} \in \mathfrak{S}_n$.

Next, making use of (1.1) and (1.7), we have for any regime (J, E),

$$\begin{bmatrix} J \\ \hdashline J^* \end{bmatrix} = \tilde{A}(D)\left(\begin{bmatrix} E \\ \hdashline 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \hdashline LJ_0 \end{bmatrix} \delta_0 - \begin{bmatrix} 0 \\ \hdashline Sq_0 \end{bmatrix} H_0\right),$$

so that by (1.14),

$$(1.15) \qquad J = A^*_{11}(D)\, E + (A_{11}(D)\, X'_1 + A^*_{12}(D)\, X'_2)\,(LJ_0\delta_0 - Sq_0H_0)\,.$$

Thus, according to the definition, A^*_{11} is the admittance matrix $A(p)$ of N.

From the assumption $A(p) = A^*_{11}(p) \in \mathfrak{R}_n$ it follows that

$$A^*_{11} = \sum_{jk} A_{jk}(p - \alpha_j)^{-k}\,, \quad k \geqq 1\,, \quad \operatorname{Re} \alpha_j < 0\,,$$

and consequently,

$$(1.16) \qquad a(t) = A^*_{11}(D)\, \delta_0 = \sum_{j=1}^{l} P_j(t)\, H_0(t) \exp \alpha_j t\,,$$

where $P_j(t)$ are matrix polynomials in t. Hence, constants $K, \lambda > 0$ exist such that (1.10) holds.

On the other hand, as $A^* \in \mathfrak{S}_m$, we have by the expansion theorem (see [3], p. 31),

(1.17)

$$\begin{bmatrix} A^*_{11} & \vdots & A^*_{12} \\ \hdashline A^{*'}_{12} & \vdots & A^*_{22} \end{bmatrix} = p \begin{bmatrix} H^{(\infty)}_{11} & \vdots & H^{(\infty)}_{12} \\ \hdashline H^{(\infty)'}_{12} & \vdots & H^{(\infty)}_{22} \end{bmatrix} + \sum_{k=1}^{s} \frac{p}{p^2 + \omega_k^2} \begin{bmatrix} H^{(k)}_{11} & \vdots & H^{(k)}_{12} \\ \hdashline H^{(k)'}_{12} & \vdots & H^{(k)}_{22} \end{bmatrix} + \begin{bmatrix} B_{11} & \vdots & B_{12} \\ \hdashline B'_{12} & \vdots & B_{22} \end{bmatrix},$$

where $0 \leqq \omega_1 < \omega_2 < \ldots < \omega_s$, the matrices $\begin{bmatrix} H^{(k)}_{11} & \vdots & H^{(k)}_{12} \\ \hdashline H^{(k)'}_{12} & \vdots & H^{(k)}_{22} \end{bmatrix}$ are positive semidefinite for $k = 1, 2, \ldots, s, \infty$, the matrix $\begin{bmatrix} B_{11} & \vdots & B_{12} \\ \hdashline B'_{12} & \vdots & B_{22} \end{bmatrix}$ has no poles in the closed right halfplane (including ∞), belongs to $\mathfrak{S}_m$ and $\begin{bmatrix} B_{11}(\infty) & \vdots & B_{12}(\infty) \\ \hdashline B'_{12}(\infty) & \vdots & B_{22}(\infty) \end{bmatrix}$ is positive semidefinite.

However, since $A^*_{11} \in \mathfrak{R}_n$, we have from (1.17) that necessarily $H^{(k)}_{11} = 0$, $k = 1, 2, \ldots, s, \infty$ and $B_{11}(\infty) = 0$. From this it follows that also $H^{(k)}_{12} = 0$, $k = 1, 2, \ldots, s, \infty$, and $B_{12}(\infty) = 0$, since otherwise positive semidefinitness would be violated. Hence the matrix $A^*_{12}(p)$ has no poles in the closed right half-plane and $A^*_{12}(\infty) = 0$.

Next, from the identity $A^*(\tilde{X}'\tilde{Z}\tilde{X}) = I$ we have by (1.12), (1.2) and (1.5) that

$$\begin{bmatrix} A^*_{11} & \vdots & A^*_{12} \\ \hdashline A^{*'}_{12} & \vdots & A^*_{22} \end{bmatrix} \begin{bmatrix} I & \vdots & X'_1 \\ \hdashline 0 & \vdots & X'_2 \end{bmatrix} \begin{bmatrix} 0 & \vdots & 0 \\ \hdashline 0 & \vdots & Z \end{bmatrix} \begin{bmatrix} I & \vdots & 0 \\ \hdashline X_1 & \vdots & X_2 \end{bmatrix} = \begin{bmatrix} I & \vdots & 0 \\ \hdashline 0 & \vdots & I \end{bmatrix},$$

and consequently, on performing the multiplication,

$$(1.18)\qquad A_{11}^{*}X_1'ZX_1 + A_{12}^{*}X_2'ZX_1 = I\,,\quad A_{11}^{*}X_1'ZX_2 + A_{12}^{*}X_2'ZX_2 = 0\,,$$

where $Z = Lp + R + Sp^{-1}$.

Multiplying (1.18) by p and then putting $p = 0$, we get

$$(1.19)\qquad \begin{aligned} A_{11}^{*}(0)\,X_1'SX_1 + A_{12}^{*}(0)\,X_2'SX_1 &= 0\,,\\ A_{11}^{*}(0)\,X_1'SX_2 + A_{12}^{*}(0)\,X_2'SX_2 &= 0\,. \end{aligned}$$

Postmultiplying the first equality (1.19) by $A_{11}^{*}(0)$, premultiplying the second by $A_{12}^{*\prime}(0)$ and summing, we get

$$(A_{11}^{*}(0)\,X_1' + A_{12}^{*}(0)\,X_2')\,S(X_1A_{11}^{*}(0) + X_2A_{12}^{*\prime}(0)) = 0\,.$$

Since S is positive semidefinite, this yields,

$$(1.20)\qquad (A_{11}^{*}(0)\,X_1' + A_{12}^{*}(0)\,X_2')\,S = 0\,.$$

Returning to (1.15), we have (see Appendix 2)

$$J = \int_0^t a(t-\tau)\,E(\tau)\,\mathrm{d}\tau + i_0$$

with $i_0 = i_1 + i_2$, where

$$(1.21)\qquad \begin{aligned} i_1 &= (A_{11}^{*}(D)\,X_1' + A_{12}^{*}(D)\,X_2')\,LJ_0\delta_0\,,\\ i_0 &= -(A_{11}^{*}(D)\,X_1' + A_{12}^{*}(D)\,X_2')\,Sq_0H_0\,. \end{aligned}$$

Putting $(A_{11}^{*}(D)\,X_1' + A_{11}^{*}(D)\,X_2')\,\delta_0 = \tilde{a}$, it follows easily from the properties of A_{11}^{*}, A_{12}^{*} that there is a $\bar{K} > 0$ such that $\|\tilde{a}(t)\| \leqq \bar{K}\exp(-\lambda t)$, $t \geqq 0$. Hence $\|i_1\| \leqq \bar{K}_1\|J_0\|\exp(-\lambda t)$. Next, we have

$$i_2 = -\int_0^t \tilde{a}(\tau)\,\mathrm{d}\tau\,.\,Sq_0\,.$$

Consequently, $i_2(\infty)$ exists and by Tauber's theorem we have (see Appendix 3)

$$i_2(\infty) = -(A_{11}^{*}(0)\,X_1' + A_{12}^{*}(0)\,X_2')\,Sq_0\,.$$

Thus by (1.20), $i_2(\infty) = 0$, and consequently there is a $\bar{K}_2 > 0$ such that $\|i_2\| \leqq \bar{K}_2\|q_0\|\exp(-\lambda t)$, $t \geqq 0$.

Hence the first inequality (1.11) is established. The second follows immediately from the fact that both i_1 and i_2 have the form

$$\sum_j Q_j(t)\, H_0 \exp(\alpha_j t)\,,$$

where $Q_j(t)$ are polynomials and $\operatorname{Re} \alpha_j < 0$. The last statement of Lemma 1.1 is a direct consequence of Th. 2.1 – 5 and Th. 2.1 – 6 in [3]. The assertion dual to Lemma 1.1 reads as follows:

Lemma 1.2. *Let N be a V-regular n-port such that its impedance matrix $Z(p)$ belongs to $\mathfrak{R}_n$. Then, for any regime (V, I) on N corresponding to the initial state (V_0, F_0), we have*

$$V(t) = \int_0^t z(t - \tau)\, I(\tau)\, \mathrm{d}\tau + v_0(t)\,, \tag{1.22}$$

where $z = Z(D)\, \delta_0$ and $v_0(t)$ is independent of $I(t)$.

Moreover, there are positive constants $\bar{K}$, $\bar{K}_1$, $\bar{K}_2$, λ independent of I, V_0, F_0 such that

$$\|z(t)\|,\ \|z'(t)\| \leqq \bar{K} \exp(-\lambda t)\,, \tag{1.23}$$

$$\|v_0(t)\|,\ \|v_0'(t)\| \leqq (\bar{K}_1 \|V_0\| + \bar{K}_2 \|F_0\|) \exp(-\lambda t) \tag{1.24}$$

for every $t \geqq 0$.

Furthermore, if $\tilde{N}$ is a V-network, then the positive definiteness of matrices $\tilde{d}'\tilde{C}\tilde{d}$, $\tilde{d}'\tilde{A}\tilde{d}$ implies that N is V-regular and $Z(p) \in \mathfrak{R}_n$.

Proof. Recalling the assumption that all the terminals u_i^*, u_k, $i, k = 1, 2, \ldots, n$ are distinct, it can be easily verified that, with a suitable numeration of columns of the matrix $\tilde{a}$, we may set

$$\tilde{d} = \left[\begin{array}{c|c} I & 0 \\ \hline d_1 & d_2 \end{array}\right],$$

where I is the $n \times n$ unit matrix. Starting from (1.3) and following exactly the same pattern as in Lemma 1.1, we conclude the proof.

Lemma 1.3. *Let N be a J-regular n-port and let its admittance matrix $A(p)$ satisfy the condition*

$$A(p) = p^{-1}\gamma + B(p)\,, \tag{1.25}$$

where $\gamma \neq 0$ and $B(p) \in \mathfrak{R}_n$. Then, for any regime (J, E) on N corresponding to the initial state (J_0, q_0), we have

$$J(t) = \int_0^t (\gamma + b(t - \tau))\, E(\tau)\, \mathrm{d}\tau + i_0(t)\,, \tag{1.26}$$

where $b(t) = B(D)\, \delta_0$ and $i_0(t)$ is independent of $E(t)$.

Moreover, $i_0(\infty)$ exists and there are positive constants $K, K_1, \ldots, K_4, \lambda$ such that

$$(1.27) \qquad \|b(t)\|, \|b'(t)\| \leqq K \exp(-\lambda t), \quad t \geqq 0,$$

$$(1.28) \qquad \|i_0(t) - i_0(\infty)\|, \|i_0'(t)\| \leqq (K_1\|J_0\| + K_2\|q_0\|) \exp(-\lambda t), \quad t \geqq 0,$$

$$(1.29) \qquad \|i_0(\infty)\| \leqq K_3\|J_0\| + K_4\|q_0\|.$$

Proof. Defining $\tilde{X}$ again by (1.12), let $A^*, \tilde{A}$ be given by (1.13) and (1.14), respectively. Since $A_{11}^* = A(p) = p^{-1}\gamma + B(p)$, we conclude as before that A_{12}^* can be expanded as $A_{12}^* = p^{-1}\gamma_{12} + B_{12}(p)$, where B_{12} has no poles in the closed right halfplane and $B_{12}(\infty) = 0$.

Next, by (1.15) we have

$$(1.30) \quad J = (\gamma D^{-1} + B(D)) E + (\gamma X_1' D^{-1} + \gamma_{12} X_2' D^{-1})(LJ_0\delta_0 - Sq_0H_0) + \\ + (B(D) X_1' + B_{12}(D) X_2')(LJ_0\delta_0 - Sq_0H_0).$$

Hence (1.26) holds and we have $i_0 = i_1 + i_2 + i_3 + i_4$ with

$$(1.31) \quad i_1 = (\gamma X_1' + \gamma_{12}X_2') LJ_0H_0, \qquad i_2 = -(\gamma X_1' + \gamma_{12}X_2') Sq_0 . H_0 t, \\ i_3 = (b(t) X_1' + b_{12}(t) X_2') LJ_0, \quad i_4 = -(b(t) X_1' + b_{12}(t) X_2') * H_0 . Sq_0,$$

where $b_{12}(t) = B_{12}(D)\,\delta_0$ and the symbol $*$ denotes the convolution product.

According to the properties of B, B_{12} there is a $\tilde{K} > 0$ and $\lambda > 0$ such that $\|b(t)\|$, $\|b'(t)\|$, $\|b_{12}(t)\| \leqq \tilde{K} \exp(-\lambda t)$, $t \geqq 0$; consequently $\|i_3\| \leqq K_5\|J_0\| \exp(-\lambda t)$, $i_4(\infty)$ exists and by Tauber's theorem,

$$(1.32) \qquad i_4(\infty) = -(B(0) X_1' + B_{12}(0) X_2') Sq_0.$$

Thus

$$(1.33) \qquad \|i_4(\infty)\| \leqq K_6\|q_0\|.$$

Also,

$$\|i_4 - i_4(\infty)\| \leqq K_7\|q_0\| \exp(-\lambda t),$$

and

$$(1.34) \qquad \|i_1\| \leqq K_8\|J_0\|.$$

On the other hand, multiplying the identities (1.18) by p^2 and then putting $p = 0$, we get

$$(1.35) \qquad \gamma X_1' S X_1 + \gamma_{12} X_2' S X_1 = 0, \quad \gamma X_1' S X_2 + \gamma_{12} X_2' S X_2 = 0.$$

From this we obtain, as before,

$$(1.36) \qquad (\gamma X_1' + \gamma_{12} X_2') S = 0.$$

Hence $i_2 = 0$.

Thus we have $i_0(\infty) = i_1 + i_4(\infty)$, so that in view of (1.34) and (1.33) the inequality (1.29) is true. Moreover, $i_0 - i_0(\infty) = i_3 + i_4 - i_4(\infty)$, so that $\|i_0 - i_0(\infty)\| \leqq (K_5\|J_0\| + K_7\|q_0\|) \exp(-\lambda t)$; the remaining inequality (1.28) follows from the particular form of i_0. Hence, Lemma 1.3 is proved.

In a completely analogous manner a similar statement for V-regular n-ports with impedance matrix $Z(p) = p^{-1}\gamma + C(p)$ may be proven. As its wording is the same as that in Lemma 1.3, the explicit formulation is omitted.

Note 3. It can be easily verified that every n-port obtained from a J-regular n-port with admittance matrix $B(p) \in \mathfrak{R}_n$ by shunting its ports with positive inductances l_i, possesses the admittance matrix $p^{-1} \operatorname{diag}(l_1^{-1}, \ldots, l_n^{-1}) + B(p)$. Similarly, if an n-port has the impedance matrix $B(p)$ and if in each of its ports a capacity $c_i > 0$ is inserted in series, then the resulting n-port has the impedance matrix $p^{-1} \operatorname{diag}(c_1^{-1}, \ldots, c_n^{-1}) + B(p)$.

Lemma 1.4. *Let $h > 0$ and let $A(p) \in \mathfrak{R}_n$; then there is a $q > 0$ such that*

$$\operatorname{Re} \bar{\eta}' \left\{ (1 + i\omega q) A(i\omega) + \frac{1}{h} I \right\} \eta \geqq 0 \tag{1.37}$$

for every real ω and complex n-vector η.

Proof. From $A(p) \in \mathfrak{R}_n$ we have

$$A(p) = \sum_{jk} A_{jk}(p - \alpha_j)^{-k}, \quad k \geqq 1, \quad \operatorname{Re} \alpha_j < 0. \tag{1.38}$$

It is obvious that there are constants $\varkappa_{jk} > 0$ such that

$$|\omega(i\omega - \alpha_j)^{-k}| \leqq \varkappa_{jk} \quad \text{for every real } \omega. \tag{1.39}$$

Thus, we have

$$|\operatorname{Re} \bar{\eta}' i\omega A(i\omega) \eta| \leqq |\bar{\eta}' i\omega A(i\omega) \eta| \leqq \sum_{jk} \|\eta\|^2 \|A_{jk}\| \varkappa_{jk} = K\|\eta\|^2. \tag{1.40}$$

On the other hand, $\operatorname{Re} 1/h \,.\, \bar{\eta}' I \eta = 1/h \,.\, \|\eta\|^2$. Thus, choosing $0 < q < h^{-1}K^{-1}$, we have by (1.40)

$$|\operatorname{Re} \bar{\eta}' i\omega q A(i\omega) \eta| \leqq h^{-1}\|\eta\|^2,$$

and consequently

$$\operatorname{Re} \bar{\eta}' \left(i\omega q A(i\omega) + \frac{1}{h} I \right) \eta \geqq 0$$

for every real ω and η. Since also $A(p) \in \mathfrak{S}_n$, we have $\operatorname{Re} \bar{\eta}' A(i\omega) \eta \geqq 0$ for every ω real and η. This proves (1.37) as required.

2. Let us now turn our attention to systems formed by a parallel connection of a linear passive n-port with a nonlinear n-port.

Any system N with n terminal-pairs (u_i^*, u_i), $i = 1, \ldots, n$, on which the concepts of current J_i and voltage E_i are defined will be called a (general) n-port, if there is given a set $\mathbf{R}$ of pairs (J, E) with $J' = [J_1, \ldots, J_n]$, $E' = [E_1, \ldots, E_n]$. A pair $(J, E) \in \mathbf{R}$ will also be called a regime on N.

Let N_1, N_2 be two n-ports with sets of regimes $\mathbf{R}_1$, $\mathbf{R}_2$ respectively; a system characterized by a set $\mathbf{S}$ of common regimes (J, E), i.e. $(J, E) \in \mathbf{S}$ if and only if there are pairs $(J_1, E_1) \in \mathbf{R}_1$, $(J_2, E_2) \in \mathbf{R}_2$ such that $J = J_1 = -J_2$, $E = E_1 = E_2$, will be called a parallel connection of N_1 and N_2.

The physical meaning of the concepts defined is straightforward and needs no comment; cf. Fig. 2.

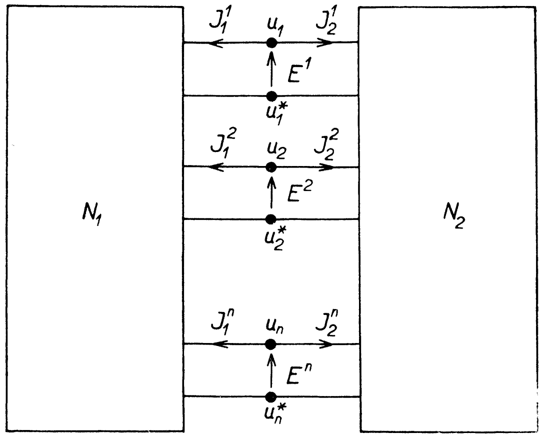


Fig. 2.

It is also clear that the J-regular and V-regular n-ports introduced in the previous section are particular cases of a general n-port.

Let $f(x)$ be an n-vector function of the n-vector argument x defined on the entire n-dimensional space; the n-port N will be called resistive if for each $(J, E) \in \mathbf{R}$ we have $E = f(J)$, and conductive if $J = f(E)$.

Moreover, introduce the following notation:

Let $\mathfrak{N}_n^*$ be the set of all real continuous n-vector functions $f(x)$ of the n-vector x which fulfil the condition

$$(2.1) \qquad f'(x)\, x > 0 \quad \text{for} \quad x \neq 0\,.$$

Let $0 < h_1 < h_2$ and let $\mathfrak{N}_n(h_1, h_2)$ be the set of all n-vector functions $f(x)$ which have the following properties:

a) For every $f \in \mathfrak{N}_n(h_1, h_2)$ there is a real scalar function $U(x)$ possessing continuous first partial derivates everywhere and such that $f(x) = \operatorname{grad} U(x)$, i.e. for the i-th component $f_i(x)$ of $f(x)$ we have

$$(2.2) \qquad f_i(x) = \frac{\partial U(x)}{\partial x_i}\,, \quad i = 1, \ldots, n\,.$$

b) $$h_1 \|x\|^2 \leqq f'(x)\, x\,, \quad \|f(x)\| \leqq h_2 \|x\|$$

for every x.

Note 4. It can be easily verified that if $f(x)$ is continuous and such that $f_i(x) = \varphi_i(x_i)$, $h_1 x_i^2 \leqq \varphi_i(x_i)\, x_i \leqq h_2 x_i^2$ for every x_i and $i = 1, \ldots, n$, then $f \in \mathfrak{N}_n(h_1, h_2)$.

Note 5. If a resistive n-port N with $E = f(J)$ is such that f belongs either to $\mathfrak{N}_n^*$ or to $\mathfrak{N}_n(h_1, h_2)$, then $E'J \geqq 0$ for any regime (J, E). Hence N cannot be a source of energy. An analogous statement is true for a conductive n-port with $f \in \mathfrak{N}_n^*$ (or $\in \mathfrak{N}_n(h_1, h_2)$).

Now, we have the following assertion:

Theorem 2.1. *Let N_1 be a J-regular n-port whose admittance matrix $A(p)$ belongs to $\mathfrak{N}_n$; furthermore, let N_2 be a resistive n-port such that $E_2 = f(J_2)$ and $f \in \mathfrak{N}_n(h_1, h_2)$. Then there exist a continuous function $\Phi(\xi)$ vanishing at $\xi = 0$ and positive constants K_1, K_2 such that, for any common regime (J, E) on the parallel connection of N_1 and N_2, we have*

$$\|J(t)\| \leqq \Phi(K_1 \|J_0\| + K_2 \|q_0\|) \tag{2.3}$$

for every $t \geqq 0$, and $J(t) \to 0$ as $t \to \infty$, where (J_0, q_0) is the initial state of N_1.

From the assertion already stated we have the following conclusion:

Corollary. *The equilibrium state of the parallel connection of N_1 and N_2, i.e. the common regime $(0, 0)$, is stable and asymptotically stable with respect to the initial state of N_1. Moreover, the region of stability is the entire space.*

Actually, it is clear that $(0, 0)$ is a regime on N_1 corresponding to the initial state $(0, 0)$; on the other hand, from (2.2) b) we have $f(0) = 0$, so that $(0, 0)$ is a regime on N_2, and consequently, by definition, $(0, 0)$ is a common regime on the connection of N_1 and N_2. Hence, by Th. 2.1, for every $\varepsilon > 0$ there is a $\delta > 0$ such that for the common regime (J, E) corresponding to the initial state (J_0, q_0) we have $\|J(t)\| < \varepsilon$ whenever $\|J_0\| < \delta$ and $\|q_0\| < \delta$.

In addition, we have $J(t) \to 0$ as $t \to \infty$.

Note 6. As by definition of a common regime (J, E) we have $E = E_2 = f(J_2)$, it is clear that Th. 2.1 may be augmented as follows: *There is a continuous function $\tilde{\Phi}(\xi)$ vanishing at $\xi = 0$ such that*

$$\|E(t)\| \leqq \tilde{\Phi}(K_1 \|J_0\| + K_2 \|q_0\|)\,, \quad t \geqq 0\,, \tag{2.4}$$

and $E(t) \to 0$ as $t \to \infty$.

For the proof of Th. 2.1 we shall make use of the following Theorem 3.1 given in [1]:

Let $k(t)$ be a real $n \times n$ matrix satisfying the inequalities $\|k(t)\|$, $\|k'(t)\| \leqq C\,.\,\exp(-\lambda t)$, $t \geqq 0$, with $\lambda > 0$, $w(t)$ a real n-vector satisfying $\|w(t)\|$, $\|w'(t)\| \leqq W \exp(-\tilde{\lambda} t)$, $t \geqq 0$ with a fixed $\tilde{\lambda} > 0$. Moreover, let $f \in \mathfrak{N}_n(h_1, h_2)$, $h > h_2^2 h_1^{-1}$ and let

$$\tilde{k}(\omega) = \int_0^\infty \mathrm{e}^{i\omega t}\, k(t)\, \mathrm{d}t\,. \tag{2.5}$$

If there is a $q > 0$ *such that*

$$\operatorname{Re} \bar{\eta}'\{(1 + i\omega q)\,\tilde{k}(\omega) + h^{-1}I\}\,\eta \geqq 0 \tag{2.6}$$

for every real ω *and every* η, *then there is a continuous function* $\Phi(\xi)$ *vanishing at* $\xi = 0$ *and such that for the solution* $x(t)$ *of*

$$x(t) = w(t) - \int_0^t k(t - \tau)\,f(x(\tau))\,\mathrm{d}\tau \tag{2.7}$$

we have $\|x(t)\| \leqq \Phi(W)$ *for* $t \geqq 0$, *and* $x(t) \to 0$ *as* $t \to \infty$.

Proof of Th. 2.1. By Lemma 1.1 we have, for a regime (J_1, E_1) on N_1 corresponding to the initial state (J_0, q_0),

$$J_1(t) = \int_0^t a(t - \tau)\,E_1(\tau)\,\mathrm{d}\tau + i_0(t)\,, \tag{2.8}$$

and at the same time, (1.10), (1.11) hold.

On the other hand, for N_2 we have $E_2 = f(J_2)$. Thus, by definition of a common regime (J, E), $J = J_1 = -J_2$, $E = E_1 = E_2$, so that by (2.8),

$$J_2(t) = -i_0(t) - \int_0^t a(t - \tau)\,f(J_2(\tau))\,\mathrm{d}\tau\,, \tag{2.9}$$

i.e. equation (2.7).

Furthermore, using the notation of the auxiliary assertion, it is evident that

$$\tilde{k}(\omega) = \int_0^\infty \mathrm{e}^{i\omega t}\,a(t)\,\mathrm{d}t = A(i\omega)$$

(see also App. 2); hence, as $A(p) \in \mathfrak{R}_n$, it follows from Lemma 1.4 that there is a $q > 0$ such that (2.6) is satisfied for a chosen $h > h_2^2 h_1^{-1}$. Consequently, there is a function $\Phi(\xi)$ with the properties stated in the auxiliary assertion and such that $\|J_2(t)\| \leqq \Phi(W)$ for $t \geqq 0$, where the meaning of W is given by $\|i_0(t)\| \leqq W \exp(-\tilde{\lambda} t)$. But by (1.11) we may set $W = K_1\|J_0\| + K_2\|q_0\|$. Since $\|J(t)\| = \|J_0(t)\|$ and $J_2(t) \to 0$ as $t \to \infty$, (2.3) and $J(t) \to 0$ follow immediately. Hence, Th. 2.1 is proved.

Making use of Lemma 1.2 and of the fact that in the case of a V-regular n-port N_1 we may set $E_1 = V$, $J_1 = I$ for a regime (V, I) on N_1, and repeating the previous proof, we obtain the following assertion, which is dual to Th. 2.1.

Theorem 2.2. *Let* N_1 *be a* V-*regular* n-*port whose impedance matrix* $Z(p)$ *belongs to* $\mathfrak{R}_n$; *further, let* N_2 *be a conductive* n-*port such that* $J_2 = f(E_2)$ *and* $f \in \mathfrak{R}_n(h_1, h_2)$. *Then there exist a continuous function* $\tilde{\Phi}(\xi)$ *vanishing at* $\xi = 0$ *and positive constant* $\tilde{K}_1$, $\tilde{K}_2$ *such that, for any common regime* (J, E) *on the parallel connection of* N_1

and N_2, we have

$$\|E(t)\| \leqq \tilde{\Phi}(\tilde{K}_1\|V_0\| + \tilde{K}_2\|F_0\|)$$

for every $t \geqq 0$, and $E(t) \to 0$ as $t \to \infty$, where (V_0, F_0) is the initial state of N_1.

Observe that, under the assumptions of Th. 2.2, the Corollary following Th. 2.1 and a note analogous to Note 6 hold.

Next we have the following assertion:

Theorem 2.3. *Let N_1 be a J-regular n-port whose admittance matrix $A(p)$ satisfies the condition $A(p) = p^{-1}\gamma + B(p)$, where $B(p) \in \mathfrak{N}_n$ and γ is positive definite; further, let N_2 be a resistive n-port such that $E_2 = f(J_2)$ and $f \in \mathfrak{N}_n^*$. Then there exists a continuous function $\Psi(\xi, \eta)$ vanishing for $\xi = \eta = 0$ and such that, for a common regime (J, E) on the parallel connection of N_1 and N_2, we have*

$$\|J(t)\| \leqq \Psi(\|J_0\|, \|q_0\|) \tag{2.10}$$

for every $t \geqq 0$, and $J(t) \to 0$ as $t \to \infty$, where (J_0, q_0) is the initial state of N_1.

The proof of Th. 2.3 is based on the following assertion which follows immediately from Th. 1.4 in [1]:

Let $k(t)$ be a real $n \times n$ matrix fulfilling the inequalities $\|k(t)\|, \|k'(t)\| \leqq \leqq C \exp(-\tilde{\lambda}t)$, $t \geqq 0$, $C, \tilde{\lambda} > 0$, $w(t)$ a real n-vector fulfilling the inequalities $\|w(t)\|, \|w'(t)\| \leqq W \exp(-\tilde{\lambda}t)$, γ a symmetric positive definite $n \times n$ matrix and c a real constant vector with $\|c\| \leqq K$. Moreover, let $f \in \mathfrak{N}_n^*$ and let $\tilde{k}(\omega)$ be defined by (2.5). If

$$\operatorname{Re} \bar{\eta}' \tilde{k}(\omega)\, \eta \geqq 0 \tag{2.11}$$

for every real ω and every η, then there is a continuous function $\tilde{\Psi}(\alpha, \beta)$ vanishing at $\alpha = \beta = 0$ such that, for the solution $x(t)$ of

$$x(t) = C + w(t) - \int_0^t (\gamma + k(t - \tau)) f(x(\tau))\, d\tau\,, \tag{2.12}$$

we have $\|x(t)\| \leqq \tilde{\Psi}(W, K)$ for every $t \geqq 0$, and $x(t) \to 0$ as $t \to \infty$.

Proof of Th. 2.3. By Lemma 1.3 we have, for a regime (J_1, E_1) on N_1 corresponding to the initial state (J_0, q_0),

$$J_1(t) = \int_0^t (\gamma + b(t - \tau))\, E_1(\tau)\, d\tau + i_0(t)\,, \tag{2.13}$$

and, at the same time, (1.27), (1.28) and (1.29) hold. On the other hand, for a regime (J_2, E_2) on N_2, we have $E_2 = f(J_2)$. By definition of a common regime on the parallel connection of N_1 and N_2, $J = J_1 = -J_2$, $E = E_1 = E_2$; consequently, by (2.13),

$$J_2(t) = (i_0(\infty) - i_0(t)) - i_0(\infty) - \int_0^t (\gamma + b(t - \tau)) f(J_2(\tau))\, d\tau\,, \tag{2.14}$$

which is an equation of the type (2.12).

Moreover, as $b = B(D)\,\delta_0$, we have

$$B(i\omega) = \int_0^\infty \exp(i\omega t)\, b(t)\, \mathrm{d}t\,;$$

but $B(p) \in \mathfrak{S}_n$ so that (2.11) is satisfied with $\tilde{k}(\omega) = B(i\omega)$. Thus, by the auxiliary assertion there is a continuous function $\widetilde{\Psi}(\alpha, \beta)$ vanishing at the origin such that, with regard to (1.28), (1.29),

$$\|J_2(t)\| \leqq \widetilde{\Psi}(K_1\|J_0\| + K_2\|q_0\|,\, K_3\|J_0\| + K_4\|q_0\|)\,.$$

At the same time, $J_2(t) \to 0$ as $t \to \infty$. Finally, setting $\Psi(u, v) = \widetilde{\Psi}(K_1 u + K_2 v, K_3 u + K_4 v)$ and recalling the equality $\|J(t)\| = \|J_2(t)\|$, (2.10) follows immediately. Hence, Th. 2.3 is proved.

Making use of the dual assertion of Lemma 1.3 and of the auxiliary theorem, we obtain the following counterpart to Th. 2.3:

Theorem 2.4. *Let N_1 be a V-regular n-port whose impedance matrix $Z(p)$ satisfies the condition $Z(p) = p^{-1}\gamma + C(p)$, where $C(p) \in \mathfrak{R}_n$ and γ is positive definite; further, let N_2 be a conductive n-port such that $J_2 = f(E_2)$ and $f \in \mathfrak{N}_n^*$. Then there exists a continuous function $\Psi(\xi, \eta)$ vanishing at $\xi = \eta = 0$ such that, for a common regime (J, E) on the parallel connection of N_1 and N_2, we have*

$$\|E(t)\| \leqq \Psi(\|V_0\|, \|F_0\|) \tag{2.15}$$

for every $t \geqq 0$, and $E(t) \to 0$ as $t \to \infty$, where (V_0, F_0) is the initial state of N_1.

Note also that under the assumptions of either Th. 2.3 or Th. 2.4, the Corollary following Th. 2.1 and Note 6 hold.

Note 7. The criteria of stability obtained in theorems 2.1 to 2.4 are advantageous particularly due to the fact that in practical cases it is not necessary to set up the system of nonlinear equations describing the behavior of the electrical system in question, but only to establish the admittance (impedance) matrix of the linear part, which may be performed by well-known methods directly from the structure of the circuit, and then check whether the admittance (impedance) matrix obtained belongs to $\mathfrak{R}_n$ or has the form considered in theorems 2.3 and 2.4.

Note 8. The criteria developed above may also be used for checking the stability of nonzero constant equilibria. Indeed, consider for example the situation dealt with in Th. 2.1, and assume in addition that N_1 contains some inner constant voltage sources such that a nonzero constant common regime (J^*, E^*) exists for an initial state (J_0^*, q_0^*) of N_1. Then instead of (2.9) we have

$$J_2^* = -i_0^*(t) - \int_0^t a(t - \tau) f(J_2^*)\, \mathrm{d}\tau - \int_0^t a_{12}(t - \tau)\, e\, \mathrm{d}\tau\,,$$

where the vector e represents the inner voltage sources. (Also see (1.15).) Since, for any common regime (J, E) corresponding to an initial state (J_0, q_0) and to the same vector e, the equation

$$J_2(t) = -i_0(t) - \int_0^t a(t-\tau) f(J_2(\tau))\,\mathrm{d}\tau - \int_0^t a_{12}(t-\tau)\,e\,\mathrm{d}\tau$$

holds, we have for $x(t) = J_2(t) - J_2^*$,

$$x(t) = -\tilde{i}_0(t) - \int_0^t a(t-\tau)\,g(x(\tau))\,\mathrm{d}\tau\,,$$

where $g(x) = f(x + J_2^*) - f(J_2^*)$. Thus the previous results are applicable to this case provided the vector function g has the required properties.

Concluding this paper let us present some applications of the results given above.

Example 1. Consider the oscillator with a tunnel diode schematically plotted in Fig. 3, and investigate the circuit behavior for $t \geqq 0$, provided the battery is switched off at $t = 0$. Assuming for the first approximation that the tunnel diode behaves as a parallel connection of a constant capacity K with a nonlinear resistance (see Fig. 4a) whose characteristic is plotted in Fig. 4b, we may consider the circuit as a parallel connection of a linear passive dipole (1-port) with a nonlinear one. (See Fig. 5.) The initial state of the linear dipole is given by the voltages V_0, $\tilde{V}_0$ across the capacity C and K, respectively, and by the flux F_0 of the coil.

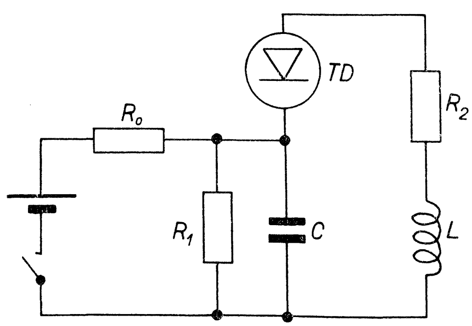


Fig. 3.

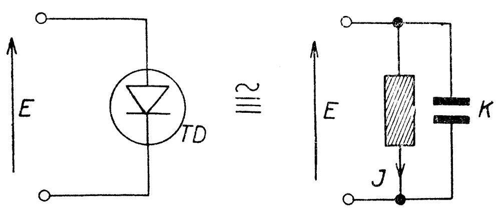


Fig. 4a.

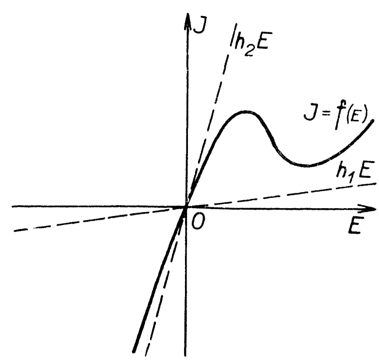


Fig. 4b.

From Fig. 4b it is clear that for $J = f(E)$ we have $f \in \mathfrak{N}_1(h_1, h_2)$; moreover, for the impedance $Z(p)$ of the linear dipole we obtain

$$Z^{-1}(p) = Kp + A_1(p)\,, \tag{2.16}$$

where

$$A_1(p) = \frac{1 + R_1Cp}{R_1 + (1 + R_1Cp)(R_2 + Lp)}\,, \tag{2.17}$$

so that

$$Z(p) = \frac{R_1 + (1 + R_1Cp)(R_2 + Lp)}{1 + R_1(C + K)\,p + (1 + R_1Cp)(R_2 + Lp)\,Kp}\,. \tag{2.18}$$

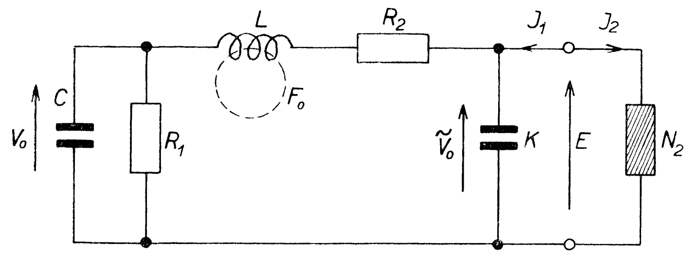


Fig. 5.

Since Kp is pure imaginary on the imaginary axis and $\operatorname{Re} A_1(i\omega) > 0$ for every real ω as can be easily verified from (2.17), we have $Z^{-1}(i\omega) \neq 0$; hence $Z(p)$ has no poles on the imaginary axis, and in addition, by (2.18), $Z(\infty) = 0$. Since also $Z(p) \in \mathfrak{S}_1$, we have $Z(p) \in \mathfrak{N}_1$. Thus by Th. 2.2, there is a continuous function $\tilde{\Phi}(x)$ with $\tilde{\Phi}(0) = 0$ and constants $\tilde{K}_1, \tilde{K}_2 > 0$ such that the voltage $E(t)$ accross the tunnel diode fulfils the inequality $|E(t)| \leqq \tilde{\Phi}(\tilde{K}_1(V_0 + \tilde{V}_0)^{\frac{1}{2}} + \tilde{K}_2|F_0|)$, $t \geqq 0$, and simultaneously, $E(t) \to 0$ as $t \to \infty$. Hence, after switching off the battery at $t = 0$ the circuit always returns to the "dead" state independently of the state at $t = 0$.

Example 2. Consider a demodulator circuit with a silicon diode plotted in Fig. 6. Our task is to establish the behavior of the circuit if the exterior excitation e disappears.

Following the ideas developed above, consider the circuit in question as a parallel connection of a linear passive dipole with a nonlinear one. (See Fig. 7a.) The silicon diode behaves as a conductive dipole, whose characteristic is shown in Fig. 7b. Thus $J = f(E)$ with $f \in \mathfrak{N}_1(h_1, h_2)$. On the other hand, for the impedance of the linear dipole we have

$$Z(p) = (Z_1^{-1} + Z_2^{-1})^{-1} \tag{2.19}$$

with

$$(2.20)\qquad Z_1 = R_1 + \frac{R_2}{1 + R_2C_3p}, \qquad Z_2 = \frac{1}{C_2p} + \frac{\varrho + Lp}{1 + \varrho C_1p + LC_1p^2}.$$

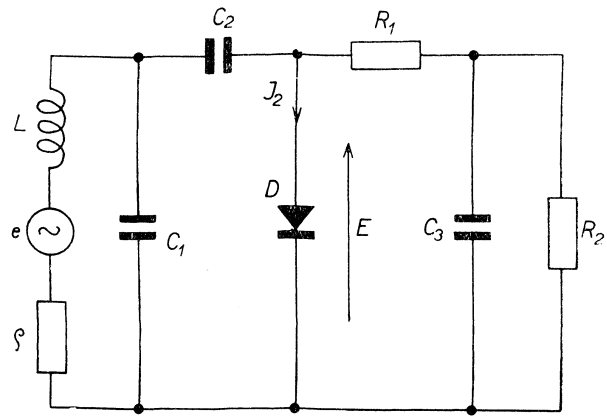
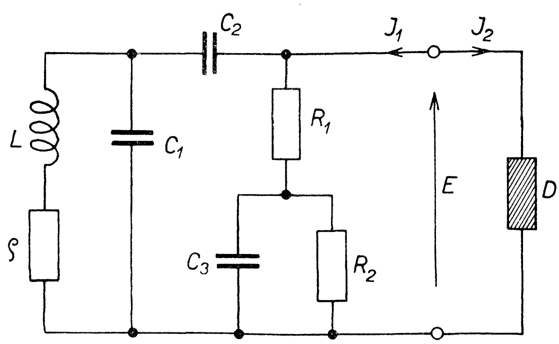


Fig. 6.



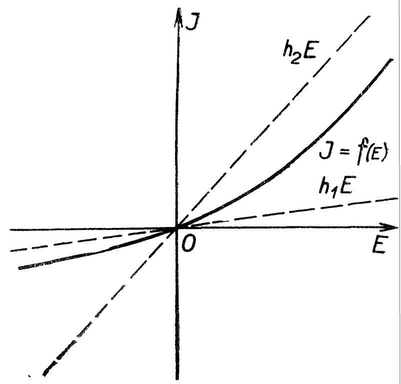


Fig. 7a. Fig. 7b.

From the expression for Z_1^{-1} it is readily seen that $\operatorname{Re} Z_1^{-1}(i\omega) > 0$ for every ω. As $Z_2^{-1}(p) \in \mathfrak{S}_1$, we have $\operatorname{Re} Z_2^{-1}(i\omega) \geqq 0$, and consequently, $Z_1^{-1} + Z_2^{-1} \neq 0$ on the imaginary axis. Hence by (2.19), $Z(p)$ has no poles on the imaginary axis. Moreover, directly from Fig. 7a it is obvious that $Z(\infty) = 0$. As $Z(p) \in \mathfrak{S}_1$, we have $Z(p) \in \mathfrak{R}_1$.

Thus, using Theorem 2.2, there is a continuous function $\tilde{\Phi}(x)$ vanishing at $x = 0$ and constants $\tilde{K}_1, \tilde{K}_2 > 0$ such that $|E(t)| \leqq \tilde{\Phi}(\tilde{K}_1\|V_0\| + \tilde{K}_2\|F_0\|)$ for every $t \geqq 0$, where V_0 is a 3-vector of initial voltages on the capacitors C_1, C_2, C_3, and F_0 the initial flux of the coil. Moreover, $E(t) \to 0$ as $t \to \infty$. From this it also follows that each current in the circuit tends to zero as $t \to \infty$.

Appendix 1.

Let us present the physical background of eq. (1.3). Consider a "branch" h_k consisting of a parallel connection of capacity C_k, conductance A_k, inductance $1/M_k$ and current source I_k (see Fig. 8). Then for the voltage V_k across h_k we have $V_k' = C_k^{-1}(i_1)^{(-1)} + V_{k0}H_0$, where V_{k0} denotes the initial value of V_k due to the initial condenser charge. Furthermore, $i_2 = A_kV_k$ and $V_k = M_k^{-1}i_3'$. Thus $i_1 = C_k(V_k' - V_{k0}\delta_0)$ and $i_3 = M_kV_k^{-(1)} + M_kF_{k0}H_0$, where F_{k0} is the initial magnetic flux of the coil. Consequently, for the total branch current $\tilde{I}_k$ we have

$$\tilde{I}_k = (C_kD + A_k + M_kD^{-1})\,V_k - C_kV_{k0}\delta_0 + M_kF_{k0}H_0 - I_k\,.$$

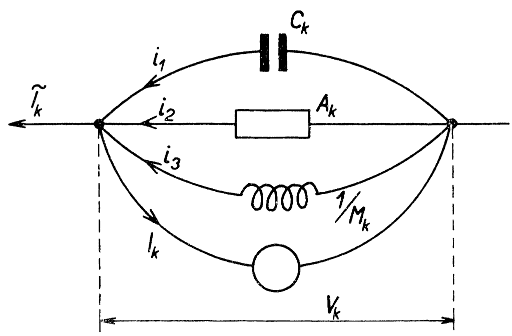


Fig. 8.

Next, if a network is formed of branches h_k, $k = 1, \ldots, r$, already discussed whoses tructure is given by an oriented graph G with branch-node incidence matrix a, then by the second Kirchhoff law we have

$$\text{(A.1)} \qquad a'\{(CD + A + MD^{-1})\,V - CV_0\delta_0 + MF_0H_0 - I\} = 0\,,$$

where C, A, M are $r \times r$ matrices with elements C_k, A_k, M_k, respectively, and V, I, V_0, F_0 are r-vectors formed of the corresponding elements V_k, I_k, V_{k0}, F_{k0}, respectively.

On the other hand, by the first Kirchhoff law we have

$$\text{(A.2)} \qquad X'V = 0\,,$$

where X has the usual meaning. Since $a'\text{A} = 0$, i.e. $X'a = 0$, the columns of d (a complete set of linearly independent columns of a) constitute a complete set of solutions of $X'\xi = 0$. Thus, from (A.2), there is a vector w such that $V = dw$. Moreover, since (A.1) is equivalent with $d'\{\ldots\} = 0$, we get

$$d'\,Y(D)\,dw = d'\{I + CV_0\delta_0 - MF_0H_0\}$$

with $Y(D) = CD + A + MD^{-1}$. From this (1.3) follows immediately, provided $\det d'\,Y(p)\,d \not\equiv 0$.

Appendix 2.

In the text we have used the formula $A(D)\,E = (A(D)\,\delta_0) * E$ without comment; to this end, one has the following assertion:

a) *Let f, g be distributions vanishing on $(-\infty, 0)$, $r(p)$ a rational function; then*

$$r(D)(f * g) = (r(D) f) * g . \tag{A.3}$$

In particular,

$$r(D) f = (r(D)\,\delta_0) * f . \tag{A.4}$$

b) *If in addition f possesses a Laplace transform $\mathscr{L}(f)$, then $r(D) f$ also does, and*

$$\mathscr{L}(r(D) f) = r(p)\,\mathscr{L}(f) . \tag{A.5}$$

Proof. Since $r(D)$ is a Heaviside operator, it can always be expanded as (see [3], p. 175)

$$r(D) = Q(D) + \sum_{i,k} \lambda_{ik}(D + \alpha_i)^{-k} , \quad k \geqq 1 , \tag{A.6}$$

where $Q(p)$ is a polynomial. Since $(f * g)^n = f^n * g$ for any integer n (cf. [3], p. 60), it is sufficient to prove (A.3) for $r(D) = (D + a)^{-1}$. Thus, putting $x = ((D + a)^{-1} f) * g$, we have

$$(D + a)\,x = x' + ax = ((D + a)^{-1} f)' * g + a((D + a)^{-1} f) * g =$$
$$= (D(D + a)^{-1} f + a(D + a)^{-1} f) * g = f * g .$$

Hence, $x = (D + a)^{-1} (f * g)$ as required.

Formula (A.4) follows immediately from (A.3) by using the fact that $\delta_0 * f = f$ (cf. [3], p. 60).

The assertion b) is given in Th. 5.5 – 2 in [3], p. 176.

Appendix 3.

In the proofs of Lemma 1.1 and 1.3 we have used the following version of Tauber's theorem:

Let $r(p)$ be a rational function with no poles in the closed right half-plane (∞ included); then $r(D)\,H_0 = f$ is a regular distribution and $f(\infty) = r(0)$.

Proof. Let $r(p) = r_0 + \tilde{r}(p)$ with $\tilde{r}(\infty) = 0$. Then

$$f = r_0 H_0 + \tilde{r}(D)\,H_0 = r_0 H_0 + D^{-1}(\tilde{r}(D)\,\delta_0) = r_0 H_0 + D^{-1}\varrho ,$$

where $\varrho = \tilde{r}(D)\,\delta_0$ is obviously regular and of exponential type. By (A.5) we also have $\mathscr{L}(\varrho) = \tilde{r}(p)$. Thus, for $t \geqq 0$,

$$f(t) = r_0 + \int_0^t \varrho(\tau)\,\mathrm{d}\tau ,$$

and consequently,

$$f(\infty) = r_0 + \int_0^t \varrho(\tau)\,\mathrm{d}\tau = r_0 + \tilde{r}(0)$$

which finishes the proof.

References

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[2] *Doležal V. — Vorel Z.:* Theory of Kirchhoff's Networks; Čas. pěst. matem. 87 (1962), 440—476.

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Výtah

APLIKACE POPOVOVY METODY V TEORII ELEKTRICKÝCH SÍTÍ

VÁCLAV DOLEŽAL

V článku je dokázáno několik vět o stabilitě klidového stavu nelineárních elektrických sítí, které jsou vytvořeny spojením lineárního $2n$-pólu s konstantními soustředěnými prvky a nelineárního $2n$-pólu, který má odporový charakter.

Nejprve jsou vyšetřeny jisté souvislosti mezi admitační (impedanční) maticí lineárního $2n$-pólu a vlastnostmi přechodových režimů na svorkách. Na tyto výsledky jsou pak aplikovány Popovovy metody, rozšíření na vektorové nelinearity [1], čímž jsou získány podmínky absolutní stability režimu na společných svorkách spojení společně s odhady pro normu režimu v závislosti na počátečním stavu lineární části.

Odvozená kritéria jsou prakticky výhodná z toho důvodu, že je lze v konkrétních případech vyhodnotit přímo ze struktury a hodnot prvků sítě, takže není nutno sestavovat nelineární soustavu rovnic, popisující dynamiku sítě.

Použití odvozených výsledků je ilustrováno na příkladě oscilátoru s tunelovou diodou a demodulátoru s křemíkovou diodou.

Резюме

ПРИМЕНЕНИЕ МЕТОДА ПОПОВА В ТЕОРИИ ЭЛЕКТРИЧЕСКИХ ЦЕПЕЙ

ВАЦЛАВ ДОЛЕЖАЛ (VÁCLAV DOLEŽAL)

В статье доказано несколько теорем об устойчивости состояния покоя нелинейных электрических цепей, образованных соединением линейного $2n$-полюсника с постоянными сосредоточенными элементами и нелинейного $2n$-полюсника омического характера.

Сначала исследуются некоторые связи между адмитансной (импедансной) матрицей линейного $2n$-полюсника и свойствами переходных режимов на за-

жимах. К этим результатам применяются затем методы Попова, распространенные на векторные нелинейности [1], и таким образом выводятся условия абсолютной устойчивости режима на общих зажимах соединения вместе с оценками для нормы режима в зависимости от начального состояния линейной части.

Полученные критерии выгодны на практике потому, что их можно в конкретных случаях вычислить непосредственно по структуре и значениям элементов цепи, так что нет надобности составлять нелинейную систему уравнений, описывающую динамику цепи.

Применение выведенных результатов иллюстрируется на примере осцилятора с тоннельным диодом и демодулятора с кремниевым диодом.

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