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ON SOME NEW ITERATIVE METHODS FOR SOLVING THE EIGENPROBLEM OF A LINEAR OPERATOR

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1. INTRODUCTION

In many physical applications it is necessery to determine at least the first eigenvalue of a differential or integral operator. The purpose of this paper is to demonstrate the advantage of several simple computational techniques, thus completing the work presented at the conference on Basic Problems of Numerical Analysis held in Liblice in 1964. Birger's, Kolomý's and Kellogg's methods are compared with the widely used method of steepest descent. The comparison is done from the point of view of memory requirements, routine degree and the speed of convergence. In the theoretical part the operator A is not specified in detail. The results may be used for solving integral equations as well as for matrices.

2. THE DERIVATION OF ITERATION PROCESSES

Let a linear operator equation

$$Ax - \mu x = 0 \tag{1}$$

be given, where A is a linear bounded operator in a complex Hilbert space H and μ is a real parameter. This equation will be solved by an iteration process

$$x_{k+1} = \frac{1}{\mu_{k+1}} Ax_k \,. \tag{2}$$

The parameters μ_{k+1} will be determined from the condition that the function $\|Ax - \tau x\|^2$ on the set of all real numbers $\tau \in (-\infty, \infty)$ attains the minimal value for the element x_k. We get

$$\mu_{k+1} = \frac{(Ax_k, x_k)}{(x_k, x_k)} \,. \tag{3}$$

This iteration process was derived by Kolomý [4], [5]. (Our Eq. (3) corresponds to Eq. (3) of [5], p. 36; in the latter equation as well as in the following Eq. (4) given in [5] there is obviously a misprint.) Now the following Theorem holds ([4], p. 18, 19, Theorem 1 and 2).

Theorem 1. *Let A be a positive compact operator in a complex Hilbert space H (i.e. $(Ax, x) > 0$ for $x \in H$, $x \neq 0$). Let $x_0 \in H$, $x_0 \neq 0$ be not orthogonal to the eigenspace $H_{\mu_1 *}$ corresponding to the first eigenvalue μ_1^* of (1). Then the sequence $\{\mu_k\}$ defined by (3), (2) is monotone, increasing and it converges to μ_1^*. The sequence $\{x_k\}$ defined by (2), (3) is convergent in H to one of the eigenvectors corresponding to μ_1^*.*

I. A. Birger [1] introduced a similar method, but without any conditions and without a proof of convergence. It differs from the process (2), (3) in determing the parameters μ_{k+1} and thus from the condition that the function $\|(1/\tau)\, Ax - x\|^2$ attains the minimal value on the set of all real numbers $\tau \in (-\infty, \infty)$. Here, we have

$$\mu_{k+1} = \frac{(Ax_k, Ax_k)}{(Ax_k, x_k)}. \tag{4}$$

Under the same assumptions as in Theorem 1 in [5], p. 43, Theorem 4 supplies the convergence proof for this method. If the operator A is not symmetric, the method (2), (3) can be used with a certain modification provided the operator A is symmetrizable (see Theorem 3, [5], p. 42). Marek [9], p. 53 has shown that the process (2), (3) can be generalized for linear unsymmetrizable operators, too.

In paper [6] the assumptions of Theorem 1 are not so strong. The assumption that the operator A is compact is omitted and it is shown that the sequence $\{\mu_k\}$ converges also in the case that the largest element of the spectrum $\sigma(A)$ is not an eigenvalue of the operator A. It is not even supposed that μ_1^* is an isolated point of the spectrum $\sigma(A)$. It is shown that the above mentioned methods can be used for finding the extreme values of the spectrum $\sigma(A)$.

The following assertion on the speed of convergence is true: Suppose that A is a linear self-adjoint positive mapping of a real Hilbert space H into H. Let $\tilde{\mu}_1$ be the largest and m the smallest element of the spectrum ($m \leqq \mu \leqq M < \tilde{\mu}_1$). Then $\tilde{\mu}_1$ is an eigenvalue of A. Let us denote $H_{\tilde{\mu}_1}$ the eigenspace corresponding to $\tilde{\mu}_1$ and $e(\|e\| = 1)$ the projection of $x_0 \notin H_2$, $x_0 \neq 0$ where H_2 is the orthogonal complement of $H_{\tilde{\mu}_1}$. Then $H = H_{\tilde{\mu}_1} \oplus H_2$ and for every x_k, $(k = 0, 1, 2, \ldots)$ defined by (2) we have a unique decomposition $x_k = \xi_k e + h_k$, where $h_k \in H_2$ and $(e, h_k) = 0$. Furthemore, there exist numbers q_k ([7], theorem 3) defined by

$$q_k = [1 - \xi_k^2 \|x_k\|^{-2} (1 - M\tilde{\mu}_1^{-1})]^{1/2}, \quad k = 0, 1, 2, \ldots,$$

such that

$$q_{k-1} < q_{k-2} < \ldots < q_0 < 1$$

so that

$$\tilde{\mu}_1 - \mu_k < q_{k-1}^2 \cdot q_{k-2}^2 \dots q_0^2 (\tilde{\mu}_1 - m) \, \|h_0\|^2 \cdot \|x_0\|^{-2} .$$

Kellogg's process is different ([12], [13]); here

$$x_{k+1} = \frac{Ax_k}{\|Ax_k\|} , \quad \mu_{k+1} = \frac{\|Ax_k\|}{\|x_k\|} . \tag{5}$$

All these iteration methods can be summarized in one class (Marek [8], [9], [10]).

Let X be a complex Banach space, X' be the adjoint space of continuous linear forms on X and X_1 the space of bounded linear transformations mapping X into X. Let $A \in X_1$ and let $R(\lambda, A) = (\lambda I - A)^{-1}$ be the resolvent of the operator A at the point λ of the complex plane Π. The resolvent $R(\lambda, A)$ can be developed into a Laurent series

$$R(\lambda, A) = \sum_{i=0}^{\infty} (\lambda - \mu_0)^i A_i + \sum_{i=0}^{\infty} (\lambda - \mu_0)^{-i} B_i ,$$

where

$$B_1 = \frac{1}{2\pi i} \int_{C_0} R(\lambda, A) \, \mathrm{d}\lambda , \quad B_{i+1} = (A - \mu_0 I) B_i , \quad i = 1, 2, \dots$$

and C_0 is a positively oriented circle with center at μ_0 and such that no points of the spectrum $\sigma(A)$ except μ_0 lie on or inside the circle C_0. Further we will call the point $\mu_0 \in \sigma(A)$ the dominant point of the spectrum of the operator A, if $|\lambda| < |\mu_0|$ for every point $\lambda \in \sigma(A)$, $\lambda \neq \mu_0$.

Let the following assumptions be satisfied:

a) The operator A is a linear bounded operator mapping the space X into X.

b) The value μ_0 is the dominant point of the spectrum of the operator A.

The symbol o will signify the zero-vector in the space X.

Let $\{x_k'\}, \{y_k'\}, \{z_k'\}$ be sequences of linear forms mapping X into Π. Let the forms $x' \in X'$, $y' \in X'$ exist such that

$$x'(x) = \lim_{k \to \infty} x_k'(x) , \quad y'(x) = \lim_{k \to \infty} y_k'(x) = \lim_{k \to \infty} z_k'(x) \tag{6}$$

for every vector $x \in X$. Let $x^{(0)} \in X$ be a definite fixed vector such that $B_1 x^{(0)} \neq o$ so that there is an index $s (1 \leqq s)$ with

$$B_s x^{(0)} \neq o , \quad B_{s+1} x^{(0)} = o . \tag{7}$$

Further let

$$x'(B_s x^{(0)}) \neq o , \quad y'(B_s x^{(0)}) \neq o \tag{8}$$

hold and let us put

$$x_{(0)} = \frac{B_s x^{(0)}}{x'(B_s x^{(0)})} . \tag{9}$$

The Kellogg's iterations are constructed according to the following formulas

$$x^{(k)} = Ax^{(k-1)}, \quad x_{(k)} = \frac{x^{(k)}}{x'(x^{(k)})}, \tag{10}$$

$$\mu_{(k)} = \frac{z_k'(x^{(k+1)})}{y_k'(x^{(k)})}. \tag{11}$$

We have ([8], theorem 2, p. 540).

Theorem 2. *Let* (6) *hold for the forms* x_k', y_k', z_k', x', y'. *Let* $x^{(0)} \in X$ *be a vector such that* (7) *and* (8) *hold. Then*

$$\lim_{k\to\infty} x_{(k)} = x_0 \tag{12}$$

for sequence (10) *in the norm of the space* X *and*

$$\lim_{k\to\infty} \mu_{(k)} = \mu_0 \tag{13}$$

for the numerical sequence (11). *The vector* x_0 *is the eigenvector of the operator* A *corresponding to the value* μ_0.

If μ_0 is a real simple dominant eigenvalue of the operator $A \in X_1$, we can write the iteration process (10), (11) in a simpler form ([9], p. 54) as

$$x_{(k)} = \frac{A^k x^{(0)}}{y_k'(A^k x^{(0)})}, \quad \mu_{(k)} = \frac{y_k'(A^{k+1}x^{(0)})}{y_k'(A^k x^{(0)})}. \tag{14}$$

The iterations determined by formulas (14), where $x^{(0)} \in X$ is a suitable vector, converge to an eigenvector x_0 of the operator A corresponding to the eigenvalue μ_0. Let

$$x_{k+1} = \lambda_{(k)} A x_k, \quad x_0 = x^{(0)}, \tag{15}$$

where

$$\lambda_{(k)} = \frac{y_k'(x_k)}{y_k'(Ax_k)}. \tag{16}$$

Then the relations $x_k \to x_0$, $\lambda_{(k)} \to \lambda_0 = \mu_0^{-1}$ hold according to Theorem 3, [8].

The form of the functionals $y_k' \in X'$ is almost arbitrary and therefore we shall ask which functionals are the most effective ones. The concept of the process effectiveness of an iteration type will be expressed in terms of some extremal properties. Let A' be the adjoint operator of A. Let a real function of the real variable

$$\psi_k(\tau) = l_k'(Ax_k - \tau x_k), \quad \text{where} \quad l_k' = A'x_k' - \tau x_k'$$

be given. Let the sequence $x'_k \in X'$ be defined by the formula

$$x'_{k+1} = \lambda_{(k)} A' x'_k, \quad x'_0 = x'^{(0)} \in X', \tag{17}$$

where the parameters $\lambda_{(k)}$ are to be determined from the condition that the function ψ_k attains the minimal value. Then

$$\frac{1}{\lambda_{(k)}} = \frac{y'_k(A^{k+1}x_0)}{y'_k(A^k x_0)} = \frac{y'_0(A^{2k+1}x_0)}{y'_0(A^{2k}x_0)},$$

from what we can infer that the "effective" process gives the approximation of degree $2k$ which is equal to the approximation of degree k given by the usual iteration process (14) with $x'_k = x'_0$ for $k = 0, 1, \ldots$ This property can be succesfully used for the symmetric operators in a Hilbert space. In this case process (17) is identical to that given by (15); therefore a half of computations falls off. We have

$$y'_k(x) = (x, x'_k) = (x, x_k)$$

and

$$\psi_k(\tau) = (Ax_k - \tau x_k, Ax_k - \tau x_k),$$

this being the Kolomý's process.

Note. The process mentioned above can be utilized without any difficulty for constructing the eigenelements of equations of the type $Lx = Bx + \lambda Cx$, $L'x' = B'x' + \lambda C'x'$, where L, B, C are, in general, unbounded linear operators mapping the domains $\mathscr{D}(L)$, $\mathscr{D}(B)$, $\mathscr{D}(C)$ into X (see [9], p. 57).

In a Hilbert space iteration methods, in which Schwarz constants of the type (3), (4) appear, are used for determining the eigenvalues of symmetric compact operators. The iterations given in [1] and [4] are analogous to this method; moreover, they provide the instruction (2) for constructing the eigenvectors, which differs from the Kellogg's original formula (5). The letter can be obtained by putting $y'_k(x) = z'_k(x) = \|x\|$ in formula (10).

If we choose the sequences of forms $\{y'_k\}$, $\{z'_k\}$ in a specific way, other well known iteration processes are obtained.

Let us assume that the operator A has a positive dominant eigenvalue; let H be a Hilbert space with the inner product (x, y). Let the assumptions of Theorem 2 be fulfilled with $\{x'_k\}$ being an arbitrary sequence for which (6) holds. Let the sequences $\{y'_k\}$, $\{z'_k\}$ be defined by one of the formulas

$$y'_k(x) = z'_k(x) = (x, x_{(k)}), \tag{18}$$

$$y'_k(x) = z'_k(x) = (x, Ax_{(k)}). \tag{19}$$

Then, according to Theorem 2, $\lim_{k\to\infty} (x^{(k+1)}, x^{(k)})/(x^{(k)}, x^{(k)}) = \mu_0$ for the case (18), $\lim_{k\to\infty} (x^{(k+1)}, x^{(k+1)})/(x^{(k+1)}, x^{(k)}) = \mu_0$ for the case (19).

Similar processes can be used for finding the eigenvalues of nonlinear bounded operators, too. For example, H. F. Bueckner [2] used the process

$$\lambda_{(k)} = \frac{\operatorname{Re}(x_k, Ax_k)}{(Ax_k, Ax_k)}$$

for solving the nonlinear Nekrasov's integral equation, determined the sufficient conditions and proved its convergence.

The method of steepest descent [3] was developed in a quite different way. The idea of this method is to look for the maximum of the functional

$$L(x) = \frac{(Ax, x)}{(x, x)},$$

which coincides with the eigenvalue μ_1^* of A and for the element x_1^* which maximizes $L(x)$ and is the eigenelement corresponding to μ_1^*.

We shall choose arbitrarily the first approximation $x_0 \in H$ and form $L(x_0 + \varepsilon r)$. The direction of the steepest descent is given by the element $-r_0 = \mu_0 x_0 - Ax_0$ and the functional attains its maximum in this direction for $\varepsilon = \varepsilon_0$, where

$$\varepsilon_0 = \frac{(r_0, r_0)}{(Ar_0, r_0) - \mu_0(r_0, r_0)}.$$

Therefore the next approximation will be realized in the form $x_1 = x_0 + \varepsilon_0 r_0$; similarly we proceed in next steps. The following Theorem on corvengence holds ([3], Theorem 1 (2, XV), p. 550).

Theorem 3. *Let A be a symmetric positive definite compact operator. If x_0 is not orthogonal to the eigenspace $H_{\mu_1 *}$, then the sequence $\{\mu_k\}$ converges to the largest eigenvalue μ_1^* of A and the sequence $\{x_k\}$ converges strongly to one of eigenvectors corresponding to μ_1^*. The speed of convergence is geometric.*

3. DESCRIPTION OF METHODS

We shall use the method mentioned in part 2 for the calculation of the first characteristic value of the integral operator given by the integral equation

$$y(x) = \lambda \int_0^1 G(x, s)\, y(s)\, \mathrm{d}s, \tag{20}$$

which will be written symbolically in the form $y = \lambda Gy$. Under certain assumption we have $\mu = 1/\lambda$, where λ is a characteristic value of the operator G and μ an eigenvalue of the corresponding differential operator. Eigenvectors and characteristic vectors are equal.

The inner product in the real Hilbert space $L_{2\langle 0,1\rangle}$ is defined by the formula $(y, v) = \int_0^1 y(s)\, v(s)\, ds$. The iteration processes for the respective methods have the following forms:

Birger's method (B),

$$y^{(k+1)} = \lambda^{(k)} G y^{(k)}, \tag{21}$$

$$\lambda^{(k)} = \frac{(y^{(k)}, Gy^{(k)})}{(Gy^{(k)}, Gy^{(k)})}. \tag{22}$$

Kolomý's method (K) differ from the method mentioned above in the calculation of numbers $\lambda^{(k)}$, i.e. here

$$\lambda^{(k)} = \frac{(y^{(k)}, y^{(k)})}{(y^{(k)}, Gy^{(k)})}. \tag{23}$$

In the same way the characteric number is calculated by the method of steepest descent (M). The corresponding eigenvector, however, will be obtained by a more complicated process,

$$y^{(k+1)} = y^{(k)} + a^{(k)} r^{(k)}, \tag{24}$$

$$r^{(k)} = \frac{1}{\lambda^{(k)}} y^{(k)} - Gy^{(k)}, \tag{25}$$

$$a^{(k)} = \frac{(r^{(k)}, r^{(k)})}{(r^{(k)}, Gr^{(k)}) - \frac{1}{\lambda^{(k)}} (r^{(k)}, r^{(k)})}. \tag{26}$$

Kellogg's method (L) uses the fomulas

$$y^{(k+1)} = \frac{Gy^{(k)}}{\|Gy^{(k)}\|}, \tag{27}$$

$$\lambda^{(k)} = \frac{\|y^{(k)}\|}{\|Gy^{(k)}\|}. \tag{28}$$

The first approximation in all methods is arbitrary, the number $k = 0, 1, 2, \ldots$ designates the succession number of iteration.

The Birger's book [1] contains also the instruction for the calculation of the second characteristic value of the operator G solving the equation $y = \lambda Fy$, the first characteristic value of which is the second characteristic value of the operator G. This property is exhibited by an equation with the kernel of the following form

$$G(x, s) - \frac{v(x)\, v(s)}{\lambda^*} = F(x, s);$$

hence, in our case we have

$$Fy = Gy - \frac{v}{\lambda^*}\frac{(y, v)}{(v, v)}.$$

However, this operator insures only the orthogonality of the first approximation to the eigenfunction v. It is not suitable for practical calculation. We shall construct the F-operator in such a way that the orthogonality of all successive approximations to the first eigenvector v is insured independently of the choice of the first approximation. We put

$$Fy = Gy - v\frac{(Gy, v)}{(v, v)}.$$

We can easily prove that $(y^{(k+1)}, v) = 0$, because

$$(y^{(k+1)}, v) = \lambda^{(k)}(Fy^{(k)}, v) = \lambda^{(k)}(Gy^{(k)}, v) - \lambda^{(k)}\frac{(Gy^{(k)}, v)}{(v, v)}(v, v) = 0.$$

This way, however, also did not prove to be successful for the computer calculation probably as a consequence of orthogonality destruction by rounding errors. The computation of the second characteristic value also failed even in the case that the first value was computed with relative error 10^{-6} (kernel G_1).

4. DISCRETIZATION

For the calculation on a computer the following discretization was made. In the interval $\langle 0, 1\rangle$ $n + 1$ meshpoints were taken. Denote $h = 1/n$, $x_i = i \,.\, h$, $s_j = j \,.\, k$, $y_i = y(x_i)$, $G_{ij} = G(x_i, s_j)$, etc. Equation (20) will be replaced by a system of linear equations

$$y_i - \lambda \sum_{j=0}^{n} A_{ij}\, G_{ij} y_j = 0 \quad (i = 0, 1, \ldots, n),$$

where A_{ij} are constants obtained by replacing the integral by a finite sum. The inner product will be defined by

$$(y, v) = \sum_{j=0}^{n} A_j\, y_j v_j,$$

where A_j are again the constants of numerical integration. The norm will be defined as usual, i.e. $\|v\| = \sqrt{[(v, v)]}$.

The numerical integration was performed by the trapezoid and the Simpson's rule. In examples, where the function $G(x, s)$ had a discontinuous first derivative on the diagonal, the modified Simpson's rule was used. Although the symmetry of the operator G, which is an essential condition for the method of steepest descent but not for the other ones, was violated, the convergence remained intact. Although the

trapezoid rule is form the programming point of view the simplest one, the results are, however, almost by one decimal rank worse than those obtained by the Simpson's rule. Therefore the Simpson's rule should be preferred. Together with the accuracy of calculation of characteristic values, the accuracy and speed of the convergence of eigenvectors was followed. The values

$$\frac{1}{n}\sum_{j=0}^{n}|g_j|\,,\quad \max_{0\leqq j\leqq n}|g_j|,\ \|g\|$$

were calculated, where

$$g_j = \frac{y_j}{v_j} - \frac{y_{n/2}}{v_{n/2}}$$

and v denotes the exact eigenvector of integral equation (20).

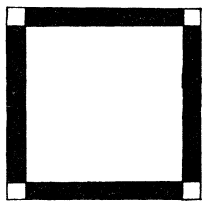

Fig. 1a. The trapezoid rule (LCH)

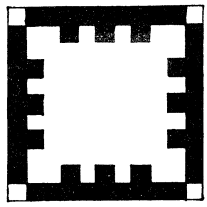

Fig. 1b. The modified Simpson's rule (MS)

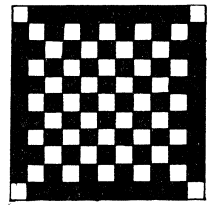

Fig. 1c. The Simpson's rule (S)

The calculations were made on the floating point on computer URAL 2 in the Institute for Computation Technique of the Czechoslovak Academy of Sciences. Mantissa has 8 figures, the maximum spread of decimal exponents is ± 19. The program was made in the machine code. The termination of the iteration process in the program was chosen for $\|y^{(k+1)} - y^{(k)}\| < \varepsilon$ in view of the fact that after the termination it was possible to go on by switching off the key 2 and by starting an infinite process, except for the method of steepest descent, which terminated with a division by zero for

$$(r, Gr) - \frac{1}{\lambda}(r, r) < 10^{-19}\,.$$

5. RESULTS

The methods mentioned above were tested on six integral equations with kernels

$$G_1(x, s) = \begin{cases} x(1-s), & x \leqq s\,, \\ s(1-x), & s \leqq x\,, \end{cases}$$

$$G_2(x, s) = (1 - \sqrt{x})(1 - \sqrt{s}),$$

$$G_3(x, s) = \sqrt{(x)}(s + 10),$$

$$G_4(x, s) = |x - s|,$$

$$G_5(x, s) = -\sqrt{(xs)} \begin{cases} \lg x, & x \leqq s, \\ \lg s, & s \leqq x, \end{cases}$$

$$G_6(x, s) = \sqrt{[(1 + x)(1 + s)]} \cdot G_1(x, s).$$

The exact solution of the eigenvalue problem was known for some of them only, i.e.

Table 1

Kernel	λ^*	$v(x)$
$G_1(x, s)$	$\pi^2 = 9{\cdot}8696040$	$\sin \pi x$
$G_2(x, s)$	$6{\cdot}0000000$	$2(1 - \sqrt{x})$
$G_3(x, s)$	$15/106 = 0{\cdot}14150943$	$\sqrt{x}$
$G_5(x, s)$	$5{\cdot}78318$	—

The estimate $2.87833 \leqq \lambda^* \leqq 2.87846$ is known for the function G_4, for the function G_5 it will be obtained from the equation $\mathcal{J}_0(\sqrt{\lambda}) = 0$, where $\mathcal{J}_0$ is the Bessel function.

The number n of meshpoints was between 10 and 500 in computing. One iteration step carried out by the method of steepest descent took 50 minutes for $n = 500$. In further examples, where the calculation of the functional value lasted longer, the number of meshpoints used did not exceed 200.

Let us now present the results of computation. The course of the computation of the characteristic value in a single process is represented for the kernels G_1 and G_5 in Fig. 2 to 9 as well as in Table 6.

Table 2

n		10	20	50	100
δ	MS	$3 \cdot 10^{-3}$	$4 \cdot 10^{-4}$	$2{\cdot}6 \cdot 10^{-5}$	$3 \cdot 10^{-6}$
	LCH	$8 \cdot 10^{-3}$	$2 \cdot 10^{-3}$	$3{\cdot}3 \cdot 10^{-4}$	$8 \cdot 10^{-5}$

Table 3

Kernel	n		d	δ	$\|g\|$	max $\|g_j\|$
G_2	10	MS	$3 . 10^{-1}$	$4 . 10^{-2}$	$1{\cdot}3 . 10^{-2}$	$2{\cdot}5 . 10^{-2}$
		S	$2 . 10^{-1}$	$3 . 10^{-2}$	$3 . 10^{-9}$	$7 . 10^{-9}$
	100	S	$6 . 10^{-3}$	$1 . 10^{-3}$	$6 . 10^{-9}$	$3 . 10^{-3}$
	200	MS	$2 . 10^{-3}$	$4 . 10^{-4}$	—	—
G_3	10	MS	$8 . 10^{-4}$	$5 . 10^{-3}$	$2 . 10^{-3}$	$4 . 10^{-3}$
		S	$1{\cdot}4 . 10^{-5}$	$1 . 10^{-4}$	10^{-9}	$5 . 10^{-9}$
	100	MS	$6 . 10^{-5}$	$1 . 10^{-4}$	$1 . 10^{-3}$	$1 . 10^{-3}$
		S	$1{\cdot}6 . 10^{-5}$	$1{\cdot}1 . 10^{-4}$	$1{\cdot}5 . 10^{-9}$	$4 . 10^{-9}$

Table 4

n	d	δ	$\|g\|$	max $\|g_j\|$
10	$3{\cdot}0 . 10^{-2}$	$3{\cdot}0 . 10^{-3}$	$2 . 10^{-2}$	$2 . 10^{-2}$
20	$4{\cdot}0 . 10^{-3}$	$4{\cdot}0 . 10^{-4}$	$3 . 10^{-3}$	$7 . 10^{-3}$
50	$2{\cdot}6 . 10^{-4}$	$2{\cdot}6 . 10^{-5}$	$2 . 10^{-4}$	$8 . 10^{-4}$
100	$3{\cdot}0 . 10^{-5}$	$3{\cdot}0 . 10^{-5}$	$4 . 10^{-5}$	$2 . 10^{-4}$
200	$4{\cdot}0 . 10^{-6}$	$4{\cdot}0 . 10^{-7}$	$7 . 10^{-6}$	$6 . 10^{-5}$
500	$2{\cdot}0 . 10^{-7}$	$2{\cdot}0 . 10^{-8}$	$3 . 10^{-6}$ *)	$2 . 10^{-5}$ *)

Table 4 contains the data concerning the error of computation of the characteristic value and of the eigenvector for the kernel G_1. Table 3 contains the same values for the kernels G_2 and G_3 separately for various integration rules used for the computation. Table 2 gives the relations between the error in the computation of the characteristic number and the mode of the numerical intergration for the kernel G_1. In all the tables the results are distinguished according to the number of meshpoints.

In spite of the difference in the algorithms of the various methods the speed of convergence as well as the accuracy of the characteristic value obtained were equal. This is demonstrated in Figures 2 to 6 for kernel G_1 and for $n = 10, 20, 50, 100, 200$, in Figures 7 to 9 for the kernel G_5 and $n = 10, 100, 200$, namely for the process of steepest descent by a line, for the Kolomý's process by a dashed line, for the Birger's process by a dotted line and for the Kellogg's process by a dot-and-dashed line. In order to compare the individual methods as much independently of the problem discretization as possible, we relate the value $\lambda^{(k)}$ calculated in the k-th step of the process to the value λ attained in the steady-state of the process. This value is always

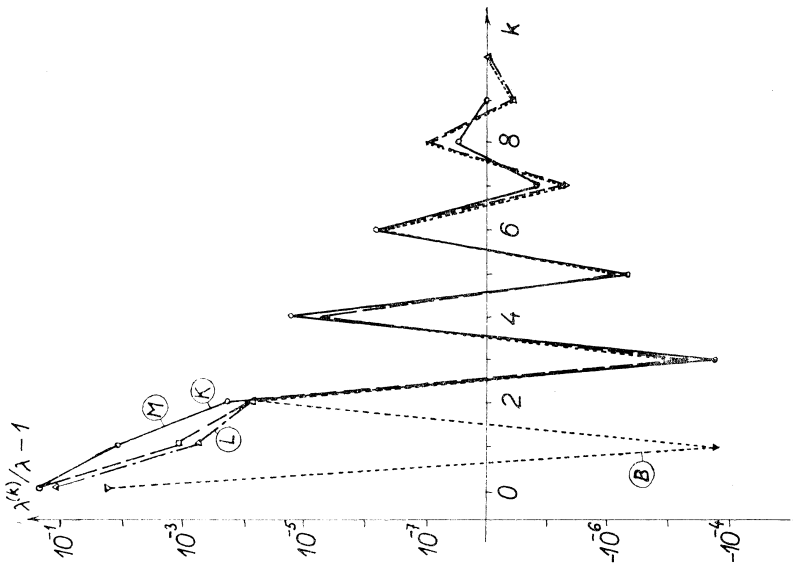
$\lambda^{(k)}/\lambda - 1$
k
M
K
L
B
10^{-1}
10^{-3}
10^{-5}
10^{-7}
-10^{-6}
-10^{-4}
0
2
4
6
8

Fig. 3,

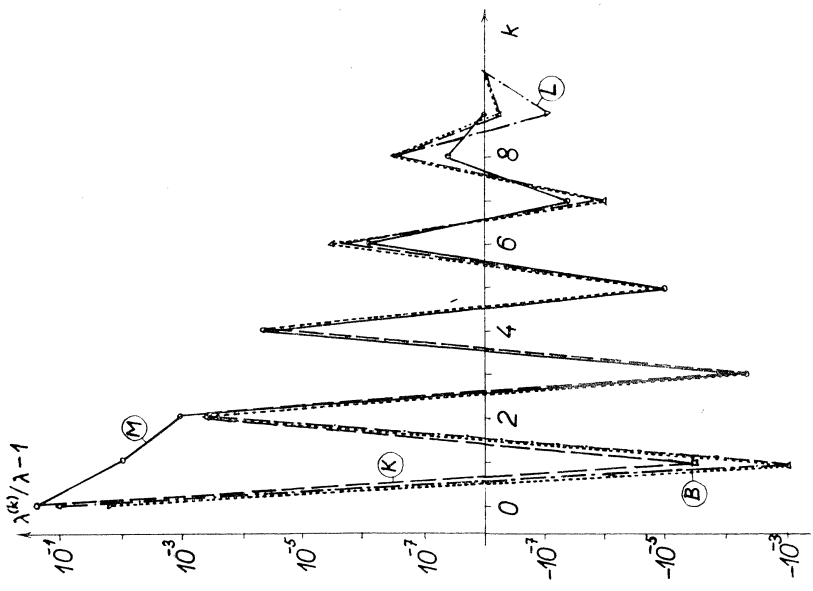
$\lambda^{(k)}/\lambda - 1$
k
M
K
L
B
10^{-1}
10^{-3}
10^{-5}
10^{-7}
-10^{-7}
-10^{-5}
-10^{-3}
0
2
4
6
8

Fig. 2,

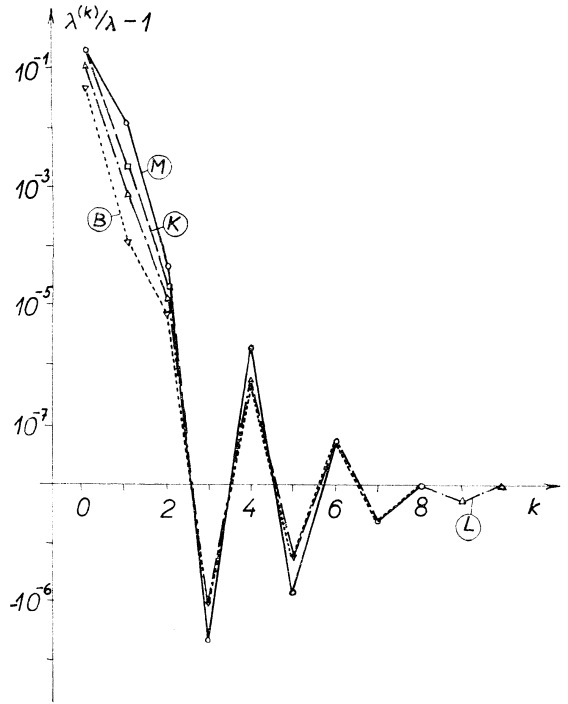
λ(k)/λ − 1
10⁻¹
10⁻³
10⁻⁵
10⁻⁷
-10⁻⁶
M
B
K
L
0
2
4
6
8
k

Fig. 4.

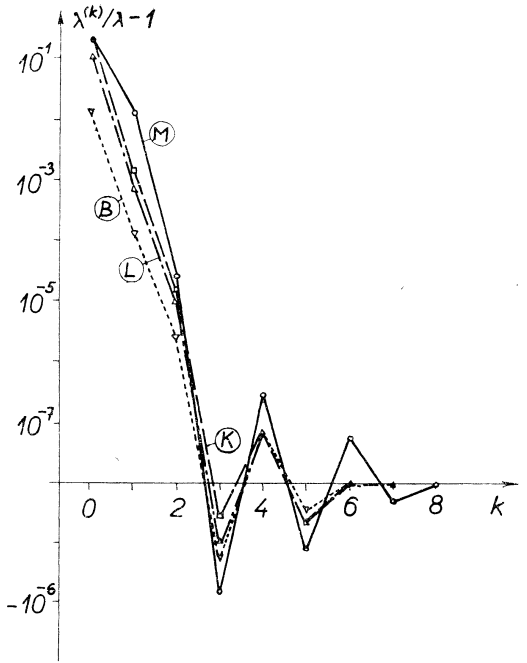
λ(k)/λ − 1
10⁻¹
10⁻³
10⁻⁵
10⁻⁷
-10⁻⁶
M
B
L
K
0
2
4
6
8
k

Fig. 5.

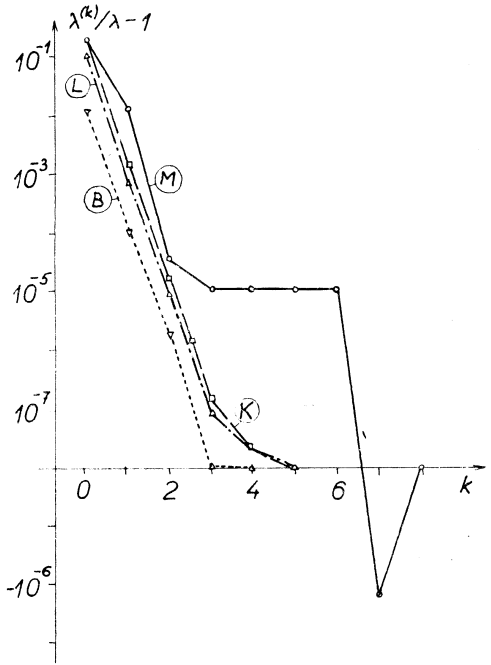
λ(k)/λ − 1
10⁻¹
10⁻³
10⁻⁵
10⁻⁷
-10⁻⁶
L
M
B
K
0
2
4
6
k

Fig. 6.

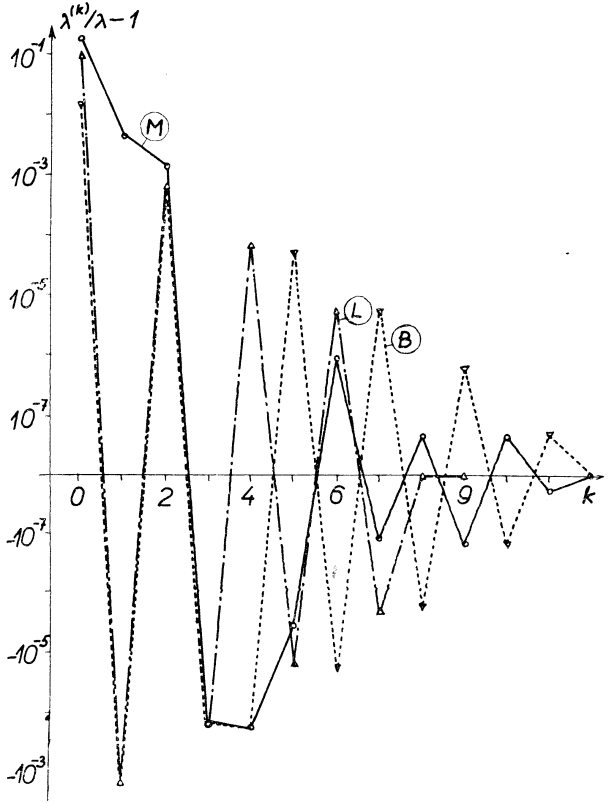
λ(k)/λ − 1
10⁻¹
10⁻³
10⁻⁵
10⁻⁷
-10⁻⁷
-10⁻⁵
-10⁻³
M
L
B
0
2
4
6
9
k

Fig. 7.

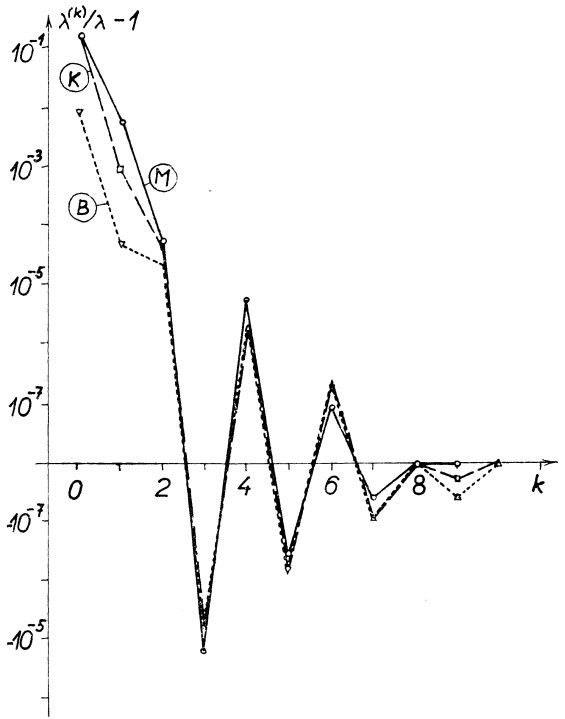
λ(k)/λ −1
10-1
10-3
10-5
10-7
-10-7
-10-5
K
M
B
0
2
4
6
8
k

Fig. 8.

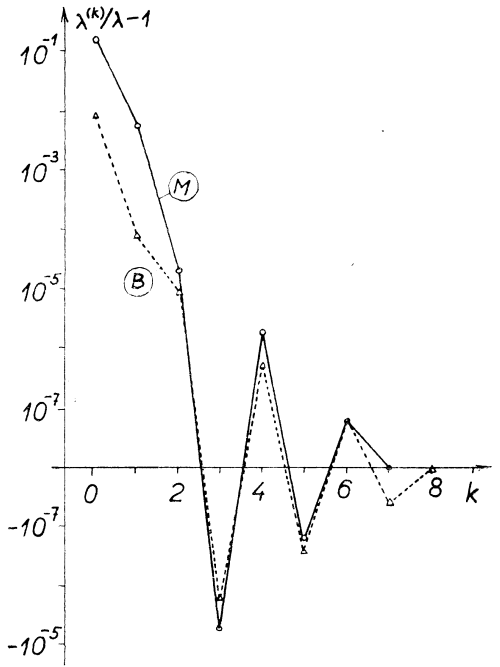
λ(k)/λ−1
10-1
10-3
10-5
10-7
-10-7
-10-5
M
B
0
2
4
6
8
k

Fig. 9.

the same for all four methods. The ratio of $\lambda^{(k)}$ and λ diminished by 1 is plotted on the vertical, the iteration step number on the horizontal axis.

It is obvious that the convergence of all the mentioned methods is almost equal. Although the integral operators are symmetric, they are not monotone; it is caused by the breaking the symmetry of the operator in consequence of the discretization according to the Simpson's rule modified in the odd lines so, that the point on the diagonal fell in the meshpoint, i.e. the following way: the Simpson's rule is used in the interval $\langle h, 1-h\rangle$ the trapezium rule is used in the intervals $\langle 0, h\rangle$, $\langle 1-h, 1\rangle$ so that the constants A_{ij} have the following form

$$A_{2j,j} = \tfrac{1}{3}h[1, 4, 2, 4, \ldots, 4, 2, 4, 1]$$

$$A_{2l-1,j} = \tfrac{1}{2}h, \tfrac{5}{6}h, \tfrac{1}{3}h[4, 2, 4, \ldots, 4, 2, 4], \tfrac{5}{6}h, \tfrac{1}{2}h\,.$$

Thereby the symmetry is broken in the points indicated in the picture 1b). For the large n this breaking is negligable. If we compute according to the trapezium rule (LCH), the symmetry is not broken at all (expecting the bounds, see picture 1a), where the function $G(x, s)$ is often nought), the obtained results, however, are worse, e.g. for the kernel $G_1(x, s)$ it makes half of the decimal rank in several cases even the whole rank the convergence being monotone. The calculation according to the Simpson's rule (S) can be carried out only in the case that the function $G(x, s)$ has a continuous first derivative. The breaking of the symmetry is considerable large here, see picture 1.c). The results are slightly better than those obtained by the modified Simpson's rule. Moreover they are obtained earlier. It is more important, however, that the eigenvector is calculated considerably more accurate, its accuracy being including all the 8 figures.

The results for the kernels $G_2(x, s)$ and $G_3(x, s)$ referring to all four methods are given in Table 3, d denotes the absolute, δ the relative error of computation of the characteristic value, $\|g\|$ and $\max|g_j|$ denotes the error of computation of the eigenvector. For the kernel $G_1(x, s)$ the eigenvector is obtained for all eight figures if the trapezoid rule is used. If the Simpson's rule is used, the absolute error of calculation of the eigenvector is usually equal to the absolute error of calculation of the characteristic value. The corresponding results are given in Table 4.

The values $\|g\|$ and $\max|g_j|$ for $n = 500$ refer to the Kolomý's method; for the method of steepest descent $\|g\| = 5 \cdot 10^{-5}$, $\max|g_j| = 4 \cdot 10^{-4}$ which may be caused by rounding errors in case of a large number of operations needed by this method.

Comparing the last two columns of Table 4 we see that the error of the eigenvector is spread rather proportionally, its maximum being attained at the boundary points. This follows from the definition of the vector g.

Furthemore, let us pay attention to the routine degree of the calculation and to memory requirements. Although the calculation showed that the method of steepest descent can be used with the same success as the other methods even in the nonsym-

metric case (i.e. when the operator in not symmetric and positive definitive), other disadvantage appear while carrying out the comparison. It needs 1.5 time more working places and twice the number of operations than the Kolomý's, Birger's, and Kellogg's methods do at every iteration step. Also the program for the calculation by the method of steepest descent is considerably longer.

Suppose that the values of the function $G(x, s)$ are computed at each step of the process by some subroutine (n^2 words in the storage will be saved). Denote the number of operations needed for computation of every value $G(x, s)$ by the letter G and the number of operations needed for computation of the coefficient of numerical integration by the letter A. The corresponding comparison is given in Table 5.

Table 5

Method	Memory	Number of operations
M	$3n$	$2n^2(3 + A + G) + n(4A + 2)$
K	$2n$	$n^2(3 + A + G) + n(2A + 1)$
B		
L	$2n$	$n^2(3 + A + G) + n(2A + 1) + \sqrt{}$

As to the Kellogg's method a number of operations needed for the calculation of the root should be added to the number of operations.

Table 6 gives the values $\lambda^{(k)}$ at the respective iteration steps. For $n = 500$ and the kernel $G_1(x, s)$ the maximal accuracy has been reached and the error of the last valid figure is due to the rounding errors.

Table 6

k	M	K
0	12·000001	12·000001
1	9·9904303	9·8823527
2	9·8698419	9·8697539
3	9·8696050	9·8696061
4	9·8696043	9·8696043
5	9·8696042	9·8696042

6. CONCLUSION

From the theoretical point of view the method of steepest descent can be used in the case of a symmetric and positive definitive operator only. The practical calculations showed that this method gives in fact good results even if some of the assumptions are not fulfilled (e.g. the operator with the kernel $G_3(x, s) = \sqrt{(x)}\,(s + 10)$ is not symmetric, the operator with the kernel $G_4(x, s) = |x - s|$ is not positive definite).

However, the method of steepest descent has in comparison with the Kolomý's Kellogg's and Birger's method many disadvantages; the essential one is the complexity of the algorithm and large memory requirements. The fact that the convergence speed and the accuracy are the same for all methods, sprats clearly for the new methods of Kolomý and Birger because of their algorithm simplicity, mild memory requirements and general applicability.

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Souhrn

O ITERAČNÍCH PROCESECH PRO VÝPOČET VLASTNÍCH ČÍSEL A VLASTNÍCH VEKTORŮ LINEÁRNÍHO OPERÁTORU

VĚRA MARŠÍKOVÁ

V práci jsou shrnuty některé výsledky (zejména podle prací [5], [7], [8], [9], [10]) z teorie řešení lineárních operátorových rovnic v Banachově a Hilbertově prostoru a uvedeny věty o konvergenci iteračních procesů pro výpočet vlastních čísel a vlastních vektorů.

Těchto procesů bylo použito s úspěchem při řešení homogenních lineárních integrálních rovnic

$$y(x) = \lambda \int_0^1 G(x, s)\, y(s)\, \mathrm{d}s$$

na samočinném počítači URAL 2. Integrály, vyskytující se při výpočtu, byly počítány přibližně podle lichoběžníkového a Simpsonova pravidla, v případech, že funkce $G(x, s)$ měla nespojitou derivaci na diagonále $x = s$, podle Simpsonova pravidla modifikovaného v lichých řádcích tak, aby bod na diagonále byl jedním z krajních bodů dílčích intervalů délky $2h$, v nichž se používá Simpsonova pravidla. Každý z těchto způsobů porušil symetrii operátoru, nejvíce Simpsonovo pravidlo (viz obr. 1a) – c)).

Měla-li funkce $G(x, s)$ spojitou derivaci, dalo přesnější výsledky vlastních funkcí obvyklé Simpsonovo pravidlo než modifikované, u vlastních čísel je rozdíl ve výsledcích malý (tab. 3).

Neměla-li funkce $G(x, s)$ spojitou derivaci, byla vlastní čísla spočtena téměř o řád přesněji podle modifikovaného Simpsonova pravidla než podle lichoběžníkového (tab. 2), zatímco situace ve výpočtu vlastních funkcí byla opačná. Zdvojnásobením počtu dělících bodů vzrostla přesnost výpočtu téměř o jeden řád.

Výsledky u jednotlivých metod jsou stejné (obr. 2–9), rozdíly jsou patrné jen v několika prvních iteracích, po stejném počtu kroků bylo u všech metod dosaženo téže hodnoty.

Pro srovnání byly všechny příklady počítány též metodou největšího spádu, která je však náročnější na strojní čas i paměť počítače než metody Birgerova, Kolomého a Kelloggova (tab. 5). Přesto, že podmínka symetričnosti operátoru je pro konvergenci metody největšího spádu podstatná, lze se bez ní, alespoň v některých případech, obejít.

Резюме

О НЕКОТОРЫХ НОВЫХ ИТЕРАЦИОННЫХ МЕТОДАХ ДЛЯ РЕШЕНИЯ ПРОБЛЕМЫ СОБСТВЕННЫХ ЗНАЧЕНИЙ ЛИНЕЙНОГО ОПЕРАТОРА

ВЕРА МАРШИКОВА (Věra Maršíková)

В статье приведены некоторые результаты (именно по работам [5], [7], [8], [9], [10]) теории решения линейных операторных уравнений в пространствах Банаха и Гильберта и теоремы о сходимости итерационных процессов для решения проблемы собственных значений.

Эти процессы были успешно использованы при решении линейных однородных интегральных уравнений

$$y(x) = \lambda \int_0^1 G(x, s)\, y(s)\, \mathrm{d}s$$

на быстрдействующей вычислительной машине Урал 2. Интегралы вычислялись приближенно по правилу трапеций и Симпосона. В слючае разрывной первой производной функции $G(x, s)$ на диагонали $x = s$ применялось правило Симпсона, модифицированное в нечетных строках таким образом, чтобы точка на диагонали попала в граничную точку одного из частичных интервалов длины $2h$, в которых используется правило Симпсона. Каждый из этих способов интегрирования нарушил симметрию оператора, больше всех правило Симпсона (черт. 1a – 1c).

При использовании обычного правила Симпсона для функции с непрерывной производной были получены более точные результаты при вычислении собственных функций, чем при использовании модифицированного правила. Собственные значения были почти одинаковые (таб. 3).

Когда функция $G(x, s)$ не имела непрерывную производную, собственные значения получились почти на один разряд лучше при использовании модифицированного правила Симпсона чем при использовании правила трапеций (таб. 2), между тем как ситуация при вычислении собственных функций была обратная. В случае удвоения числа точек деления точность вычислений повысилась почти на один разряд.

Результаты отдельных методов одинаковы (черт. 2 – 9), разницу видно только в нескольких первых итерациях, после одинакового числа итераций было достигнуто того же значения у всех методов.

Для сравнения вычислялись все примеры тоже по методу скорейшего спуска, который требует больше времени и памяти чем методы Биргера, Коломого и Келлогга (таб. 5). Хотя условие симметрии оператора для сходимости метода скорейшего спуска важно, можно без него, по крайней мере в некоторых случаях, обойтись.

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