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VARIATIONAL PRINCIPLES IN THE LINEAR THEORY OF ELASTICITY FOR GENERAL BOUNDARY CONDITIONS

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INTRODUCTION. NOTATIONS AND ASSUMPTIONS, DEFINITION OF THE PROBLEM

Nearly in all monographs on mathematical theory of elasticity, theory of plates and shells, in structural analysis etc., some chapters are devoted to variational principles. However, the principles given there rarely correspond to boundary conditions of the general type, when, for example, on parts of the boundary of the given region (surface of the body) not only displacements and surface tractions are prescribed, but also contact conditions and conditions of elastic supports are given separately in the normal and tangential directions to the surface.[1]) The present paper aims both to fill in this gap and to offer a deeper mathematical view on classical variational principles with use of the methods of functional analysis, especially of the theory of Hilbert space. In this sense the paper extends the ideas contained e.g. in books of С. Г. Михлин [3], [4].

Although all this study is restricted only to the spatial problems of linear elastostatics, in the same spirit it is possible to establish and apply analogous principles in the structural theories for beams, plates and shells, if we replace the three-dimensional fields of displacements, strains and stresses by the corresponding two- or one-dimensional fields of deflections, tangential displacements, extensions and curvature changes of the middle surface or middle curve respectively and fields of generalized stresses, such as force and moments resultants across a section of the element under consideration (see [2] for an example).

In the following we deal with the principle of virtual work (section 1), principle of virtual displacements and the definition of the weak (generalized) solution to the problem (section 2), the principle of minimum potential energy (section 3), the principle of minimum complementary energy and that of virtual changes of stresses

[1]) Exceptions are presented e.g. in the works of E. REISSNER [1] or К. Ф. Черных [2].

(section 4), cases of "free bodies" (section 5), generalized principles of HU HAI-CHANG and WASHIZU, HELLINGER and REISSNER (section 6) and the estimates of errors of approximate solutions (section 7).

Let the elastic body occupy an open bounded region Ω of the three-dimensional Euclidean space with a fixed rectangular Cartesian coordinate system $(x_1, x_2, x_3) \equiv \equiv X$. We suppose that[2])

a)
$$\Omega = \bigcup_{\alpha=1}^{A} \Omega_\alpha$$

where each region Ω_α is starlike with respect to some sphere K_α (that means in each region Ω_α such sphere $K_\alpha \subset \Omega_\alpha$ exists, that each half-line, going out from any point of this sphere, intersects the boundary of Ω_α only at one point);

b) the boundary Γ of Ω has the form

$$\Gamma = \bigcup_{k=1}^{\lambda} \bar{S}_k ,$$

where each part $\bar{S}_k$ can be described by the relation

$$x_{l_k} = \psi_k(x_{m_k}, x_{n_k}) ,$$

where ψ_k is continuous, with first derivatives piecewise continuous for $X_k \equiv (x_{m_k}, x_{n_k}) \in \in \bar{g}_k$, $\bar{g}_k$ being a closed two-dimensional region, the projection of $\bar{S}_k$;

c) if we define the open cylinders $G_k'(\delta)$ and $G_k''(\delta)$ as the regions

$$G_k'(\delta) = \{X_k \in g_k,\ \psi_k - \delta < x_{l_k} < \psi_k\} ,$$
$$G_k''(\delta) = \{X_k \in g_k,\ \psi_k < x_{l_k} < \psi_k + \delta\} ,$$

then one of them belongs to Ω for sufficiently small $\delta > 0$. This cylinder will be denoted by $G_k(\delta)$;

d) there exist closed regions $\bar{g}_k^{(0)} \subset g_k$, $k = 1, 2, \dots \lambda$ such that the region

$$G(\delta) = \bigcup_{k=1}^{\lambda} G_k^{(0)}(\delta)$$

(where $G_k^{(0)}(\delta)$ is the part of $G_k(\delta)$, which has $g_k^{(0)}$ as its projection), includes a boundary layer of Ω, that means the set of all points $X \in \Omega$, the distances of which from the boundary Γ are less than a fixed $\eta > 0$.

Note that from the latter condition it follows, that the neighbouring parts of surfaces $\bar{S}_k$ may not meet only on curves, but their interiors must have a non-vacuous intersection.

In the following sections some function spaces will be used, which we introduce now through the preliminary definitions.

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[2]) These assumptions are taken over from the book [3], but they can be replaced by the definition of so called region with Lipschitz-like boundary (see e.g. [9]).

Let $L_2(\Omega)$ be the space of real functions $f(X)$, $X \in \Omega$, square-integrable in Ω in the Lebesgue sense, with the norm

$$|f|^2_{L_2(\Omega)} = \int_\Omega f^2(X) \, \mathrm{d}X .$$

$[L_2(\Omega)]^3$ is the space of vector-functions $\mathbf{f}(X)$, $X \in \Omega$ with every component $f_i \in L_2(\Omega)$ and with the norm

$$|\mathbf{f}|^2_{[L_2(\Omega)]^3} = \sum_{i=1}^{3} |f_i|^2_{L_2(\Omega)} .$$

$W_2^{(1)}(\Omega)$ denotes the space of functions of $L_2(\Omega)$, which have all first derivatives in the generalized sense and these also belong to $L_2(\Omega)$. The norm is given through

$$|f|^2_{W_2^{(1)}} = \int_\Omega f^2 \, \mathrm{d}X + \sum_{k=1}^{3} \int_\Omega f_{,k}^2 \, \mathrm{d}X , \quad \text{where} \quad f_{,k} \equiv \partial f / \partial x_k .$$

$[W_2^{(1)}(\Omega)]^3$ is the space of vector-functions $\mathbf{f}(X)$, $X \in \Omega$, with every component $f_i \in W_2^{(1)}(\Omega)$ and with the norm

$$|\mathbf{f}|^2_{[W_2^{(1)}]^3} = \sum_{i=1}^{3} |f_i|^2_{W_2^{(1)}} .$$

Let $\Gamma' \subset \Gamma$ be a part of the boundary Γ, containing a set $\mathcal{O}$, which is open in Γ – i.e. for each point $X^0 \in \mathcal{O}$ such positive $\varepsilon(X^0)$ exists, that also each point $X \in \Gamma$, for which the distance

$$\left(\sum_{i=1}^{3} (x_i - x_i^0)^2\right)^{1/2} < \varepsilon(X^0)$$

belongs to $\mathcal{O}$. We define $L_2(\Gamma')$ as the space of real functions $f(X)$, $X \in \Gamma'$, with the norm

$$|f|^2_{L_2(\Gamma')} = \sum_{k=1}^{\lambda} \int_{g'_k} f^2(X_k, \psi_k(X_k)) \, \mathrm{d}X_k < \infty ,$$

where g'_k means the projection of the intersection $\Gamma' \cap S_k$. Note that g'_k may be even vacuous for some k.

Let us state the mixed boundary-value problem in the linear quasi-static theory of elasticity. Suppose that the strain-displacement relations

(0.1) $$\varepsilon_{ik} = \tfrac{1}{2}(u_{i,k} + u_{k,i}) ,$$

stress-strain relations (generalized Hooke's law)

(0.2) $$\tau_{ik} = c_{iklm} \varepsilon_{lm} \quad \text{or (inverse)}$$
$$\varepsilon_{ik} = a_{iklm} \tau_{lm}$$

respectively and the stress equations of equilibrium

(0.3) $$\tau_{ik,k} + K_i = 0$$

hold on Ω. Here u_i, ε_{ik}, τ_{ik} and K_i denote the cartesian components of the displacement vector $\boldsymbol{u}$, strain tensor, stress tensor and body force vector respectively. We use the convention that a repeated suffix implies summation over the range 1, 2, 3. The elastic coefficients $c_{iklm}(X)$, $a_{iklm}(X)$ satisfy the well-known symmetry relations

$$c_{iklm} = c_{kilm} = c_{lmik}\,, \quad a_{iklm} = a_{kilm} = a_{lmik}$$

and the inequality

$$c_{iklm}\varepsilon_{ik}\varepsilon_{lm} \geqq \mu_0 \sum_{i,k=1}^{3} \varepsilon_{ik}^2 \tag{0.4}$$

for every symmetric tensor ε_{ik} at each point $X \in \bar{\Omega}$, μ_0 being a positive constant.

On the boundary Γ of the region Ω the boundary conditions are prescribed in the form of linear combinations of the displacements and surface tractions components

$$\begin{aligned} A_n u_n + B_n T_n &= C_n\,, \\ A_t \mathbf{u}_t + B_t \mathbf{T}_t &= \mathbf{C}_t\,, \end{aligned} \tag{0.5}$$

where the suffixes n or t denote the normal or tangential components of vectors u_i and $T_i = \tau_{ik} n_k$ into the direction of the unit outward normal n_k or of the tangential plane to Γ respectively, defined through

$$\begin{aligned} u_n &= u_k n_k\,, \qquad (\mathbf{u}_t)_j = (\mathbf{u} - u_n \mathbf{n})_j = u_j - u_i n_i n_j\,, \\ T_n &= \tau_{ik} n_i n_k\,, \quad (\mathbf{T}_t)_j = \tau_{jk} n_k - \tau_{ik} n_i n_k n_j\,. \end{aligned} \tag{0.6}$$

A_n, A_t are piecewise constant functions on Γ, the values of which are 0 or 1 only. B_n and B_t are such bounded measurable functions on Γ, that

$$\begin{aligned} B_n &\geqq \beta_n > 0 \quad \text{or} \quad B_n = 0\,, \\ B_t &\geqq \beta_t > 0 \quad \text{or} \quad B_t = 0\,. \end{aligned}$$

For every point $X \in \Gamma$ we have

$$A_n + B_n > 0\,, \quad A_t + B_t > 0\,. \tag{0.7}$$

Let us denote the following sets of points on Γ:

$$\begin{aligned} \mathscr{A}_n &= \{X \in \Gamma,\ B_n = 0\}\,, \\ \mathscr{A}_t &= \{X \in \Gamma,\ B_t = 0\}\,, \\ \mathscr{B}_n &= \{X \in \Gamma,\ B_n > 0\}\,, \quad \Gamma_n = \{X \in \Gamma,\ A_n B_n > 0\}\,, \\ \mathscr{B}_t &= \{X \in \Gamma,\ B_t > 0\}\,, \quad \Gamma_t = \{X \in \Gamma,\ A_t B_t > 0\}\,. \end{aligned}$$

Suppose that the sets $\mathscr{A}$, $\mathscr{B}$, Γ, $\mathscr{B} \dot{-} \Gamma$ (difference of the sets) with subscripts n or t are either vacuous or contain an open set in Γ (see the definition of $L_2(\Gamma')$).

Let a vector-function $\bar{\boldsymbol{u}} \in [W_2^{(1)}(\Omega)]^3$ be given, which defines functions $C_n \in L_2(\mathscr{A}_n)$ and $(\boldsymbol{C}_t)_j \in L_2(\mathscr{A}_t)$ on $\mathscr{A}_n$ and $\mathscr{A}_t$ by means of the relations

$$\bar{u}_n = C_n \quad \text{on} \quad \mathscr{A}_n\,, \quad \bar{\boldsymbol{u}}_t = \boldsymbol{C}_t \quad \text{on} \quad \mathscr{A}_t$$

as the linear combinations (0.6) of the traces of its components.

Let $C_n/B_n \equiv P_n \in L_2(\mathscr{B}_n)$ and $(\boldsymbol{C}_t)_j/B_t \equiv (\boldsymbol{P}_t)_j \in L_2(\mathscr{B}_t)$, $(j = 1, 2, 3)$, $\boldsymbol{K} \in [L_2(\Omega)]^3$, c_{iklm} and a_{iklm} be bounded measurable functions on $\bar{\Omega} = \Omega \cup \Gamma$. P_n and $\boldsymbol{P}_t$ are prescribed components of surface tractions.

Moreover, we suppose that if $\bar{\boldsymbol{u}} \equiv 0$, through the boundary conditions on $\mathscr{A}_n \cup \mathscr{A}_t$ and $v_n = 0$ on Γ_n, $\boldsymbol{v}_t = 0$ on Γ_t any rigid body displacements and small rotations are eliminated. Then in the linear manifold $\boldsymbol{M}$ of vector-functions $\boldsymbol{v}$, which have continuously differentiable components on $\bar{\Omega}$ and satisfy the above-mentioned homogeneous boundary conditions on $\mathscr{A}_n \cup \mathscr{A}_t$, the generalized Korn's inequality (see [12])

$$(0.8) \quad \sum_{i,k=1}^{3} \int_\Omega (v_{i,k} + v_{k,i})^2 \, \mathrm{d}X + \int_{\Gamma_n} \frac{1}{B_n} v_n^2 \, \mathrm{d}S + \int_{\Gamma_t} \frac{1}{B_t} \boldsymbol{v}_t^2 \, \mathrm{d}S \geqq C \, |\boldsymbol{v}|^2_{W_2^{(1)}[\Omega]^3}$$

holds. If these assumptions about the properties of $\mathscr{A}_n$, $\mathscr{A}_t$, Γ_n, Γ_t are not fulfilled, then further conditions, relating the loads P_n, $\boldsymbol{P}_t$ and the body forces $\boldsymbol{K}$ and some relations for the functions $\boldsymbol{u}$ of displacements and $\boldsymbol{v} \in \boldsymbol{M}$ are needed to guarantee the total equilibrium of the body, the uniqueness of solution and the validity of inequality of Korn (0.8). Cases of this type will be called cases of "free bodies" and some of them dealt with in section 5.

The proofs of Korn's inequality and of (0.8) are published for example in the book [3] of С. Г. Михлин for various cases of "fixed" and "free" bodies as well. General results concerning these problems were reached by J. Nečas ([9], [10]).[3])

1. THE PRINCIPLE OF VIRTUAL WORK

Let $\boldsymbol{W}$ be the set of such symmetric tensor-functions of stress $\boldsymbol{T}(X)$ with components $\tau_{ik} = \tau_{ki} \in W_2^{(1)}(\Omega)$, which meet the equations of equilibrium (0.3) (in the sense of elements of $L_2(\Omega)$) and the "pure static" boundary conditions (0.5) on the parts $\mathscr{B}_n \dot{-} \Gamma_n = \mathscr{D}_n$ and $\mathscr{B}_t \dot{-} \Gamma_t = \mathscr{D}_t$ (in the sense of $L_2(\mathscr{D}_n)$ and $L_2(\mathscr{D}_t)$ respectively). $\boldsymbol{W}$ will be called the set of statically admissible stress-fields.

Let $\boldsymbol{U}$ be the set of vector-functions of displacement $\boldsymbol{u} \in [W_2^{(1)}(\Omega)]^3$, which satisfy the "pure geometric" boundary conditions on $\mathscr{A}_n \cup \mathscr{A}_t$ (in the sense of traces) and moreover, if $\Gamma_n \cup \Gamma_t$ is non-vacuous, the conditions of elastic supports on $\Gamma_n \cup \Gamma_t$ in the form (0.5) as well, where $T_n = T_n(\boldsymbol{u})$ and $\boldsymbol{T}_t = \boldsymbol{T}_t(\boldsymbol{u})$ represent the components

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[3]) Starting with the theorems of [10], J. Nečas and the author attempted to create a more systematic work on this field (see [12]).

of surface tractions, derived from $\boldsymbol{u}$ on the base of (0.6) and (0.1), (0.2) (in the sense of elements of $L_2(\Gamma_n)$ and $L_2(\Gamma_t)$ respectively). Therefore, if $\Gamma_n \cup \Gamma_t$ is non-vacuous, we must suppose, that the corresponding stresses $\tau_{ik}(\boldsymbol{u}) \in W_2^{(1)}(\Omega)$. $\boldsymbol{U}$ will be called the set of geometrically admissible displacement-fields.

If $\boldsymbol{T} \in \boldsymbol{W}$ and $\boldsymbol{u} \in \boldsymbol{U}$, by virtue of divergence theorem

$$\int_\Omega (\tau_{ik,k} + K_i)\, u_i \,\mathrm{d}X = -\int_\Omega \tau_{ik} u_{i,k} \,\mathrm{d}X + \int_\Omega K_i u_i \,\mathrm{d}X + \int_\Gamma \tau_{ik} n_k u_i \,\mathrm{d}S = 0$$

and using decompositions

$$\Gamma = \mathscr{A}_n \cup \mathscr{D}_n \cup \Gamma_n\,, \quad \mathscr{A}_n \cap \mathscr{D}_n = \mathscr{A}_n \cap \Gamma_n = \mathscr{D}_n \cap \Gamma_n = \emptyset\,,$$
$$\Gamma = \mathscr{A}_t \cup \mathscr{D}_t \cup \Gamma_t\,, \quad \mathscr{A}_t \cap \mathscr{D}_t = \mathscr{A}_t \cap \Gamma_t = \mathscr{D}_t \cap \Gamma_t = \emptyset\,,$$

we derive

$$\int_\Omega \tau_{ik}\varepsilon_{ik}(\boldsymbol{u}) \,\mathrm{d}X = \int_\Omega K_i u_i \,\mathrm{d}X + \int_{\mathscr{A}_n} T_n \bar{u}_n \,\mathrm{d}S + \int_{\mathscr{A}_t} \boldsymbol{T}_t \,.\, \bar{\boldsymbol{u}}_t \,\mathrm{d}S +$$
$$+ \int_{\mathscr{D}_n} P_n u_n \,\mathrm{d}S + \int_{\mathscr{D}_t} \boldsymbol{P}_t \,.\, \boldsymbol{u}_t \,\mathrm{d}S + \int_{\Gamma_n} (C_n - B_n\, T_n(\boldsymbol{u}))\, T_n \,\mathrm{d}S + \int_{\Gamma_t} (\boldsymbol{C}_t - B_t\, \boldsymbol{T}_t(\boldsymbol{u}))\,.\,\boldsymbol{T}_t \,\mathrm{d}S\,. \tag{1.1}$$

This relation represents the principle of virtual work, which is often interpreted in the following way: "the virtual work of internal forces

$$\int_\Omega \tau_{ik}\varepsilon_{ik}(\boldsymbol{u}) \,\mathrm{d}X + \int_{\Gamma_n} B_n T_n\, T_n(\boldsymbol{u}) \,\mathrm{d}S + \int_{\Gamma_t} B_t \boldsymbol{T}_t \,.\, \boldsymbol{T}_t(\boldsymbol{u}) \,\mathrm{d}S \tag{1.2}$$

is equal to the virtual work of external forces"

$$\int_\Omega K_i u_i \,\mathrm{d}X + \int_{\mathscr{A}_n} T_n \bar{u}_n \,\mathrm{d}S + \int_{\mathscr{A}_t} \boldsymbol{T}_t \,.\, \bar{\boldsymbol{u}}_t \,\mathrm{d}S + \int_{\mathscr{D}_n} P_n u_n \,\mathrm{d}S + \int_{\mathscr{D}_t} \boldsymbol{P}_t \,.\, \boldsymbol{u}_t \,\mathrm{d}S +$$
$$+ \int_{\Gamma_n} C_n T_n \,\mathrm{d}S + \int_{\Gamma_t} \boldsymbol{C}_t \,.\, \boldsymbol{T}_t \,\mathrm{d}S\,. \tag{1.3}$$

Consequently, the work of "elastic supports" is comprised by the work of internal forces and the work of "yielding of supports" $\bar{\boldsymbol{u}}$, C_n, $\boldsymbol{C}_t$ by the work of external forces. Sometimes, the virtual work of internal forces is defined with negative sign, so that the principle asserts: "the sum of virtual works of internal and external forces is equal to zero".

Let us emphasize that in (1.1) the statically admissible stress field and the geometrically admissible displacement field are in general mutually independent. Any of them may be replaced by the real stress field and real displacement field respectively (provided the assumptions of definitions of $\boldsymbol{W}$ and $\boldsymbol{U}$ respectively are satisfied) and the other field remains virtual, hypothetic. Such an application of the principle is usual in structural analysis.

The principle of virtual work in the above-mentioned physical interpretation holds only within the range of infinitesimal deformations, when the strain-displacement relations are given through (0.1).

In case that the elastic coefficients $c_{iklm}(X)$ are piecewise continuous with jump discontinuities on a finite number of surfaces, then $\boldsymbol{T} \in \boldsymbol{W}$ will be defined as such stress-tensor, which satisfies (0.3) only in a piecewise manner on every subregion Ω_j with continuous c_{iklm} in the sense of $L_2(\Omega_j)$, consequently $\tau_{ik} \in W_2^{(1)}(\Omega_j)$. Furthermore, let $\tau'_{ik}n'_k = -\tau''_{ik}n''_k$ on the surfaces of discontinuities. Then integrating by parts we obtain again (1.1), because the corresponding surface integrals cancel out.

Similarly, in this case $\boldsymbol{u} \in \boldsymbol{U}$ means only $\tau_{ik}(\boldsymbol{u}) \in W_2^{(1)}(\Omega_j)$ instead of $\tau_{ik}(\boldsymbol{u}) \in W_2^{(1)}(\Omega)$, for $\Gamma_n \cup \Gamma_t$ non-vacuous.

In order to illustrate the general form of the principle, we apply it to several examples of mixed boundary-value problems.

1. Let $\Gamma = \Gamma_u \cup \Gamma_P \cup \Gamma_K$ be the mutually disjoint decomposition of the boundary, where each part involves an open set (according to the definition of $L_2(\Gamma')$). Moreover prescribe by $\bar{\boldsymbol{u}}$, $\boldsymbol{P}$ and $u_n = 0$, $\boldsymbol{T}_t = 0$ the displacements on Γ_u, the surface tractions on Γ_P and the conditions of contact support on Γ_K, respectively. Therefore on Γ_K $B_n = C_n = 0$, $A_t = 0$, $\boldsymbol{C}_t = \boldsymbol{0}$; $\mathscr{A}_n = \Gamma_u \cup \Gamma_K$, $\mathscr{A}_t = \Gamma_u$, $\mathscr{B}_n = \Gamma_P$, $\mathscr{B}_t = \Gamma_P \cup \Gamma_K$, $\Gamma_n = \Gamma_t = \emptyset$. The principle of virtual work takes the form

$$\int_\Omega \tau_{ik}\varepsilon_{ik}(u)\,\mathrm{d}X = \int_\Omega K_i u_i\,\mathrm{d}X + \int_{\Gamma_u} (T_n\bar{u}_n + \boldsymbol{T}_t\,.\,\bar{\boldsymbol{u}}_t)\,\mathrm{d}S + \int_{\Gamma_P} (P_n u_n + \boldsymbol{P}_t\,.\,\boldsymbol{u}_t)\,\mathrm{d}S\,.$$

2. Let $\Gamma = \Gamma_u \cup \Gamma_P \cup \Gamma_V$ be the mutually disjoint decomposition of Γ, where on Γ_V the conditions, corresponding to the torsion problem of a cylinder on its end are prescribed: $T_n = 0$, $\boldsymbol{u}_t = \bar{\boldsymbol{u}}_t$. Consequently $A_n = C_n = B_t = 0$, $\mathscr{A}_n = \Gamma_u$, $\mathscr{A}_t = \Gamma_u \cup \Gamma_V$, $\mathscr{B}_n = \Gamma_P \cup \Gamma_V$, $\mathscr{B}_t = \Gamma_P$, $\Gamma_n = \Gamma_t = \emptyset$ and the principle takes the form

$$\int_\Omega \tau_{ik}\varepsilon_{ik}(\boldsymbol{u})\,\mathrm{d}X = \int_\Omega K_i u_i\,\mathrm{d}X + \int_{\Gamma_u} \tau_{ik}n_k\bar{u}_i\,\mathrm{d}S + \int_{\Gamma_P} P_i u_i\,\mathrm{d}S + \int_{\Gamma_V} \boldsymbol{T}_t\,.\,\bar{\boldsymbol{u}}_t\,\mathrm{d}S\,.$$

3. Let $\Gamma = \Gamma_u \cup \Gamma_P \cup \Gamma_Q$ be the mutually disjoint decomposition, where on Γ_Q the conditions of elastic supports

$$u_n = -B_n T_n\,, \quad \boldsymbol{u}_t = -B_t\boldsymbol{T}_t$$

are prescribed. Consequently

$$C_n = 0\,, \quad \boldsymbol{C}_t = \boldsymbol{0} \quad \text{on} \quad \Gamma_Q\,, \quad \mathscr{A}_n = \Gamma_u\,, \quad \mathscr{A}_t = \Gamma_u\,, \quad \Gamma_n = \Gamma_t = \Gamma_Q\,,$$
$$\mathscr{D}_n = \mathscr{D}_t = \Gamma_P\,,$$

and the principle takes the form

$$\int_\Omega \tau_{ik}\varepsilon_{ik}(\boldsymbol{u})\,\mathrm{d}X = \int_\Omega K_i u_i\,\mathrm{d}X + \int_{\Gamma_u} \tau_{ik}n_k\bar{u}_i\,\mathrm{d}S + \int_{\Gamma_P} P_i u_i\,\mathrm{d}S -$$
$$- \int_{\Gamma_Q} (B_n T_n\,T_n(\boldsymbol{u}) + B_t\boldsymbol{T}_t\,.\,\boldsymbol{T}_t(\boldsymbol{u}))\,\mathrm{d}S\,.$$

If $\Gamma_n \cup \Gamma_t$ is non-vacuous, we can state the principle of virtual work with alternative definitions of the sets $\mathbf{W}$ and $\mathbf{U}$. Thus let $\tilde{\mathbf{W}}$ be the set of such symmetric tensor-functions of stress $\mathbf{T}(X)$ with components $\tau_{ik} = \tau_{ki} \in W_2^{(1)}(\Omega)$ (or $W_2^{(1)}(\Omega_j)$ in case of jump discontinuities of $c_{iklm}(X)$ — see the definition of $\mathbf{W}$), which meet the equations of equilibrium (0.3) (in the sense of $L_2(\Omega)$ or $L_2(\Omega_j)$ respectively), the "pure statical" boundary conditions (0.5) on $\mathscr{D}_n \cup \mathscr{D}_t$ (in $L_2(\mathscr{D}_n)$ and $L_2(\mathscr{D}_t)$) and the conditions of "elastic supports" in the form (0.5) on Γ_n and Γ_t in the following manner: such vector-function $\tilde{\mathbf{u}} \in [W_2^{(1)}(\Omega_j)]^3$ exists, that

$$T_n = T_n(\tilde{\mathbf{u}}) = (C_n - \tilde{u}_n)/B_n \quad (\text{in } L_2(\Gamma_n))$$
$$\mathbf{T}_t = \mathbf{T}_t(\tilde{\mathbf{u}}) = (\mathbf{C}_t - \tilde{\mathbf{u}}_t)/B_t \quad (\text{in } L_2(\Gamma_t)) \, ,$$

where $T_n(\tilde{\mathbf{u}})$ and $\mathbf{T}_t(\tilde{\mathbf{u}})$ are derived from $\tilde{\mathbf{u}}$ on the base of (0.6) and (0.1), (0.2). Again, $\tilde{\mathbf{W}}$ will be reffered to as the set of statically admissible stress fields.

Let $\tilde{\mathbf{U}}$ be the set of vector-functions of displacements $\mathbf{u} \in [W_2^{(1)}(\Omega)]^3$, which satisfy the "pure geometric" boundary conditions of the form (0.5) on $\mathscr{A}_n \cup \mathscr{A}_t$ (in the sense of traces, i.e. in $L_2(\mathscr{A}_n)$ and $L_2(\mathscr{A}_t)$ respectively). Again, $\tilde{\mathbf{U}}$ will be reffered to as the set of geometrically admissible displacement field.

If $\mathbf{T} \in \tilde{\mathbf{W}}$ and $\mathbf{u} \in \tilde{\mathbf{U}}$, the principle of virtual work (see the derivation of (1.1)) holds in the form

$$(1.1') \qquad \int_\Omega \tau_{ik}\varepsilon_{ik}(\mathbf{u})\,\mathrm{d}X = \int_\Omega K_i u_i \,\mathrm{d}X + \int_{\mathscr{A}_n} T_n \bar{u}_n \,\mathrm{d}S + \int_{\mathscr{A}_t} \mathbf{T}_t \,.\, \bar{\mathbf{u}}_t \,\mathrm{d}S +$$
$$+ \int_{\mathscr{D}_n} P_n u_n \,\mathrm{d}S + \int_{\mathscr{D}_t} \mathbf{P}_t \,.\, \mathbf{u}_t \,\mathrm{d}S + \int_{\Gamma_n} (C_n - \tilde{u}_n)\, u_n \frac{\mathrm{d}S}{B_n} + \int_{\Gamma_t} (\mathbf{C}_t - \tilde{\mathbf{u}}_t) \,.\, \mathbf{u}_t \frac{\mathrm{d}S}{B_t}$$

which allows the following interpretation: "the virtual work of internal forces

$$(1.2') \qquad \int_\Omega \tau_{ik}\varepsilon_{ik}(\mathbf{u})\,\mathrm{d}X + \int_{\Gamma_n} \frac{1}{B_n} u_n \tilde{u}_n \,\mathrm{d}S + \int_{\Gamma_t} \frac{1}{B_t} \mathbf{u}_t \,.\, \tilde{\mathbf{u}}_t \,\mathrm{d}S$$

is equal to the virtual work of external forces"

$$(1.3') \int_\Omega K_i u_i \,\mathrm{d}X + \int_{\mathscr{A}_t} T_n \bar{u}_n \,\mathrm{d}S + \int_{\mathscr{A}_t} \mathbf{T}_t \,.\, \bar{\mathbf{u}}_t \,\mathrm{d}S + \int_{\mathscr{B}_n} P_n u_n \,\mathrm{d}S + \int_{\mathscr{B}_t} \mathbf{P}_t \,.\, \mathbf{u}_t \,\mathrm{d}S \,.$$

Here again the work of "elastic supports" is involved in the work of internal forces and the work of "yielding of supports" $\bar{\mathbf{u}}$ and that of surface tractions C_n/B_n on Γ_n and C_t/B_t on Γ_t in the work of external forces.

The fields $\mathbf{T}(X)$ and $\mathbf{u}(X)$ are in general independent. The application of the latter principle, however, is restricted in praxis to the case, when the field $\mathbf{T}(X) = \mathbf{T}(\mathring{\mathbf{u}})$ is real field, associated with the real displacement field $\mathring{\mathbf{u}}(X)$, representing the solution to the problem.

Among examples 1 till 3 only the expression in example 3 will be changed:

$$\int_\Omega \tau_{ik}\varepsilon_{ik}(\boldsymbol{u})\,\mathrm{d}X = \int_\Omega K_i u_i\,\mathrm{d}X + \int_{\Gamma_u} \tau_{ik}n_k\bar{u}_i\,\mathrm{d}S + \int_{\Gamma_P} P_i u_i\,\mathrm{d}S -$$
$$- \int_{\Gamma_Q} \left(\frac{u_n}{B_n}\,\tilde{u}_n + \frac{\boldsymbol{u}_t}{B_t}\,.\,\tilde{\boldsymbol{u}}_t\right)\mathrm{d}S\,.$$

A converse of the virtual work principle due to Dorn and Schild [5] was generalized by M. E. Gurtin in [6]. We introduce here the Gurtin's assertion:

let $\Gamma = \Gamma_u \cup \Gamma_P$ be a disjoint decomposition of Γ; let Ω be a bounded, simply-connected region, whose boundary Γ is the union of a finite number of regular surfaces, (the latter term being used in the sense of Kellog [7]), Ω convex with respect to Γ_u (that means: the straight line joining any two points $X' \in \Gamma_u$, $X'' \in \Gamma_u$ intersects Γ only at X' and X''). Let $K_i = P_i = 0$, $\bar{u}_i$ be continuous on Γ_u and let ε_{ij} be a symmetric tensor field, which is twice continuously differentiable on $\bar{\Omega}$. Then if

$$\int_\Omega \tau_{ij}\varepsilon_{ij}\,\mathrm{d}X = \int_{\Gamma_u} \tau_{ik}n_k\bar{u}_i\,\mathrm{d}S$$

holds for every symmetric tensor-function τ_{ij}, which is continuously differentiable arbitrarily often on Ω and meets the equilibrium equations (0.3) on Ω and $\tau_{ik}n_k = 0$ on Γ_P, there exists a vector-function u_i, which is continuously differentiable on $\bar{\Omega}$ and satisfies the strain-displacement relations (0.1) and the boundary conditions $u_i = \bar{u}_i$ on Γ_u.

2. THE PRINCIPLE OF VIRTUAL DISPLACEMENT AND THE DEFINITION OF A WEAK SOLUTION TO THE PROBLEM

Let the real displacement field $\mathring{u}_i \in \tilde{\boldsymbol{U}}$ and the real stress field $\mathring{\tau}_{ik} = c_{iklm}\mathring{u}_{l,m} \in \tilde{\boldsymbol{W}}$. Inserting $\mathring{\tau}_{ik}$ together with both the real displacement field $\mathring{u}_i$ and the varied field $\mathring{u}_i + \delta u_i \in \tilde{\boldsymbol{U}}$ in the principle of virtual work (1.1′) and subtracting, we obtain the principle of virtual displacements

$$\int_\Omega \mathring{\tau}_{ik}\delta u_{i,k}\,\mathrm{d}X + \int_{\Gamma_n} \frac{1}{B_n}\,\mathring{u}_n\delta u_n\,\mathrm{d}S + \int_{\Gamma_t} \frac{1}{B_t}\,\mathring{\boldsymbol{u}}_t\,.\,\delta\boldsymbol{u}_t\,\mathrm{d}S = \tag{2.1}$$
$$= \int_\Omega K_i\delta u_i\,\mathrm{d}X + \int_{\mathscr{B}_n} P_n\,.\,\delta u_n\,\mathrm{d}S + \int_{\mathscr{B}_t} \boldsymbol{P}_t\,.\,\delta\boldsymbol{u}_t\,\mathrm{d}S\,.$$

According to (2.1) the definition of weak (generalized) solution to the mixed boundary-value problem is formed: let $\boldsymbol{M}$ be the linear manifold of vector-functions $\boldsymbol{v}$, continuously differentiable on $\bar{\Omega}$, (see the Introduction), which satisfy the homogeneous boundary conditions (for $\bar{\boldsymbol{u}} \equiv 0$) on $\mathscr{A}_n$ and $\mathscr{A}_t$. The weak solution to the mixed boundary-value problem is defined as a vector-function $\mathring{\mathbf{u}} \in \tilde{\mathbf{U}}$ such that for every $\mathbf{v} \in \mathbf{M}$

$$(2.2)\qquad \int_\Omega c_{iklm}\mathring{u}_{l,m}v_{i,k}\,\mathrm{d}X = \int_\Omega K_i v_i\,\mathrm{d}X + \int_{\mathscr{B}_n} P_n v_n\,\mathrm{d}S + \int_{\mathscr{B}_t} \mathbf{P}_t\,.\,\mathbf{v}_t\,\mathrm{d}S - \\ - \int_{\Gamma_n} \frac{1}{B_n}\mathring{u}_n v_n\,\mathrm{d}S - \int_{\Gamma_t}\frac{1}{B_t}\mathring{\mathbf{u}}_t\,.\,\mathbf{v}_t\,\mathrm{d}S\,.$$

Note that this definition differs from the definition of the set $\tilde{\mathbf{W}}$ by the fact, that the stress components $\mathring{\tau}_{ik}$ need not belong to the space $W_2^{(1)}(\Omega)$ or $W_2^{(1)}(\Omega_j)$, but only to $L_2(\Omega)$, and the boundary conditions on $\mathscr{B}_n$ and $\mathscr{B}_t$ have not to be met by the components $\mathring{\tau}_{ik}$ in the sense of $L_2(D_n)$, $L_2(D_t)$ and $L_2(\Gamma_n)$, $L_2(\Gamma_t)$ respectively.

Let $\mathbf{u} \in \tilde{\mathbf{U}}$ be such a displacement field that $\mathbf{T}(\mathbf{u}) \in \tilde{\mathbf{W}}$. Then (2.1) holds for $\delta\mathbf{u} \equiv \equiv \mathbf{v} \in \mathbf{M}$ and consequently $\mathbf{u}$ is a weak solution. Hence (2.2) defines a solution, which is more general than that of section 1, called "real" displacement field, for which the associated stress field belongs to $\tilde{\mathbf{W}}$.

Next we shall discuss briefly the proof of the existence and uniqueness of the weak solution. Introduce the scalar product

$$(2.3)\qquad [\mathbf{v}, \mathbf{w}] \equiv \int_\Omega c_{iklm}v_{i,k}w_{l,m}\,\mathrm{d}X + \int_{\Gamma_n}\frac{1}{B_n}v_n w_n\,\mathrm{d}S + \int_{\Gamma_t}\frac{1}{B_t}\mathbf{v}_t\,.\,\mathbf{w}_t\,\mathrm{d}S$$

on the linear manifold $\mathbf{M}$. By means of Korn's inequality (0.8) and (0.4) we can prove, that for each $\mathbf{v} \in \mathbf{M}$

$$(2.4)\qquad [\mathbf{v}, \mathbf{v}] \geqq C_2|\mathbf{v}|^2_{[W_2^{(1)}(\Omega)]^3}$$

and thus the bilinear form (2.3) has all the properties of the scalar product on $\mathbf{M}$ with the norm of $[W_2^{(1)}(\Omega)]^3$. In order to create a Hilbert (complete) space $\mathbf{H}_M$ from $\mathbf{M}$ with this scalar product, in general it is necessary to form the completion of $\mathbf{M}$ in the norm, associated with the product (2.3).

Let us write the solution in the form

$$\mathring{\mathbf{u}} = \bar{\mathbf{u}} + \mathbf{w}\,.$$

The definition (2.2) implies an equivalent definition for $\mathbf{w}$: $w_n = 0$ on $\mathscr{A}_n$, $\mathbf{w}_t = 0$ on $\mathscr{A}_t$ (in the sense of traces, i.e. in $L_2(\mathscr{A}_n)$ and $L_2(\mathscr{A}_t)$ respectively) and for each $\mathbf{v} \in \mathbf{M}$ it shall hold

$$(2.5)\qquad \int_\Omega c_{iklm}w_{l,m}v_{i,k}\,\mathrm{d}X + \int_{\Gamma_n}\frac{1}{B_n}w_n v_n\,\mathrm{d}S + \int_{\Gamma_t}\frac{1}{B_t}\mathbf{w}_t\,.\,\mathbf{v}_t\,\mathrm{d}S = \int_\Omega K_i v_i\,\mathrm{d}X - \\ - \int_\Omega c_{iklm}\bar{u}_{l,m}v_{i,k}\,\mathrm{d}X + \int_{\mathscr{B}_n}\left(P_n - \frac{A_n}{B_n}\bar{u}_n\right)v_n\,\mathrm{d}S + \int_{\mathscr{B}_t}\left(\mathbf{P}_t - \frac{A_t}{B_t}\bar{\mathbf{u}}_t\right).\,\mathbf{v}_t\,\mathrm{d}S\,.$$

It is possible to form $\boldsymbol{H}_M$ from the elements of $[W_2^{(1)}(\Omega)]^3$ and then the inequality (2.4) holds in the whole H_M. From this and from the continuity of the immersion of $W_2^{(1)}(\Omega)$ into $L_2(\mathscr{A}_n)$ and $L_2(\mathscr{A}_t)$ respectively (see e.g. [3] § 22) it follows that if $\boldsymbol{w} \in \boldsymbol{H}_M$, then $w_n = 0$ on $\mathscr{A}_n$ and $\boldsymbol{w}_t = 0$ on $\mathscr{A}_t$ in the sense of traces. Therefore it is sufficient to find $\boldsymbol{w}$ in $\boldsymbol{H}_M$. By virtue of (2.4) on $\boldsymbol{H}_M$ and the continuity of immersion of $W_2^{(1)}(\Omega)$ into $L_2(\Gamma_n)$ and $L_2(\Gamma_t)$ respectively, the definition of the scalar product (2.3) may be extended from $\boldsymbol{M}$ onto the whole $\boldsymbol{H}_M$ in accordance with the construction of $\boldsymbol{H}_M$ preserving the form (2.3). On the left-hand side of (2.5) we have therefore $[\boldsymbol{w}, \boldsymbol{v}]$. Using the assumptions on the coefficients c_{iklm}, functions $\boldsymbol{u}$ and P_n, $\boldsymbol{P}_t$, the Cauchy's inequality and continuity of immersion of $W_2^{(1)}(\Omega)$ into $L_2(\mathscr{B}_n)$ and $L_2(\mathscr{B}_t)$ respectively, we obtain, that the right-hand side of (2.5) is a continuous linear functional on $[W_2^{(1)}(\Omega)]^3$ and consequently, by virtue of (2.4), also on $\boldsymbol{H}_M$. Then Riesz-Fréchet theorem implies the existence of a unique element $\boldsymbol{w}$ in $\boldsymbol{H}_M$, which satisfies (2.5) for every $\boldsymbol{v} \in \boldsymbol{H}_M$. As $\boldsymbol{M}$ is dense in $\boldsymbol{H}_M$, the solution $\boldsymbol{w}$ according to the equivalent definition (2.5) (i.e. with restriction to $\boldsymbol{v} \in \boldsymbol{M}$) is unique in $\boldsymbol{H}_M$. Then $\mathring{\boldsymbol{u}} = \bar{\boldsymbol{u}} + \boldsymbol{w}$ *represents a weak solution to the mixed boundary problem and it is unique in the set* $\bar{\boldsymbol{u}} \oplus \boldsymbol{H}_M$ *of all sums* $\boldsymbol{u} + \boldsymbol{y}$, *where* $\boldsymbol{y} \in \boldsymbol{H}_M$.

3. THE PRINCIPLE OF MINIMUM POTENTIAL ENERGY

Let us define on $\boldsymbol{H}_M$ the quadratic functional

$$(3.1)\quad \Phi(\boldsymbol{v}) = |\boldsymbol{v}|^2_{H_M} - 2\left\{(\boldsymbol{K}, \boldsymbol{v}) - B(\bar{\boldsymbol{u}}, \boldsymbol{v}) + \left(P_n - \frac{A_n}{B_n}\bar{u}_n, v_n\right) + \left(\boldsymbol{P}_t - \frac{A_t}{B_t}\boldsymbol{u}_t, \boldsymbol{v}_t\right)\right\},$$

where the terms in curly bracket are defined by the corresponding integrals on the right-hand side of (2.5). The latter relation may be written in the form

$$[\boldsymbol{w}, \boldsymbol{v}] = (\boldsymbol{K}, \boldsymbol{v}) - B(\bar{\boldsymbol{u}}, \boldsymbol{v}) + \left(P_n - \frac{A_n}{B_n}\bar{u}_n, v_n\right) + \left(\boldsymbol{P}_t - \frac{A_t}{B_t}\bar{\boldsymbol{u}}_t, \boldsymbol{v}_t\right).$$

Therefore it holds

$$(3.2)\qquad \Phi(\boldsymbol{v}) = |\boldsymbol{v}|^2_{H_M} - 2[\boldsymbol{w}, \boldsymbol{v}] = |\boldsymbol{v} - \boldsymbol{w}|^2_{H_M} - |\boldsymbol{w}|^2_{H_M} \geqq - |\boldsymbol{w}|^2_{H_M}\,.$$

Extending the definition of the bilinear form $[\boldsymbol{v}, \boldsymbol{w}]$ from $\boldsymbol{H}_M$ onto the whole space $[W_2^{(1)}(\Omega)]^3$ with designation $\langle \boldsymbol{v}, \boldsymbol{w}\rangle$, and inserting $\boldsymbol{v} = \boldsymbol{u} - \bar{\boldsymbol{u}}$ into (3.1), we obtain

$$\Phi(\boldsymbol{v}) = \Phi(\boldsymbol{u} - \bar{\boldsymbol{u}}) = \langle \boldsymbol{u}, \boldsymbol{u}\rangle - 2\langle \boldsymbol{u}, \bar{\boldsymbol{u}}\rangle + \langle \bar{\boldsymbol{u}}, \boldsymbol{u}\rangle -$$

$$- 2\left\{(\boldsymbol{K}, \boldsymbol{u} - \bar{\boldsymbol{u}}) - B(\bar{\boldsymbol{u}}, \boldsymbol{u} - \bar{\boldsymbol{u}}) + \left(P_n - \frac{A_n}{B_n}\bar{u}_n, u_n - \bar{u}_n\right) + \left(\boldsymbol{P}_t - \frac{A_t}{B_t}\bar{\boldsymbol{u}}_t, \boldsymbol{u}_t - \bar{\boldsymbol{u}}_t\right)\right\} =$$

$$= \langle \boldsymbol{u}, \boldsymbol{u}\rangle - 2\left\{\int_\Omega K_i u_i \,\mathrm{d}X + \int_{\mathscr{B}_n} P_n u_n \,\mathrm{d}S + \int_{\mathscr{B}_t} \boldsymbol{P}_t \,.\, \boldsymbol{u}_t \,\mathrm{d}S\right\} + \Phi_1\,,$$

where Φ_1 does not depend on $\boldsymbol{u}$. Hence the principle of minimum potential energy follows: *The quadratic functional (total potential energy of internal and external forces) defined for* $\boldsymbol{u} \in [W_2^{(1)}(\Omega)]^3$ *through*

$$(3.3)\quad \mathscr{L}(\boldsymbol{u}) \equiv \tfrac{1}{2}\int_\Omega c_{iklm} u_{i,k} u_{l,m} \,\mathrm{d}X + \tfrac{1}{2}\int_{\Gamma_n} \frac{1}{B_n} u_n^2 \,\mathrm{d}S + \tfrac{1}{2}\int_{\Gamma_t} \frac{1}{B_t} \boldsymbol{u}_t^2 \,\mathrm{d}S - \int_\Omega K_i u_i \,\mathrm{d}X - \\ - \int_{\mathscr{B}_n} P_n u_n \,\mathrm{d}S - \int_{\mathscr{B}_t} \boldsymbol{P}_t \,.\, \boldsymbol{u}_t \,\mathrm{d}S$$

attains a minimum on the set $\bar{\boldsymbol{u}} \oplus \boldsymbol{H}_M \subset [W_2^{(1)}(\Omega)]^3$, *if and only if* $\boldsymbol{u} = \boldsymbol{u} = \bar{\boldsymbol{u}} + \boldsymbol{w}$ *represents a weak solution to the mixed problem.*

For illustration let us show the functional (3.3) for the example 3 of section 1:

$$\mathscr{L}(\boldsymbol{u}) \equiv \tfrac{1}{2}\int_\Omega c_{iklm} u_{i,k} u_{l,m} \,\mathrm{d}X + \tfrac{1}{2}\int_{\Gamma_Q} \left(\frac{1}{B_n} u_n^2 + \frac{1}{B_t} \boldsymbol{u}_t^2\right) \mathrm{d}S - \int_\Omega K_i u_i \,\mathrm{d}X - \int_{\Gamma_P} P_i u_i \,\mathrm{d}S .$$

4. THE PRINCIPLE OF MINIMUM COMPLEMENTARY ENERGY

We shall distinguish two kinds of mixed boundary-value problems according to the existence of the set of "elastic supports".

I. Let a non-vacuous set $\Gamma_n \cup \Gamma_t$ exist. Let $\mathscr{T}_2^{(1)}$ be the space of symmetric tensor-functions $\boldsymbol{T}(X)$ with components $\tau_{ik} = \tau_{ki} \in W_2^{(1)}(\Omega)$, (or $W_2^{(1)}(\Omega_j)$). Denote

$$|\boldsymbol{T}|_2 \equiv \Big(\sum_{i,k=1}^{3} |\tau_{ik}|^2_{L_2(\Omega)}\Big)^{1/2}$$

and introduce in $\mathscr{T}_2^{(1)}$ the scalar product

$$(4.1)\qquad ((\boldsymbol{T}', \boldsymbol{T}'')) = \int_\Omega a_{iklm} \tau'_{ik} \tau''_{lm} \,\mathrm{d}X + \int_{\Gamma_n} B_n T'_n\, T''_n \,\mathrm{d}S + \int_{\Gamma_t} B_t \boldsymbol{T}'_t \,.\, \boldsymbol{T}''_t \,\mathrm{d}S .$$

Using the relations (0.2), (0.4) and assumptions on B_n, B_t, we can easily verify, that the bilinear form (4.1) has all the properties required for the scalar product in $\mathscr{T}_2^{(1)}$ with the norm $|\boldsymbol{T}|_2$.

Suppose, that the solution $\mathring{\boldsymbol{u}} \in \boldsymbol{U}$. Then according to the definition of $\boldsymbol{U}$ the associated stress tensor $\boldsymbol{T}(\mathring{\boldsymbol{u}}) \in \mathscr{T}_2^{(1)}$, but we shall suppose even $\boldsymbol{T}_0 \equiv \boldsymbol{T}(\mathring{\boldsymbol{u}}) \in \boldsymbol{W} \subset \mathscr{T}_2^{(1)}$. The properties of the scalar product (4.1) imply that if $\boldsymbol{T} - \boldsymbol{T}_0 \in \mathscr{T}_2^{(1)}$, $|\boldsymbol{T} - \boldsymbol{T}_0|_2 > 0$, then $((\boldsymbol{T} - \boldsymbol{T}_0, \boldsymbol{T} - \boldsymbol{T}_0)) > 0$. The latter inequality may be written in the form

$$(4.2)\qquad ((\boldsymbol{T}, \boldsymbol{T})) + ((\boldsymbol{T}_0, \boldsymbol{T}_0)) > 2((\boldsymbol{T}, \boldsymbol{T}_0)) .$$

Let $\boldsymbol{T} \in \boldsymbol{W}$ and let us apply the principle of virtual work (1.1) to the fields $\boldsymbol{T}$ and $\mathring{\boldsymbol{u}}$.

We obtain

$$(4.3) \qquad ((\mathbf{T}, \mathbf{T}_0)) = \int_\Omega a_{iklm}\tau_{ik}\mathring{\tau}_{lm}\,\mathrm{d}X + \overline{\int_{\Gamma_n} B_n T_n T_{0n}\,\mathrm{d}S} = \int_\Omega K_i\mathring{u}_i\,\mathrm{d}X + $$

$$+ \overline{\int_{\mathscr{A}_n} T_n\bar{u}_n\,\mathrm{d}S} + \overline{\int_{\mathscr{D}_n} P_n\mathring{u}_n\,\mathrm{d}S} + \overline{\int_{\Gamma_n} T_n C_n\,\mathrm{d}S}\,.$$

Henceforth we use for brevity the following convention: the bar above a term indicates the fact that an analogous term, but with subscripts t instead of n, should be added.

Similarly, it holds

$$(4.4) \quad ((\mathbf{T}_0, \mathbf{T}_0)) = \int_\Omega K_i\mathring{u}_i\,\mathrm{d}X + \overline{\int_{\mathscr{A}_n} T_{0n}\bar{u}\,\mathrm{d}S} + \overline{\int_{\mathscr{D}_n} P_n\mathring{u}_n\,\mathrm{d}S} + \overline{\int_{\Gamma_n} T_{0n}C_n\,\mathrm{d}S}\,.$$

Subtracting twice the equation (4.4) from (4.2) and using (4.3), we may write

$$\tfrac{1}{2}((\mathbf{T}, \mathbf{T})) - \tfrac{1}{2}((\mathbf{T}_0, \mathbf{T}_0)) > \overline{\int_{\mathscr{A}_n} (T_n - T_{0n})\,\bar{u}_n\,\mathrm{d}S} + \overline{\int_{\Gamma_n} (T_n - T_{0n})\,C_n\,\mathrm{d}S}\,.$$

Hence the principle of minimum complementary energy (Castigliano-Menabrea) follows directly: *If* $\mathring{\mathbf{u}} \in \mathbf{U}$ *and* $\mathbf{T}_0 \equiv \mathbf{T}(\mathring{\mathbf{u}}) \in \mathbf{W}$, *then the quadratic functional (complementary energy)*

$$\mathscr{S}(\mathbf{T}) \equiv \tfrac{1}{2}\int_\Omega a_{iklm}\tau_{ik}\tau_{lm}\,\mathrm{d}X + \int_{\Gamma_n} (\tfrac{1}{2}B_n T_n^2 - C_n T_n)\,\mathrm{d}S + \int_{\Gamma_t} (\tfrac{1}{2}B_t \mathbf{T}_t^2 - \mathbf{C}_t \,.\, \mathbf{T}_t)\,\mathrm{d}S -$$

$$- \int_{\mathscr{A}_n} T_n\bar{u}_n\,\mathrm{d}S - \int_{\mathscr{A}_t} \mathbf{T}_t \,.\, \bar{u}_t\,\mathrm{d}S$$

attains a minimum on the set $\mathbf{W}$ *of statically admissible stress fields* $\mathbf{T}$, *precisely if* $|\mathbf{T} - \mathbf{T}_0|_2 = 0$.

II. Let the set $\Gamma_n \cup \Gamma_t$ be vacuous- that means no elastic supports exist, everywhere on Γ $A_n B_n = A_t B_t = 0$. Clearly, all of the part ***I.*** holds, with the only change that the integrals on Γ_n and Γ_t have to be omitted. In the present case, however, it is possible to apply the method of orthogonal projections in Hilbert space (see e.g. [4], § 54) with weaker requirements for the fields $\mathbf{T}(X)$ than in the foregoing method.

Let us define a scalar product in the space $\mathscr{T}_2$ of symmetric tensor-functions $\mathbf{T}(X)$ with components $\tau_{ik} = \tau_{ki} \in L_2(\Omega)$ and with the norm $|\mathbf{T}|_2$, through

$$(4.5) \qquad ((\mathbf{T}', \mathbf{T}'')) = \int_\Omega a_{iklm}\tau'_{ik}\tau''_{lm}\,\mathrm{d}X\,.$$

As in the part ***I.*** all the properties of the scalar product are easily verifiable for (4.5). Since $\mathscr{T}_2$ is a complete space, $\mathscr{T}_2$ with the scalar product (4.5) forms a Hilbert space $\mathscr{H}$. Note that the norms $|\mathbf{T}|_2$ and $((\mathbf{T}, \mathbf{T}))^{1/2}$ are even equivalent.

Denote $\mathscr{H}_1 \subset \mathscr{H}$ the subset of tensor-functions $\boldsymbol{T}'$, to which such vector-function $\boldsymbol{v}' \in \boldsymbol{H}_M$ exists (recall the section 2 for the definition of $\boldsymbol{H}_M$) that

$$\tau_{ik}(\boldsymbol{T}') \equiv \tau'_{ik} = c_{iklm} v'_{l,m} \, .$$

Furthermore, denote $\mathscr{H}_2 \subset \mathscr{H}$ the subset of tensor-functions $\boldsymbol{T}''$, which satisfy the homogeneous equations of equilibrium and homogeneous statical boundary-conditions in the weak sense, (see (2.1) or (2.2) for $\boldsymbol{K} = 0$, $P_n = 0$, $\boldsymbol{P}_t = 0$) i.e. it holds for every $\boldsymbol{v} \in \boldsymbol{M}$, that

$$\int_\Omega \tau_{ik}(\boldsymbol{T}'') \, v_{i,k} \, \mathrm{d}X = 0 \, . \tag{4.6}$$

Since the transition to the limit in (4.6) is possible for $v^{(n)} \in \boldsymbol{M}$,

$$\lim_{n \to \infty} \boldsymbol{v}^{(n)} = \boldsymbol{v}' \in \boldsymbol{H}_M$$

in the sense of $[W_2^{(1)}(\Omega)]^3$, we have

$$((\boldsymbol{T}', \boldsymbol{T}'')) = \int_\Omega \tau_{ik}(\boldsymbol{T}'') \, v'_{i,k} \, \mathrm{d}X = 0 \tag{4.7}$$

for any $\boldsymbol{T}' \in \mathscr{H}_1$, $\boldsymbol{T}'' \in \mathscr{H}_2$.

Next let $\boldsymbol{T} \in \mathscr{H}$ be an arbitrary stress tensor, satisfying the complete equations of equilibrium (0.3) and the statical boundary-conditions on $\mathscr{B}_n$ and $\mathscr{B}_t$ in the weak sense, i.e. it holds for every $\boldsymbol{v} \in \boldsymbol{M}$ that

$$\int_\Omega \tau_{ik} v_{i,k} \, \mathrm{d}X = \int_\Omega K_i v_i \, \mathrm{d}X + \int_{\mathscr{B}_n} P_n v_n \, \mathrm{d}S + \int_{\mathscr{B}_t} \boldsymbol{P}_t \, . \, \boldsymbol{v}_t \, \mathrm{d}S \, , \tag{4.8}$$

where τ_{ik} are components of the tensor $\boldsymbol{T}$.

Denote again $\boldsymbol{T}_0$ the tensor $\boldsymbol{T}(\mathring{\boldsymbol{u}})$ associated with the weak solution $\mathring{\boldsymbol{u}} = \bar{\boldsymbol{u}} + \boldsymbol{w}$ according to the Hooke's law (0.2). Now use (2.2), (4.8) and compare with (4.6) to deduce that $\boldsymbol{T} - \boldsymbol{T}_0 \in \mathscr{H}_2$. It holds

$$\boldsymbol{T}_0 = \boldsymbol{T}(\bar{\boldsymbol{u}}) + \boldsymbol{T}(\boldsymbol{w})$$

with $\boldsymbol{T}(\boldsymbol{w}) \in \mathscr{H}_1$. By virtue of (4.7) we have

$$|\boldsymbol{T} - \boldsymbol{T}(\bar{\boldsymbol{u}})|^2_{\mathscr{H}} = |(\boldsymbol{T} - \boldsymbol{T}_0) + \boldsymbol{T}(\boldsymbol{w})|^2_{\mathscr{H}} = |\boldsymbol{T}(\boldsymbol{w})|^2_{\mathscr{H}} + |\boldsymbol{T} - \boldsymbol{T}_0|^2_{\mathscr{H}} \geqq |\boldsymbol{T}(\boldsymbol{w})|^2_{\mathscr{H}} \, , \tag{4.9}$$

where the equality takes place precisely if $|\boldsymbol{T} - \boldsymbol{T}_0|_{\mathscr{H}} = 0$. We can rewrite

$$|\boldsymbol{T} - \boldsymbol{T}(\bar{\boldsymbol{u}})|^2_{\mathscr{H}} = ((\boldsymbol{T}, \boldsymbol{T})) - 2((\boldsymbol{T}, \boldsymbol{T}(\bar{\boldsymbol{u}})) + ((\boldsymbol{T}(\bar{\boldsymbol{u}}), \boldsymbol{T}(\bar{\boldsymbol{u}})) =$$
$$= \int_\Omega a_{iklm} \tau_{ik} \tau_{lm} \, \mathrm{d}X - 2 \int_\Omega \tau_{ik} \bar{u}_{i,k} \, \mathrm{d}X + |\boldsymbol{T}(\bar{\boldsymbol{u}})|^2_{\mathscr{H}} \, .$$

Hence the principle of minimum complementary energy follows: *The quadratic functional*

$$(4.10)\qquad \tilde{\mathscr{S}}(\boldsymbol{T}) \equiv \tfrac{1}{2}\int_{\Omega} a_{iklm}\tau_{ik}\tau_{lm}\,\mathrm{d}X - \int_{\Omega}\tau_{ik}\bar{u}_{i,k}\,\mathrm{d}X$$

attains a minimum on the set of tensor-functions $\boldsymbol{T} \in \mathscr{T}_2$, *which satisfy the equations of equilibrium and the statical boundary conditions in the sense of* (4.8), *if and only if* $|\boldsymbol{T} - \boldsymbol{T}_0|_2 = 0$.

If moreover the weak solution $\mathring{\boldsymbol{u}}$ is such that $\boldsymbol{T}(\mathring{\boldsymbol{u}}) \in \boldsymbol{W}$ (i.e. the equilibrium inside and on the surface of the body is satisfied in a stronger manner than in the sense of (4.8)), we can take for $\boldsymbol{T}$ the statically admissible stress fields $\boldsymbol{T} \in \boldsymbol{W}$ and apply the principle of virtual work (1.1) to the fields $\boldsymbol{T}$ and $\bar{\boldsymbol{u}}$. Thus we obtain

$$\int_{\Omega}\tau_{ik}\bar{u}_{i,k}\,\mathrm{d}X = \int_{\mathscr{A}_n} T_n\bar{u}_n\,\mathrm{d}S + \int_{\mathscr{B}_n} P_n\bar{u}_n\,\mathrm{d}S + \int_{\Omega} \langle\ \bar{u}_{\iota}\,\mathrm{d}S\,.$$

As the last integrals do not depend upon $\boldsymbol{T}$, we may omit them and write the principle in the usual form: *The quadratic functional*

$$\mathscr{S}(\boldsymbol{T}) = \tfrac{1}{2}\int_{\Omega} a_{iklm}\tau_{ik}\tau_{lm}\,\mathrm{d}X - \int_{\mathscr{A}_n} T_n\bar{u}_n\,\mathrm{d}S - \int_{\mathscr{A}_t} \boldsymbol{T}_t\,.\,\bar{u}_t\,\mathrm{d}S$$

attains a minimum on the set $\boldsymbol{T} \in \boldsymbol{W}$ *of statically admissible stress fields, if and only if* $|\boldsymbol{T} - \boldsymbol{T}_0|_2 = 0$.

The latter assertion corresponds to the principle of minimum complementary energy from the part ***I***. The relation

$$(4.11)\qquad \tilde{\mathscr{S}}(\boldsymbol{T}) = \mathscr{S}(\boldsymbol{T}) - \int_{\Omega} K_i\bar{u}_i\,\mathrm{d}X - \int_{\mathscr{B}_n} P_n\bar{u}_n\,\mathrm{d}S - \int_{\mathscr{B}_t} \boldsymbol{P}_t\,.\,\bar{\boldsymbol{u}}_t\,\mathrm{d}S$$

holds.

Let again $\Gamma_n \cup \Gamma_t$ be a non-vacuous set. The necessary condition for the minimum of $\mathscr{S}(\boldsymbol{T})$ at $\boldsymbol{T} = \boldsymbol{T}_0$ is expressed by the principle of virtual changes of stresses: *Let the real stress field* $\boldsymbol{T}_0 = \boldsymbol{T}(\mathring{\boldsymbol{u}}) \in \boldsymbol{W}$. *Then it holds*

$$(4.12)\qquad \delta\mathscr{S}(\boldsymbol{T}_0) \equiv \int_{\Omega} a_{iklm}\mathring{\tau}_{ik}\delta\tau_{lm}\,\mathrm{d}X - \int_{\mathscr{A}_n}\bar{u}_n\delta T_n\,\mathrm{d}S - \int_{\mathscr{A}_t}\bar{\boldsymbol{u}}_t\,.\,\delta\boldsymbol{T}_t\,\mathrm{d}S +$$
$$+ \int_{\Gamma_n}(B_nT_{0n} - C_n)\,\delta T_n\,\mathrm{d}S + \int_{\Gamma_t}(B_t\boldsymbol{T}_{0t} - \boldsymbol{C}_t)\,.\,\delta\boldsymbol{T}_t\,\mathrm{d}S = 0\,,$$

where the variations $\delta\tau_{ik}$ *satisfy (in the sense of* $L_2(\Omega)$ *or* $L_2(\Omega_j)$*) the homogeneous equations of equilibrium*

$$(4.13)\qquad \delta\tau_{ik,k} = 0$$

and (in the sense of $L_2(\mathscr{D}_n)$ and $L_2(\mathscr{D}_t)$) the "pure statical" homogeneous boundary-conditions on $\mathscr{D}_n$ and $\mathscr{D}_t$

(4.14) $$\delta T_n = 0 \quad \text{and} \quad \delta \mathbf{T}_t = \mathbf{0}$$

respectively.

If $\Gamma_n = \Gamma_t = \emptyset$, then $\mathbf{T}(\mathring{u}) \in \mathscr{T}_2$ for the weak solution $\mathring{\mathbf{u}}$ and the necessary condition for the minimum of $\widetilde{\mathscr{S}}(\mathbf{T})$ is expressed by the principle of virtual changes of stresses in the form

$$\delta \widetilde{\mathscr{S}}(\mathbf{T}_0) \equiv \int_\Omega a_{iklm} \mathring{\tau}_{ik} \delta\tau_{lm} \, \mathrm{d}X - \int_\Omega \bar{u}_{i,k} \delta\tau_{ik} \, \mathrm{d}X = 0 \,,$$

where the variations $\delta\tau_{ik}$ satisfy the homogeneous equations of equilibrium and the statical boundary-conditions in the sense of (4.6) (*i.e.* $\delta\mathbf{T} \in \mathscr{H}_2$).

In case, that $\mathbf{T}(\bar{\mathbf{u}}) \in \mathbf{W}$, the principle holds in the form (4.12) with side conditions (4.13), (4.14), the integrals on Γ_n and Γ_t being omitted.

5. CASES OF "FREE BODIES"

In the present section we draw attention to such cases, for which some of assumptions of the introduction fail to hold, namely when all $\mathscr{A}_n$, $\mathscr{A}_t$, Γ_n, Γ_t are vacuous or the conditions $\bar{\mathbf{u}} \equiv \mathbf{0}$, $v_n = 0$ on Γ_n, $\mathbf{v}_t = 0$ on Γ_t do not eliminate the rigid body motions. Let us consider several important cases of that type:

1. Let the surface tractions be prescribed on the whole boundary, i.e. $\Gamma = \Gamma_P = \mathscr{B}_n = \mathscr{B}_t$, $\mathscr{A}_n = \mathscr{A}_t = \Gamma_n = \Gamma_t = \emptyset$. Then the necessary conditions of static equilibrium are

(5.1) $$\int_\Omega \mathbf{K} \, \mathrm{d}X + \int_\Gamma \mathbf{P} \, \mathrm{d}S = \mathbf{0} \,,$$

(5.2) $$\int_\Omega \mathbf{r} \times \mathbf{K} \, \mathrm{d}X + \int_\Gamma \mathbf{r} \times \mathbf{P} \, \mathrm{d}S = \mathbf{0} \,.$$ [4]

In order to guarantee the uniqueness of the solution $\mathring{\mathbf{u}}$, we choose for example the following complementary conditions

(5.3) $$\int_\Omega \mathbf{u} \, \mathrm{d}X = \mathbf{0} \,, \quad \int_\Omega \operatorname{rot} \mathring{\mathbf{u}} \, \mathrm{d}X = \mathbf{0} \,.$$

Note that the necessary equilibrium conditions (5.1), (5.2) follow also from the principle of virtual work (1.1). Indeed, inserting $u_j = \delta_{ji}$, ($\delta_{ij} = 1$ for $i = j$, $\delta_{ij} = 0$ for $i \neq j$), we obtain (5.1) and inserting three admissible small rotations

$$\mathbf{u}^{(l)} = \mathbf{b}^{(l)} \times \mathbf{r} \,, \quad (l = 1, 2, 3)$$

where $b_j^{(l)} = \delta_{lj}$, we obtain (5.2).

[4]) $\mathbf{r}$ denotes the radius-vector, $\times$ the vector product.

2. Let $\Gamma = \mathscr{A}_n \cup \Gamma_P \cup \mathscr{B}_t$, $\mathscr{A}_t = \Gamma_n = \Gamma_t = \emptyset$, $\mathscr{B}_n \subset \Gamma_P$, and let $\mathscr{A}_n$ be such that $\bar{\boldsymbol{u}} = \boldsymbol{0}$ does not exclude the possibility of rigid body motions. Thus for example let

a) $\mathscr{A}_n$ consist of surfaces of rotation having a common axis of rotation x_3. Then the external forces must satisfy the equation of moments

(5.4)
$$\int_\Omega (x_1 K_2 - x_2 K_1)\, \mathrm{d}X + \int_{\Gamma_P} (x_1 P_2 - x_2 P_1)\, \mathrm{d}S + \int_{\mathscr{B}_t \dot{-} \Gamma_P} (x_1(\mathbf{P}_t)_2 - x_2(\mathbf{P}_t)_1\, \mathrm{d}S = 0$$

and we choose the complementary condition

$$\int_\Omega (u_{2,1} - u_{1,2})\, \mathrm{d}X = 0\,. \tag{5.5}$$

b) Let $\mathscr{A}_n$ be composed of portions of cylindrical surfaces parallel, for example, with the axis x_1. The external forces must satisfy the equilibrium condition in the direction of x_1:

$$\int_\Omega K_1\, \mathrm{d}X + \int_{\Gamma_P} P_1\, \mathrm{d}S + \int_{\mathscr{B}_t \dot{-} \Gamma_P} (\mathbf{P}_t)_1\, \mathrm{d}S = 0$$

and the complementary condition may be chosen in the form

$$\int_\Omega u_1\, \mathrm{d}X = 0\,. \tag{5.6}$$

c) Let $\mathscr{A}_n$ reduce to portions of planes, which are parallel, for example, with the plane $x_3 = 0$. The necessary conditions of equilibrium are (5.4) and

$$\int_\Omega K_i\, \mathrm{d}X + \int_{\Gamma_P} P_i\, \mathrm{d}S + \int_{\mathscr{B}_t \dot{-} \Gamma_P} (\mathbf{P}_t)_i\, \mathrm{d}S = 0 \quad (i = 1, 2)\,.$$

The complementary condition may be chosen in the form (5.5) and

$$\int_\Omega u_i\, \mathrm{d}X = 0 \quad (i = 1, 2)\,.$$

d) Let $\mathscr{A}_n$ consist of portions of concentric spherical surfaces. The necessary condition of equilibrium is

$$\int_\Omega (\mathbf{r} \times \mathbf{K})\, \mathrm{d}X + \int_{\Gamma_P} (\mathbf{r} \times \mathbf{P})\, \mathrm{d}S + \int_{\mathscr{B}_t \dot{-} \Gamma_P} (\mathbf{r} \times \mathbf{P}_t)\, \mathrm{d}S = \boldsymbol{0}$$

and the complementary condition may be chosen in the form

$$\int_\Omega \operatorname{rot} \boldsymbol{u}\, \mathrm{d}X = \boldsymbol{0}\,.$$

e) Let $\mathscr{A}_n$ be composed of portions of the helical surface

$$x_1 = \varrho \cos \omega \,, \quad x_2 = \varrho \sin \omega \,, \quad x_3 = f(\varrho) + k\omega \,.$$

The necessary condition of equilibrium is

$$\int_\Omega (x_1 K_2 - x_2 K_1)\, \mathrm{d}X + \int_{\Gamma_P} (x_1 P_2 - x_2 P_1)\, \mathrm{d}S + \int_{\mathscr{B}_t \dot{-} \Gamma_P} (x_1 (P_t)_2 -$$

$$- x_2 (P_t)_1)\, \mathrm{d}S + h \left(\int_\Omega K_3\, \mathrm{d}X + \int_{\Gamma_P} P_3\, \mathrm{d}S + \int_{\mathscr{B}_t \dot{-} \Gamma_P} (P_t)_3\, \mathrm{d}S \right) = 0$$

and as the complementary condition (5.5) may be imposed.

In all cases mentioned above the inequality of Korn (0.8) holds on the linear manifold $\boldsymbol{M}$ of vector-functions, continuously differentiable on $\bar{\Omega}$ and satisfying both the homogeneous boundary-conditions $u_n = 0$ on $\mathscr{A}_n$ and the complementary conditions (see e.g. [3] § 43 or [12] for the proof of this assertion).

3. Let $\Gamma_n \cup \Gamma_t$ be non-vacuous, $\mathscr{A}_t = \emptyset$ and $\mathscr{A}_n = \emptyset$ vacuous or let $\mathscr{A}_n$ be of some of the types 2a–2e discussed above. In these cases we have equilibrium conditions, relating the external forces to the displacements on $\Gamma_n \cup \Gamma_t$. The appropriate choice of complementary conditions is shown in [12].

6. THE GENERALIZED VARIATIONAL PRINCIPLES

Using the method suggested in [11], the generalized variational principle of Hu-Hai-Chang and Washizu may be derived from the principle of minimum potential energy (3.3). Thus the condition

$$\delta\, \mathscr{J}(u_i, \varepsilon_{ik}, \tau_{ik}) = 0 \,,$$

where

$$(6.1) \quad \mathscr{J}(u_i, \varepsilon_{ik}, \tau_{ik}) \equiv \int_\Omega \{\tfrac{1}{2} c_{iklm} \varepsilon_{ik} \varepsilon_{lm} - K_i u_i - \tau_{ik} \varepsilon_{ik} + \tfrac{1}{2}(u_{i,k} + u_{k,i})\, \tau_{ik}\}\, \mathrm{d}X +$$

$$+ \overline{\int_{\mathscr{A}_n} T_n (\bar{u}_n - u_n)\, \mathrm{d}S} - \overline{\int_{\mathscr{D}_n} P_n u_n\, \mathrm{d}S} + \overline{\int_{\Gamma_n} \frac{u_n}{B_n} (\tfrac{1}{2} u_n - C_n)\, \mathrm{d}S}$$

implies both the constitutive equations of the elastostatics inside the body (strain-displacement and stress-strain relations and equilibrium equations) and all the boundary conditions (0.5) as Euler's equations and natural boundary conditions respectively.

Similarly, on the base of the principle of minimum complementary energy, the generalized variational principle of Hellinger and Reissner may be derived. This principle asserts: *From the condition*

$$\delta \mathscr{R}(u_i, \tau_{ik}) = 0 \,,$$

where

$$(6.2) \qquad \mathscr{R}(u_i, \tau_{ik}) \equiv \int_\Omega \{-\tfrac{1}{2}a_{iklm}\tau_{ik}\tau_{lm} + \tfrac{1}{2}(u_{i,k} + u_{k,i})\,\tau_{ik} - K_i u_i\}\,\mathrm{d}X +$$

$$+ \int_{\mathscr{A}_n} T_n(\bar{u}_n - u_n)\,\mathrm{d}S - \int_{\mathscr{D}_n} P_n u_n\,\mathrm{d}S + \int_{\Gamma_n} T_n(C_n - \tfrac{1}{2}B_n T_n - u_n)\,\mathrm{d}S$$

the relations

$$a_{iklm}\tau_{lm} = \tfrac{1}{2}(u_{i,k} + u_{k,i})$$

and equations of equilibrium inside Ω and the boundary-conditions (0.5) *follow as Euler's conditions and natural boundary conditions respectively.*

From the foregoing two principles a group of special variational theorems may be derived choosing variously some side conditions among the constitutive relations (see e.g. [11]). Here we present only the Reissner's theorem "for boundary conditions":

From $\delta\mathscr{R}'(u_i) = 0$, where

$$(6.3) \qquad \mathscr{R}'(u_i) \equiv -\tfrac{1}{2}\int_\Omega K_i u_i\,\mathrm{d}X + \overline{\int_{\mathscr{A}_n} T_n(\mathbf{u})\,(\bar{u}_n - \tfrac{1}{2}u_n)\,\mathrm{d}S} +$$

$$+ \overline{\int_{\mathscr{D}_n} u_n(\tfrac{1}{2}T_n(\mathbf{u}) - P_n)\,\mathrm{d}S} + \overline{\int_{\Gamma_n} T_n(\mathbf{u})\,(C_n - \tfrac{1}{2}B_n T_n(\mathbf{u}) - \tfrac{1}{2}u_n)\,\mathrm{d}S}\,,$$

$T_n(\mathbf{u})$, $\mathbf{T}_t(\mathbf{u})$ *are defined through* (0.1), (0.2) *and* (0.6) *by means of* $\mathbf{u}$ *and* $\tau_{ik}(\mathbf{u})$ *satisfy in Ω also the equations of equilibrium, the boundary-conditions* (0.5) *follow as natural conditions.*

Remark 1. Inserting the stress-strain relations (0.2) into the functional (6.1) we obtain a functional equivalent to the functional (6.2) (in the sense of the principle $\delta\mathscr{R} = 0$) and having the integrals

$$\overline{\int_{\Gamma_n} \frac{u_n}{B_n}(\tfrac{1}{2}u_n - C_n)\,\mathrm{d}S}$$

instead of the last two integrals (over Γ_n and Γ_t) in (6.2). This variant may be obtained also from the Reissner's principle in [1], if we choose the potentials there as follows:

$$(6.4) \qquad \overline{\psi_s^{(n)}(u_n)} = \overline{-\frac{C_n}{B_n}u_n + \frac{A_n}{2B_n}u_n^2} \quad \text{on} \quad \mathscr{B}_n(\text{or } \mathscr{B}_t)\,,$$

$$\overline{\psi_d^{(n)}(T_n)} = \overline{-T_n\bar{u}_n} \quad \text{on} \quad \mathscr{A}_n(\text{or } \mathscr{A}_t)\,.$$

To the latter variant the functional $\mathscr{L}(\mathbf{u})$ corresponds, having been derived on the assumption that $\mathbf{u}$ satisfies only "pure geometric" conditions on $\mathscr{A}_n \cup \mathscr{A}_t$.

The variant (6.2) follows, if we choose in [1]

$$(6.5) \qquad \overline{\psi_s^{(n)}(u_n)} = -\overline{\frac{C_n}{B_n}} u_n \quad \text{on} \quad \mathscr{D}_n(\text{or } \mathscr{D}_t),$$

$$\overline{\psi_d^{(n)}(T_n)} = \overline{-C_n T_n + \tfrac{1}{2} B_n T_n^2} \quad \text{on} \quad \mathscr{A}_n \cup \Gamma_n(\text{or } \mathscr{A}_t \cup \Gamma_t).$$

To the variant (6.2) the functional $\mathscr{S}(\boldsymbol{T})$ corresponds, having been derived on the assumption that $\boldsymbol{T}$ satisfies only "pure statical" conditions on $\mathscr{D}_n$, $\mathscr{D}_t$.

It is easy to verify that both the variants (6.4) and (6.5) of potentials imply the same boundary conditions (0.5) by means of the relations (see [1])

$$\overline{T_n + \partial \psi_s^{(n)}/\partial u_n} = 0, \quad \overline{u_n + \partial \psi_d^{(n)}/\partial T_n} = 0.$$

Analogically in (6.3) we may replace the last two integrals by

$$-\int_{\Gamma_n} \overline{\frac{u_n}{B_n}(C_n - \tfrac{1}{2} B_n T_n(\boldsymbol{u}) - \tfrac{1}{2} u_n)\, \mathrm{d}S},$$

which corresponds to the theorem "for boundary conditions" in [1] when the potentials are chosen according to (6.4).

Remark 2. The functional $\mathscr{S}(\boldsymbol{T})$ of the complementary energy follows from (6.2) by inserting the conditions of equilibrium and the static boundary-conditions. If we use the other variant of $\mathscr{R}$ (according to Remark 1), we should obtain a functional, depending not only on the stress field $\boldsymbol{T}$ but also on the displacement field $\boldsymbol{u}$.

7. ESTIMATES OF ERRORS OF THE APPROXIMATE SOLUTIONS, OBTAINED FROM THE PRINCIPLES OF MINIMUM POTENTIAL AND MINIMUM COMPLEMENTARY ENERGY

I. As in section 4 let us consider at first the problems with non-vacuous part $\Gamma_n \cup \Gamma_t$ of the boundary. We can show, that if the solution $\mathring{\boldsymbol{u}} \in \boldsymbol{U}$ and the associated $\boldsymbol{T}_0 \equiv \equiv \boldsymbol{T}(\mathring{\boldsymbol{u}}) \in \boldsymbol{W}$, we have

$$(7.1) \qquad -\mathscr{S}(\boldsymbol{T}_0) = \mathscr{L}(\mathring{\boldsymbol{u}}) + \Delta,$$

where

$$\Delta = \int_{\Gamma_n} \overline{C_n^2/2B_n \, \mathrm{d}S}.$$

Indeed, according to the definition of $\boldsymbol{U}$

$$(7.2) \qquad \mathring{u}_n + B_n T_{0n} = C_n \quad \text{on} \quad \Gamma_n,$$

$$\mathring{\boldsymbol{u}}_t + B_t \boldsymbol{T}_{0t} = \boldsymbol{C}_t \quad \text{on} \quad \Gamma_t.$$

Furthermore it holds

$$-\mathscr{S}(\mathbf{T}_0) = -\tfrac{1}{2}\int_\Omega \mathring{u}_{i,k}\mathring{\tau}_{ik}\,\mathrm{d}X + \overline{\int_{\mathscr{A}_n} T_{0n}\bar{u}_n\,\mathrm{d}S} + \overline{\int_{\Gamma_n} (T_{0n}C_n - \tfrac{1}{2}B_nT_{0n}^2)\,\mathrm{d}S}\,,$$

$$\mathscr{L}(\mathring{\mathbf{u}}) = \int_\Omega (\tfrac{1}{2}\mathring{u}_{i,k}\mathring{\tau}_{ik} - K_i\mathring{u}_i)\,\mathrm{d}X - \overline{\int_{\mathscr{D}_n} P_n\mathring{u}_n\,\mathrm{d}S} + \overline{\int_{\Gamma_n}\left(\frac{\mathring{u}_n^2}{2B_n} - P_n\mathring{u}_n\right)\mathrm{d}S} =$$

$$= -\tfrac{1}{2}\int_\Omega \mathring{u}_{i,k}\mathring{\tau}_{ik}\,\mathrm{d}X + \overline{\int_{\mathscr{A}_n} T_{0n}\bar{u}_n\,\mathrm{d}S} + \overline{\int_{\Gamma_n}\left(\frac{\mathring{u}_n^2}{2B_n} - P_n\mathring{u}_n + T_{0n}\mathring{u}_n\right)\mathrm{d}S}\,.$$

Consequently

$$\mathscr{L}(\mathring{\mathbf{u}}) + \mathscr{S}(\mathbf{T}_0) = \overline{\int_{\Gamma_n}\left(\frac{\mathring{u}_n^2}{2B_n} - \frac{C_n}{B_n}\mathring{u}_n + T_{0n}\mathring{u}_n\right)\mathrm{d}S} - \overline{\int_{\Gamma_n}(T_{0n}C_n - \tfrac{1}{2}B_nT_{0n}^2)\,\mathrm{d}S}\,.$$

Inserting

$$\overline{T_{0n} = (C_n - \mathring{u}_n)/B_n}$$

according to (7.2) into the two integrals, we obtain

$$\mathscr{L}(\mathring{\mathbf{u}}) + \mathscr{S}(\mathbf{T}_0) = \overline{\int_{\Gamma_n} -\frac{\mathring{u}_n^2}{2B_n}\,\mathrm{d}S} - \overline{\int_{\Gamma_n}\frac{1}{2B_n}(C_n^2 - \mathring{u}_n^2)\,\mathrm{d}S} = -\overline{\int_{\Gamma_n}\frac{C_n^2}{2B_n}\,\mathrm{d}S}\,,$$

which is precisely (7.1). Note that

$$\mathscr{L}(\mathring{\mathbf{u}}) = -\tfrac{1}{2}\int_\Omega c_{iklm}\mathring{u}_{i,k}\mathring{u}_{l,m}\,\mathrm{d}X - \tfrac{1}{2}\overline{\int_{\Gamma_n}\frac{\mathring{u}_n^2}{B_n}\,\mathrm{d}S} = -\tfrac{1}{2}\langle\mathring{\mathbf{u}}, \mathring{\mathbf{u}}\rangle\,.$$

Next let $\mathbf{u}^{(n)}$ be a term of a sequence, minimizing the functional $\mathscr{L}(\mathbf{u})$ and let $\mathbf{T}^{(m)}$ be a term of a sequence, minimizing the functional $\mathscr{S}(\mathbf{T})$.

The error $\mathscr{E}(\mathbf{u}^{(n)})$ of the approximation $\mathbf{u}^{(n)}$ may be measured by the difference

$$\mathscr{E}(\mathbf{u}^{(n)}) = \mathscr{L}(\mathbf{u}^{(n)}) - \mathscr{L}(\mathring{\mathbf{u}})\,.$$

Then it holds

$$-\mathscr{S}(\mathbf{T}^{(m)}) \leqq -\mathscr{S}(\mathbf{T}_0) = \mathscr{L}(\mathring{\mathbf{u}}) + \Delta\,,$$

$$\mathscr{E}(\mathbf{u}^{(n)}) \leqq \mathscr{L}(\mathbf{u}^{(n)}) + \mathscr{S}(\mathbf{T}^{(m)}) + \Delta \tag{7.3}$$

by virtue of (7.1) and the principle of minimum complementary energy. Similarly, the error $\mathscr{F}(\mathbf{T}^{(m)})$ of the approximation $\mathbf{T}^{(m)}$ may be measured by the difference $\mathscr{F}(\mathbf{T}^{(m)}) = \mathscr{S}(\mathbf{T}^{(m)}) - \mathscr{S}(\mathbf{T}_0)$. Then by virtue of (7.1) and the principle of minimum potential energy it holds

$$-\mathscr{S}(\mathbf{T}_0) \leqq \mathscr{L}(\mathbf{u}^{(n)}) + \Delta\,,$$

$$\mathscr{F}(\mathbf{T}^{(m)}) \leqq \mathscr{L}(\mathbf{u}^{(n)}) + \mathscr{S}(\mathbf{T}^{(m)}) + \Delta\,. \tag{7.4}$$

II. Next let $\Gamma_n \cup \Gamma_t = \emptyset$ as in the section 4/***II.*** The estimates of errors of $\boldsymbol{u}^{(n)}$ and $\boldsymbol{T}^{(m)}$ may be derived again by the method of Hilbert space. According to (3.2) we have (for $\boldsymbol{v} = \boldsymbol{w}^{(n)} = \boldsymbol{u}^{(n)} - \bar{\boldsymbol{u}} \in \boldsymbol{H}_M$)

$$(7.5) \qquad |\boldsymbol{u}^{(n)} - \mathring{\boldsymbol{u}}|_{H_M}^2 = |\boldsymbol{w}^{(n)} - \boldsymbol{w}|_{H_M}^2 = \Phi(\boldsymbol{w}^{(n)}) + |\boldsymbol{w}|_{H_M}^2 .$$

Comparison of (2.3) with (4.5) implies

$$(7.6) \qquad |\boldsymbol{w}|_{H_M}^2 = |\boldsymbol{T}(\boldsymbol{w})|_{\mathscr{H}}^2 .$$

When $\boldsymbol{T}^{(m)}$ meets (4.8), according to (4.9) we obtain

$$(7.7) \qquad |\boldsymbol{u}^{(n)} - \mathring{\boldsymbol{u}}|_{H_M}^2 \leqq \Phi(\boldsymbol{w}^{(n)}) + |\boldsymbol{T}^{(m)} - \boldsymbol{T}(\bar{\boldsymbol{u}})|_{\mathscr{H}}^2 .$$

From the formulas, following (3.2), we deduce

$$\Phi(\boldsymbol{w}^{(n)}) \equiv 2\mathscr{L}(\boldsymbol{u}^{(n)}) - \langle \bar{\boldsymbol{u}}, \bar{\boldsymbol{u}} \rangle + 2(\boldsymbol{K}, \bar{\boldsymbol{u}}) + 2\overline{(P_n, \bar{u}_n)} .$$

Similarly, from the formulas, following (4.9), we deduce

$$|\boldsymbol{T}^{(m)} - \boldsymbol{T}(\bar{\boldsymbol{u}})|_{\mathscr{H}}^2 = 2\tilde{\mathscr{S}}(\boldsymbol{T}^{(m)}) + |\boldsymbol{T}(\bar{\boldsymbol{u}})|_{\mathscr{H}}^2 .$$

As in the case under consideration

$$|\boldsymbol{T}(\bar{\boldsymbol{u}})|_{\mathscr{H}}^2 = \langle \bar{\boldsymbol{u}}, \bar{\boldsymbol{u}} \rangle ,$$

we may write finally the estimate

$$(7.8) \qquad |\boldsymbol{u}^{(n)} - \mathring{\boldsymbol{u}}|_{H_M}^2 \leqq 2[\mathscr{L}(\boldsymbol{u}^{(n)}) + \tilde{\mathscr{S}}(\boldsymbol{T}^{(m)}) + (\boldsymbol{K}, \bar{\boldsymbol{u}}) + \overline{(P_n, \bar{u}_n)}] ,$$

where

$$(\boldsymbol{K}, \bar{\boldsymbol{u}}) = \int_\Omega K_i \bar{u}_i \, dX , \quad \overline{(P_n, \bar{u}_n)} = \int_{\mathscr{B}_n} P_n \bar{u}_n \, dS + \int_{\mathscr{B}_t} \boldsymbol{P}_t \, . \, \bar{\boldsymbol{u}}_t \, \boldsymbol{dS} .$$

Remark. If moreover $\boldsymbol{T}^{(m)} \in \boldsymbol{W}$, then with respect to (4.11)

$$|\boldsymbol{u}^{(n)} - \mathring{\boldsymbol{u}}|_{H_M}^2 \leqq 2[\mathscr{L}(\boldsymbol{u}^{(n)} + \mathscr{S}(\boldsymbol{T}^{(m)})] .$$

Thus (7.8) is an estimate precisely twice as high as $\mathscr{E}(\boldsymbol{u}^{(n)})$ of the part 7/***I.***

For $|\boldsymbol{T}^{(m)} - \boldsymbol{T}_0|_{\mathscr{H}}$ we have according to (4.9)

$$|\boldsymbol{T}^{(m)} - \boldsymbol{T}_0|_{\mathscr{H}}^2 = |\boldsymbol{T}^{(m)} - \boldsymbol{T}(\bar{\boldsymbol{u}})|_{\mathscr{H}}^2 - |\boldsymbol{T}(\boldsymbol{w})|_{\mathscr{H}}^2 .$$

By virtue of (7.6) and (3.2) it holds

$$- |\boldsymbol{T}(\boldsymbol{w})|_{\mathscr{H}}^2 = - |\boldsymbol{w}|_{H_M}^2 \leqq \Phi(\boldsymbol{w}^{(n)}) .$$

Thus we obtain the estimate

$$|\boldsymbol{T}^{(m)} - \boldsymbol{T}_0|_{\mathscr{H}}^2 \leqq \Phi(\boldsymbol{w}^{(n)}) + |\boldsymbol{T}^{(m)} - \boldsymbol{T}(\bar{\boldsymbol{u}})|_{\mathscr{H}}^2$$

with the same right-hand side, as in (7.7) and consequently also the same as that in (7.8).

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VARIAČNÍ PRINCIPY LINEÁRNÍ TEORIE PRUŽNOSTI PRO OBECNÉ OKRAJOVÉ PODMÍNKY

IVAN HLAVÁČEK

Souhrn

V článku jsou probrány klasické i neklasické variační principy lineární statické teorie pružnosti pro obecný kombinovaný okrajový problém ve třech dimenzích. Na částech povrchu tělesa jsou uvažovány podmínky pro daná posunutí, dané povrchové zatížení, kontaktní podmínky a pružné podepření, a to odděleně ve směrech normály a tečné roviny k povrchu tělesa. Ukazuje se na souvislost s definicí slabého (zobecněného) řešení, přičemž se podstatně využívá metod Hilbertova prostoru. Také k odvození principu minima doplňkové energie je užita metoda ortogonálních projekcí v jistém Hilbertově prostoru tenzorových funkcí napětí. V tomto smyslu jde o rozšíření idejí z knihy С. Г. Михлина [3] a [4].

Резюме

ВАРИАЦИОННЫЕ ПРИНЦИПЫ ЛИНЕЙНОЙ ТЕОРИИ УПРУГОСТИ ДЛЯ ОБЩИХ КРАЕВЫХ УСЛОВИЙ

ИВАН ГЛАВАЧЕК (IVAN HLAVÁČEK)

Рассматривается смешанная краевая задача линейной теории упругости, в которой заданы перемещения и поверхностная нагрузка отдельно в направлениях нормальном и тангенциальном к граничной поверхности, включая условия контактной задачи и упругие опоры.

Классические принципы минимума потенциальной и минимума комплементарной энергии установлены при помощи теории гильбертова пространства. Показаны их связи с принципами виртуальных работ, виртуальных перемещений, виртуальных изменений напряженного состояния и с определением слабого решения из теории эллиптических систем дифференциальных уравнений. Приведены тоже обобщенные принципы Ху-Хай-Чанга и Вашицу [8] и Рейсснера и Хелингера [1]. Исследуется вопрос о двусторонних оценках погрешностей приближенных решений. Случаи „свободных тел", когда краевые условия для перемещений не исключают возможность жестких смещений тела, тоже подвергаются дискуссии.

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