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ON NECESSARY CONDITIONS FOR A CLASS OF SYSTEMS OF LINEAR INEQUALITIES

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In this note a class of convex polyhedral sets of functions is studied. The set of considered class is non-empty if it satisfies certain conditions (Theorem 1). Using Theorem 1 in the case of multi-index transportation problem ([1]) we obtain necessary conditions for the existence of a feasible solution to this problem. It is shown that these conditions imply the conditions stated in [1] and [2].

1° Let $X \neq 0$ be a finite set and let $\alpha(A)$, $\beta(A)$ be real functions defined on the family of all subsets of X and suppose

$$\begin{aligned} -\infty \leqq \alpha(A) < +\infty\,, &\quad -\infty < \beta(A) \leqq +\infty \\ \alpha(\emptyset) = 0\,, &\quad \beta(\emptyset) = 0\,. \end{aligned}$$

The problem is to find such an additive[1]) function φ that the inequalities

$$\alpha(A) \leqq \varphi(A) \leqq \beta(A) \tag{1}$$

hold for any $A \subset X$.

We find necessary conditions for the existence of a solution to this problem. For $A \subset X$ we define

$$\begin{aligned} & m^{(0)}(A) = \beta(A)\,, \quad M^{(0)}(A) = \alpha(A)\,, \\ & m^{(r+1)}(A) = \min\,[n^{(r+1)}(A), p^{(r+1)}(A)]\,, \\ & M^{(r+1)}(A) = \max\,[N^{(r+1)}(A), P^{(r+1)}(A)]\,, \quad r = 0, 1, \ldots \end{aligned} \tag{2}$$

where

$$\begin{aligned} n^{(r+1)}(A) &= \min\,\{m^{(r)}(A') + m^{(r)}(A'') \mid A' \cup A'' = A,\, A' \cap A'' = \emptyset\}\,, \\ p^{(r+1)}(A) &= \min\,\{m^{(r)}(A \cup A_1) - M^{(r)}(A_1) \mid A_1 \subset X - A\}\,, \\ N^{(r+1)}(A) &= \max\,\{M^{(r)}(A') + M^{(r)}(A'') \mid A' \cup A'' = A,\, A' \cap A'' = \emptyset\}\,, \\ M^{(r+1)}(A) &= \max\,\{M^{(r)}(A \cup A_1) - m^{(r)}(A_1) \mid A_1 \subset X - A\}\,. \end{aligned}$$

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[1]) A finite real function $\varphi(A)$ ($A \subset X$) is said to be an additive function if

$$A_1 \subset X\,, \quad A_2 \subset X\,, \quad A_1 \cap A_2 = \emptyset \Rightarrow \varphi(A_1 \cup A_2) = \varphi(A_1) + \varphi(A_2)$$

Theorem 1. *If there exists such an additive function that the conditions* (1) *are satisfied then for each* A, $A \subset X$ *there exist the limits*

$$\hat{\beta}(A) = \lim_{r \to \infty} m^{(r)}(A), \quad \hat{\alpha}(A) = \lim_{r \to \infty} M^{(r)}(A)$$

and

$$\hat{\alpha}(A) \leqq \hat{\beta}(A). \tag{3}$$

At the same time the functions $\hat{\alpha}(A)$, $\hat{\beta}(A)$ *satisfy the inequalities*

$$(A_1 \cap A_2 = \emptyset) \Rightarrow [\hat{\alpha}(A_1) + \hat{\alpha}(A_2) \leqq \hat{\alpha}(A_1 \cup A_2) \leqq \hat{\alpha}(A_1) + \hat{\beta}(A_2) \leqq \\ \leqq \hat{\beta}(A_1 \cup A_2) \leqq \hat{\beta}(A_1) + \hat{\beta}(A_2)]. \tag{4}$$

Proof. Let φ be an additive function satisfying (1). It is not difficult to verify by induction with respect to r that

$$M^{(0)}(A) \leqq M^{(1)}(A) \leqq \ldots \leqq \varphi(A) \leqq \ldots \leqq m^{(1)}(A) \leqq m^{(0)}(A)$$

for any subset A of X. The first half of the theorem is evident now and relations (4) follow from the definitions of the functions $m^{(r)}(A)$, $M^{(r)}(A)$.

The functions $\hat{\alpha}(A)$, $\hat{\beta}(A)$ have certain extremal properties with respect to inequalities (4).

Theorem 2. *Let* $\tilde{\alpha}(A)$, $\tilde{\beta}(A)$ *satisfy inequalities* (4) *and moreover*

$$\alpha(A) \leqq \tilde{\alpha}(A), \quad \tilde{\beta}(A) \leqq \beta(A).$$

Then

$$\alpha(A) \leqq \hat{\alpha}(A) \leqq \tilde{\alpha}(A) \leqq \tilde{\beta}(A) \leqq \hat{\beta}(A) \leqq \beta(A).$$

The proof is similar to the proof of Theorem 1.

Remark. The supposition concerning finiteness of X is clearly not essential and we can consider any algebra $\boldsymbol{A}$ of sets, $\boldsymbol{A} \subset \exp X$, instead of $\exp X$.

2° Now we apply Theorem 1 to the multi-index transportation problem of the following type: To minimize the function f of φ_{ijk}

$$f(\varphi_{ijk}) = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} C_{ijk} \varphi_{ijk}$$

subject to

$$\sum_{i=1}^{m} \varphi_{ijk} = A_{jk}, \quad \sum_{j=1}^{n} \varphi_{ijk} = B_{ik}, \quad \sum_{k=1}^{p} \varphi_{ijk} = C_{ij}, \quad \varphi_{ijk} \geqq 0 \tag{5}$$

where $A_{jk} \geqq 0$, $B_{ik} \geqq 0$, $C_{ij} \geqq 0$.

We define

$$M = \{1, 2, \ldots, m\}, \quad N = \{1, 2, \ldots, n\}, \quad P = \{1, 2, \ldots, p\},$$
$$X = M \times N \times P, \quad \varphi(R) = \sum_{(i,j,k)\in R} \varphi_{ijk}$$

where $R \subset X$. Clearly the constraints (5) can be interpreted as relations (1) for the additive function φ. Now we are going to show that the necessary conditions stated in [1] and [2] follow from Theorem 1.

The necessary conditions stated in [2] can be formulated (as it is shown in [3]) in the following way:

$$A(J, K) \leqq B(I, K) + C(\bar{I}, J) \tag{6}$$

where

$$A(J, K) = \sum_{j\in J} \sum_{k\in K} A_{jk}, \quad B(I, K) = \sum_{i\in I} \sum_{k\in K} B_{ik}, \quad C(I, J) = \sum_{i\in I} \sum_{j\in J} C_{ij},$$
$$I \subset M, \quad J \subset N, \quad K \subset P \quad \text{and} \quad \bar{I} = M - I.$$

It is evident that the conditions (6) occur in the iterative procedure (2) specified to (5).

Now we turn our attention to the conditions of Haley ([1]). These conditions can be formulated as follows:

$$\lim_{r\to\infty} M_{ijk}^{(r)} \leqq \lim_{r\to\infty} m_{ijk}^{(r)} \tag{7}$$

where

$$m_{ijk}^{(0)} = \min(A_{jk}, B_{ik}, C_{ij}), \quad M_{ijk}^{(0)} = 0,$$
$$m_{ijk}^{(r+1)} = \min\big(A_{jk} - \sum_{i', i' \neq i} M_{i'jk}^{(r)}, B_{ik} - \sum_{j', j' \neq j} M_{ij'k}^{(r)}, C_{ij} - \sum_{k', k' \neq k} M_{ijk'}^{(r)}\big),$$
$$M_{ijk}^{(r+1)} = \max\big(A_{jk} - \sum_{i', i' \neq i} m_{i'jk}^{(r)}, B_{ik} - \sum_{j', j' \neq j} m_{ij'k}^{(r)}, C_{ij} - \sum_{k', k' \neq k} m_{ijk'}^{(r)}\big).$$

To see that the conditions (7) are consequences of Theorem 1 we notice that

$$\hat{\beta}(A) = \lim_{r\to\infty} \overline{m}^{(r)}(A), \quad \hat{\alpha}(A) = \lim_{r\to\infty} \overline{M}^{(r)}(A)$$

where

$$\overline{m}^{(0)}(A) = \beta(A), \quad \overline{M}^{(0)}(A) = \alpha(A),$$
$$\overline{m}^{(r+1)}(A) = \min[\bar{n}^{(r+1)}(A), \bar{p}^{(r+1)}(A)],$$
$$\overline{M}^{(r+1)}(A) = \max[\overline{N}^{(r+1)}(A), \overline{P}^{(r+1)}(A)], \quad r = 0, 1, \ldots$$

where

$$\bar{n}^{(r+1)}(A) = \min\{\overline{m}^{(r)}(A_1) + \ldots + \overline{m}^{(r)}(A_s) \mid \bigcup_{\sigma=1}^{s} A_\sigma = A; \sigma \neq \sigma' \Rightarrow A_\sigma \cap A_{\sigma'} = \emptyset\},$$

$$\bar{p}^{(r+1)}(A) = \min \{\overline{m}^{(r)}(A \cup (\bigcup_{\sigma=1}^{s} B_\sigma)) - \overline{M}^{(r)}(B_1) - \ldots - \overline{M}^{(r)}(B_s) \mid B_\sigma \subset X - A;$$
$$\sigma \neq \sigma' \Rightarrow B_\sigma \cap B_{\sigma'} = \emptyset\},$$
$$\overline{N}^{(r+1)}(A) = \max \{\overline{M}^{(r)}(A_1) + \ldots + \overline{M}^{(r)}(A_s) \mid \bigcup_{\sigma=1}^{s} A_\sigma = A;$$
$$\sigma \neq \sigma' \Rightarrow A_\sigma \cap A_{\sigma'} = \emptyset\},$$
$$\bar{P}^{(r+1)}(A) = \max \{\overline{M}^{(r)}(A \cup (\bigcup_{\sigma=1}^{s} B_\sigma)) - \overline{m}^{(r)}(B_1) - \ldots - \overline{m}^{(r)}(B_s) \mid B_\sigma \subset X - A;$$
$$\sigma \neq \sigma' \Rightarrow B_\sigma \cap B_{\sigma'} = \emptyset\}.$$

Now it is evident that the conditions (7) follow from (3). On the contrary it follows from the example presented in [2] that the conditions (3) do not follow from (7).

Concluding remarks. 1. It seems that the necessary conditions (3) are sufficient neither in general case nor in the case of general multi-index transportation problem. On the other hand these conditions are sufficient e.g. in the case of three-index transportation problem with $p = 2$ ([3]). It would be interesting to find other cases in which these conditions are sufficient.

2. The sequences $\{m^{(r)}(A)\}_{r=1}^{\infty}$, $\{M^{(r)}(A)\}_{r=1}^{\infty}$ are stationary in the case of rational numbers $\alpha(A)$, $\beta(A)$. In the general case this question is open.

References

[1] *K. B. Haley:* The Multi-Index Problem, Opns. Res. *11*, 1963, p. 368.

[2] *J. Morávek, M. Vlach:* On the Necessary Conditions for the Existence of the Solution of *Multi-Index Transportation Problem, Opns. Res. 15* (1967).

[3] *K. B. Haley:* A Note to the paper by Moravek and Vlach, Opns. Res. *15* (1967), str. 545—546.

Výtah

O NUTNÝCH PODMÍNKÁCH ŘEŠITELNOSTI JEDNÉ TŘÍDY SOUSTAV LINEÁRNÍCH NEROVNOSTÍ

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V této poznámce se vyšetřuje jistá třída konvexních polyedrických množin funkcí definovaných lineárními nerovnostmi. K tomu, aby množina z uvažované třídy byla neprázdná, musí splňovat jisté nutné podmínky (věta 1). Použitím věty 1 v případě 3-indexového dopravního problému ([1]), dostáváme nutné podmínky existence přípustného řešení této úlohy. Ukazuje se, že z těchto podmínek již vyplývají podmínky uvedené v [1] a [2].

Резюме

О НЕОБХОДИМЫХ УСЛОВИЯХ РАЗРЕШИТЕЛЬНОСТИ ОДНОГО КЛАССА СИСТЕМ ЛИНЕЙНЫХ НЕРАВЕНСТВ

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В этой заметке рассматривается один класс выпуклых множеств функций определенных при помощи линейных неравенств. Для того, чтобы множество рассматриваемого класса было непустым, должно удовлетворять определенным условиям (Теорема 1). Применение теоремы 1 в случае трехиндексной транспортной задачи ([1]) ведет к необходимым условиям существования допустимого решения из которых уже вытекают условия приведенные в работах [1] и [2].

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