Aplikace matematiky

Rostislav Zezula

A sufficient condition for flattening of the thermal neutron flux and some related problems (in onedimensional geometries)

Aplikace matematiky, Vol. 14 (1969), No. 2, 134–145

Persistent URL: http://dml.cz/dmlcz/103216

A SUFFICIENT CONDITION FOR FLATTENING OF THE THERMAL NEUTRON FLUX AND SOME RELATED PROBLEMS (IN ONEDIMENSIONAL GEOMETRIES)

ROSTISLAV ZEZULA

(Received October 30, 1967)

Let us consider a reactor with reflector, the core of which is described by the two-groups equations (with the usual notation [1], [2], [3], [6])

$$(1)\qquad -\operatorname{div}(D \operatorname{grad} \Phi) + (\Sigma_u^a + \Sigma_M^a)\,\Phi = q \Leftrightarrow -D\,\Delta\Phi - \operatorname{grad} D \,.\, \operatorname{grad}\Phi + (\Sigma_u^a + \Sigma_M^a)\,\Phi = q$$

$$(1a)\qquad -\operatorname{div}(\tau \operatorname{grad} q) + q = k\,\Sigma_u^a\Phi \Leftrightarrow -\tau\,\Delta q - \operatorname{grad}\tau \,.\, \operatorname{grad} q + q = k\,\Sigma_U^a\Phi$$

By elimitation of the slowing-down density q we obtain from (1), (1a) the following equation for the thermal neutron flux (evidently equivalent to the system (1), (1a)):

$$(2)\qquad -\tau\,\Delta[-D\,\Delta\Phi - \operatorname{grad} D \,.\, \operatorname{grad}\Phi + (\Sigma_U^a + \Sigma_M^a)\,\Phi] - \operatorname{grad}\tau \,.\, \operatorname{grad}[-D\,\Delta\Phi - \operatorname{grad} D \,.\, \operatorname{grad}\Phi + (\Sigma_U^a + \Sigma_M^a)\,\Phi] + [-D\,\Delta\Phi - \operatorname{grad} D \,.\, \operatorname{grad}\Phi + (\Sigma_U^a + \Sigma_M^a)\,\Phi] = k\Sigma_U^a\Phi$$

Often we may assume that, for the diffusion coefficient D and the macroscopic absorption cross-section in the moderator Σ_M^a and the age τ (which are given functions), we have approximately

$$(3)\qquad D = \text{const.} \neq 0$$

$$(3a)\qquad \Sigma_M^a = \text{const.}\,; \quad \tau = \text{const}\,.$$

Therefore, it is useful to introduce the following notation for the relative fuel concentration M,

$$(4)\qquad M = \frac{\Sigma_U^a}{\Sigma_M^a} = \frac{\sigma_U^a}{\Sigma_M^a} N_U$$

(under the assumption that

(4a) $$\Sigma_M^a > 0)$$

so that for the neutron multiplication coefficient k we have the relation

(5) $$k = k(M)\,.$$

From the equation (2) there follows, under the assumption

(5a) $$\tau > 0\,,$$

the following relation between M and Φ (evidently equivalent to (2)):

(6) $$\Delta(D\,\Delta\Phi + \Delta(\mathrm{grad}\,D\,.\,\mathrm{grad}\,\Phi) - \Delta\Sigma_M^a[(M+1)\,\Phi] - \\ - \frac{D}{\tau}\,\Delta\Phi - \frac{1}{\tau}\,\mathrm{grad}\,D\,.\,\mathrm{grad}\,\Phi - \frac{\mathrm{grad}\,\tau}{\tau}\,.\,\mathrm{grad}\left[-D\,\Delta\Phi - \mathrm{grad}\,D\,.\right. \\ \left.\,.\,\mathrm{grad}\,\Phi + \Sigma_M^a(M+1)\,\Phi\right] + \frac{\Sigma_M^a}{\tau}\left[M(1-k)+1\right]\Phi = 0$$

which for the case of onedimensional core geometries (with the space coordinate x), can be transformed with the help of the well-known formulae in the following one:

(6a) $$\left\{D\,\Delta(\Delta\Phi) + \left[\Delta D + \frac{1}{\tau}\frac{\mathrm{d}\tau}{\mathrm{d}x}\frac{\mathrm{d}D}{\mathrm{d}x}\right]\Delta\Phi + \left[2\frac{\mathrm{d}D}{\mathrm{d}x} + \frac{D}{\tau}\frac{\mathrm{d}\tau}{\mathrm{d}x}\right]\frac{\mathrm{d}\Delta\Phi}{\mathrm{d}x} + \right. \\ + \Delta\left(\frac{\mathrm{d}D}{\mathrm{d}x}\frac{\mathrm{d}\Phi}{\mathrm{d}x}\right) - \left[(M+1)\,\Sigma_M^a + \frac{D}{\tau}\right]\Delta\Phi - 2\frac{\mathrm{d}}{\mathrm{d}x}\left[\Sigma_M^a(M+1)\right]\frac{\mathrm{d}\Phi}{\mathrm{d}x} - \\ \left. - \frac{1}{\tau}\frac{\mathrm{d}D}{\mathrm{d}x}\frac{\mathrm{d}\Phi}{\mathrm{d}x} - \frac{1}{\tau}\,\Sigma_M^a(M+1)\,.\,\frac{\mathrm{d}\tau}{\mathrm{d}x}\frac{\mathrm{d}\Phi}{\mathrm{d}x} + \frac{1}{\tau}\frac{\mathrm{d}\tau}{\mathrm{d}x}\left(\frac{\mathrm{d}D}{\mathrm{d}x}\frac{\mathrm{d}^2\Phi}{\mathrm{d}x^2} + \frac{\mathrm{d}\Phi}{\mathrm{d}x}\frac{\mathrm{d}^2D}{\mathrm{d}x^2}\right)\right\} + \\ + \frac{\Phi}{\tau}\left\{\Sigma_M^a[(M(1-k)+1] - \Sigma_M^a\,.\,\frac{\mathrm{d}\tau}{\mathrm{d}x}\frac{\mathrm{d}M}{\mathrm{d}x} - (M+1)\frac{\mathrm{d}\tau}{\mathrm{d}x}\frac{\mathrm{d}\Sigma_M^a}{\mathrm{d}x} - \right. \\ \left. - \tau\Delta[\Sigma_M^a(M+1)]\right\} = 0$$

From the equation (6a) there follows the validity of the implication

(7) $$\Phi = \Phi_0 = \mathrm{const.} \Rightarrow \Sigma_M^a[M(1-k(M))+1] - \Sigma_M^a\frac{\mathrm{d}\tau}{\mathrm{d}x}\frac{\mathrm{d}M}{\mathrm{d}x} - \\ - (M+1)\frac{\mathrm{d}\tau}{\mathrm{d}x}\frac{\mathrm{d}\Sigma_M^a}{\mathrm{d}x} - \tau\Delta[\Sigma_M^a(M+1)] = 0\,,$$

which gives the following necessary condition for the relative fuel concentration $M(x)$

(7a)
$$\Delta M + \left[\frac{2}{\Sigma_M^a}\frac{\mathrm{d}\Sigma_M^a}{\mathrm{d}x} + \frac{\mathrm{d}\tau}{\mathrm{d}x}\right]\frac{\mathrm{d}M}{\mathrm{d}x} + \left[\frac{k(M)-1}{\tau} + \frac{1}{\Sigma_M^a}\left(\frac{\mathrm{d}\tau}{\mathrm{d}x}\frac{\mathrm{d}\Sigma_M^a}{\mathrm{d}x} + \Delta\Sigma_M^a\right)\right] M = \left[\frac{1}{\tau} - \frac{1}{\Sigma_M^a}\left(\frac{\mathrm{d}\tau}{\mathrm{d}x}\frac{\mathrm{d}\Sigma_M^a}{\mathrm{d}x} + \Delta\Sigma_M^a\right)\right];$$

thus, we see (with respect to the spatial symmetry of the problem considered) that the Cauchy problem, given by the equation (7a) and the initial conditions

(8)
$$M'(0) = \left.\frac{\mathrm{d}M(x)}{\mathrm{d}x}\right|_{x=0} = 0\,; \quad M(0) = M_0\,; \quad (M_0 \geqq 0)$$

determines a onedimensional spatial distribution $M = M(x)$ of the relative fuel concentration, which is necessary for attaining the flattened thermal neutron flux $\Phi = \Phi_0 = \text{const.} \neq 0$ in the reactor core described by the eq. (1), (1a). If the assumption (3a) is fulfilled, then the equation (7a) for the fuel distribution $M(x)$ results into the well-known Goertzel's equation

(8a)
$$\Delta M + \frac{k(M)-1}{\tau} M = \frac{1}{\tau},$$

which together with the initial conditions (8) defines the Cauchy problem, giving the necessary condition for the fuel distribution in the reactor core with the flattened thermal neutron flux, considered in the paper [4].

Let us suppose now that for $M = M(x)$ the necessary condition for thermal neutron flux flattening (7a), (8) is fulfilled. Then it follows from eq. (6a) and from the spatial symmetry of the problem that the distribution of the thermal neutron flux Φ in reactor with fuel distribution $M = M(x)$ given by the relations (7a), (8) must necessarily fulfill the homogeneous equation (of third order in $\Phi' = \mathrm{d}\Phi/\mathrm{d}x$)

(9)
$$D\Delta(\Delta\Phi) + \left[3\frac{\mathrm{d}D}{\mathrm{d}x} + \frac{D}{\tau}\frac{\mathrm{d}\tau}{\mathrm{d}x}\right]\frac{\mathrm{d}\Delta\Phi}{\mathrm{d}x} + \left[\Delta D + \frac{1}{\tau}\frac{\mathrm{d}\tau}{\mathrm{d}x}\frac{\mathrm{d}D}{\mathrm{d}x} - \frac{D}{\tau} - (M+1)\,\Sigma_M^a\right]\Delta\Phi + \left[\frac{1}{\tau}\frac{\mathrm{d}\tau}{\mathrm{d}x}\frac{\mathrm{d}D}{\mathrm{d}x} + 2\frac{\mathrm{d}^2D}{\mathrm{d}x^2}\right]\frac{\mathrm{d}^2\Phi}{\mathrm{d}x^2} + \left[\left(\Delta\frac{\mathrm{d}D}{\mathrm{d}x} + \frac{1}{\tau}\frac{\mathrm{d}\tau}{\mathrm{d}x}\frac{\mathrm{d}^2D}{\mathrm{d}x^2} - \frac{1}{\tau}\frac{\mathrm{d}D}{\mathrm{d}x}\right) - \left(\frac{1}{\tau}\Sigma_M^a(M+1)\frac{\mathrm{d}\tau}{\mathrm{d}x} + 2(M+1)\frac{\mathrm{d}\Sigma_M^a}{\mathrm{d}x} + 2\,\Sigma_M^a\frac{\mathrm{d}M}{\mathrm{d}x}\right)\right]\frac{\mathrm{d}\Phi}{\mathrm{d}x} = 0,$$

which under the assumptions (3), (3a) assumes the simplified form

(9a)
$$D\Delta(\Delta\Phi) - \left[(M+1)\,\Sigma_M^a + \frac{D}{\tau}\right]\Delta\Phi - 2\,\Sigma_M^a\frac{\mathrm{d}M}{\mathrm{d}x}\frac{\mathrm{d}\Phi}{\mathrm{d}x} = 0,$$

and the boundary conditions

$$\text{(9b)} \qquad \Phi'(0) = \frac{d\Phi}{dx}\bigg|_{x=0} = 0\,; \quad \Phi'''(0) = \frac{d^3\Phi}{dx^3}\bigg|_{x=0} = 0\,; \quad \Phi'(b) = \frac{d\Phi}{dx}\bigg|_{x=b} = \Phi'_b$$

where $b \geqq b^{\min} > 0$ determines the boundary between the core and the reflector and Φ'_b is an arbitrary real parameter. Conversely, from the validity of the relations (7a), (8) and (9), (9b) there obviously follows the validity of the equation (6a).

Due to the fact that the solution $\Phi' = d\Phi/dx$ of the equation (9) is uniquely determined by the boundary conditions (9b), the sufficient condition for the thermal neutron flux flattening in the reactor core with the relative fuel concentration $M(x)$ given by (7a), (8) has the following form

$$\text{(10)} \qquad \Phi'(b) = \frac{d\Phi}{dx}\bigg|_{x=b} = \Phi'_b = 0\,.$$

From the homogeneity of the equation (9) it follows that we can set

$$\text{(11)} \qquad \Phi' = \frac{d\Phi}{dx} = \Phi'_b\psi \Rightarrow \psi(b) = \psi_b = 1\,; \quad \psi(0) = \psi_0 = 0\,;$$

$$\psi''(0) = \psi''_0 = 0\,,$$

where Φ'_b is an arbitrary norming constant, and the function ψ is uniquely determined by the relations (11), (9), (9b) and (7a), (8), so that it depends on the parameter $M(0) = M_0 \geqq 0$, i.e.

$$\text{(11a)} \qquad \psi = \psi(x; M_0)\,.$$

Since we have (for $\Phi_0 = \Phi(0) > 0$)

$$\text{(12)} \qquad \Phi(x) = \Phi_0 + \int_0^x \Phi'(x)\,dx = \Phi_0\left[1 + \frac{\Phi'_b}{\Phi_0}\int_0^x \Psi(x; M_0)\,dx\right],$$

we must also have

$$\text{(12a)} \qquad \Phi(b) = \Phi_0\left[1 + \omega\int_0^b \Psi(x; M_0)\,dx\right] = \Phi(b; \omega, M_0)$$

where

$$\text{(12b)} \qquad \omega = \frac{\Phi'_b}{\Phi_0}$$

is a real parameter replacing the parameter Φ'_b; thus we get the following sufficient condition for inducing the thermal neutron flux given by (11), (12a) in the reactor core,

(12c) $$\omega = \omega(b, M_0) = \frac{\Phi(b) - \Phi_0}{\Phi_0 \int_0^b \psi(x; M_0)\, dx} = \omega_0 = \text{const}$$

which for $\omega_0 = 0$ is evidently equivalent to the condition (10).

Let us denote now a the coordinate of the extrapolated outer boundary of the reactor. The shape of the slowing-down density q_R and of the thermal neutron flux Φ_R in the reflector, which fulfill the usual conditions on the outer boundary a,

(13) $$q_R(a) = 0\,; \quad \Phi_R(a) = 0\,, \quad (a \geqq a^{\min} > b > 0, [4])$$

are given (according to the considered geometry) by the formulae

(14) $$q_R(x; a) = A \,.\, y_1(x; a)$$

(14a) $$\Phi_R(x; a) = A \,.\, \mu_1 y_1(x; a) + B \,.\, y_2(x; a)$$

where the known functions $y_1(x; a)$, $y_2(x; a)$ (linear combinations of the fundamental solutions of the two-group equations for reflector with coefficients depending on a, which fulfill the boundary conditions (13)) depend on the geometry of the reflector and the constant μ_1 is a known function of the physical constants of the reflector. The free integration constants A, B can be determined from the usual continuity conditions of the thermal neutron flux and current on the boundary between core and reflector, i.e.

(15) $$\Phi(b) = \Phi_R(b; a)\,; \quad D\Phi'(b) = D_R \Phi'_R(b; a)\,;$$

thus it follows from (15), (as a consequence of the relations (11), (12a)) that

(16) $$A = A(\omega_0, M_0; b, a) = \Phi_0 \frac{(\alpha_1 \omega_0 + \alpha_2)}{\mu_1 W(b; a)}\,; \quad B = B(\omega_0, M_0; b, a) = -\Phi_0 \frac{(\beta_1 \omega_0 + \beta_2)}{W(b; a)}\,,$$

where

(16a) $$\alpha_1 = y'_2(b, a) \int_0^b \psi(x, M_0)\, dx - y_2(b, a) \frac{D}{D_R} = \alpha_1(b, M_0, a)\,;$$

$$\alpha_2 = y'_2(b, a)$$

(16b) $$\beta_1 = y'_1(b; a) \int_0^b \psi(x, M_0)\, dx - y_1(b; a) \frac{D}{D_R} = \beta_1(b, M_0, a)\,;$$

$$\beta_2 = y'_1(b, a)$$

(16c) $$W(b, a) = [y_1(b, a)\, y'_2(b, a) - y_2(b, a)\, y'_1(b, a)]$$

Consequently the integration constants A, B are linear functions of the parameter ω_0 but depend in a nonlinear manner on the parameter M_0 and on the values b, a.

For fulfilling the remaining two physically meaningful continuity conditions of the flux and current of fast neutrons on the boundary of the core and reflector we have

$$(17)\qquad \frac{1}{\xi\Sigma_s}\, q(b;\,\omega_0,\,M_0) = \frac{1}{(\xi\Sigma_s)_R}\, q_R(b;\,\omega_0,\,M_0,\,a)\,;$$
$$\tau q'(b;\,\omega_0,\,M_0) = \tau_R q'_R(b;\,\omega_0,\,M_0,\,a)$$

therefore we have (when a is given) two free parameters ω_0, M_0 at our disposal so that, under certain assumptions, we can fulfill these relations for an arbitrary coordinate b of the reactor core. Substituting the relations (1), (1a), (14), (14a), (16), (16a), (16b), (16c) in both to the conditions (17) and eliminating the parameter ω_0 from them we obtain an implicit function $f(b, M_0;\, a) = 0$ depending on the parameter a, i.e.

$$(18)\quad 0 = f(b, M_0;\, a) =$$
$$= \frac{\dfrac{\alpha_2}{\mu_1}\dfrac{y_1'(b;\,a)}{W(b;\,a)} - \dfrac{\tau}{\tau_R}\,\Sigma_M^a M'(b;\,M_0)}{\dfrac{\alpha_1}{\mu_1}\dfrac{y_1'(b,\,a)}{W(b;\,a)} - \dfrac{\tau}{\tau_R}\,h_1(b;\,M_0)} - \frac{\dfrac{\alpha_2}{\mu_1}\dfrac{y_1(b;\,a)}{W(b;\,a)} - \mu_2\,\Sigma_M^a[M(b,\,M) + 1]}{\dfrac{\alpha_1}{\mu_1}\dfrac{y_1(b;\,a)}{W(b;\,a)} - \mu_2\,h_2(b;\,M_0)}$$

where

$$(18a)\quad h_1(b;\,M_0) = \{\Sigma_M^a[M(x;\,M_0) + 1]\,\psi(x,\,M_0) + \Sigma_M^a M'(x;\,M_0)\,\Psi(x;\,M_0) - D[\Delta\Psi(x,\,M_0)]'\}\big|_{x=b}$$

$$(18b)\quad h_2(b;\,M_0) = \{\Sigma_M^a[M(x;\,M_0) + 1]\,\Psi(x,\,M_0) - D\,\Delta\Psi(x;\,M_0)\}\big|_{x=b}$$

$$(18c)\quad \Psi(x;\,M_0) = \int_0^x \psi(x,\,M_0)\,\mathrm{d}x\,;\quad \mu_2 = (\xi\Sigma_s)_R\,.\,(\xi\Sigma_s)^{-1}$$

If, for the function f of the variables b, M_0 given by the relation (18) on the domain $\Omega_1 \subset R_2$ the assumptions

$$(18d)\quad f \in C^{(1)}(\Omega_1)\,;\quad f_b' = \frac{\partial f}{\partial b} \neq 0 \quad \text{on} \quad \Omega_1 \equiv (M_0^{\min},\,M_0^{\max}) \times (b^{\min},\,\infty) \subset R_2$$
$$f(b^{(1)},\,M_0^{(1)};\,a) = 0 \text{ for some } (b^{(1)},\,M_0^{(1)}) \in \Omega_1 \text{ and all } a \geqq a^{\min}$$

hold for all $a \geqq a^{\min} > b > 0$, then there is defined by the relation (18) in the stability interval $\langle M_0^{\min},\,M_0^{\max}\rangle$ of $M(x;\,M_0)$ [4] the unique explicit function (the so called "criticality condition")

$$(19)\qquad b = b(M_0;\,a)$$

depending on the parameter a. Substituting the relation (19) into sufficient condition (12c) we get with the help of the first of conditions (15) and of (16), (14a), (18c) the following equation

$$0 = H(M_0, a; \omega_0) = \gamma_1 \omega_0 + \gamma_2 , \tag{20}$$

where

$$\gamma_1 = 1 - \{W[b(M_0), a]\, \Psi[b(M_0), M_0]\}^{-1} \{1 - \tag{20a}$$
$$- \alpha_1[b(M_0), M_0, a]\, y_1[b(M_0); a] + \beta_1[b(M_0), M_0, a]\, y_2[b(M_0); a]\}$$

$$\gamma_2 = \{W[b(M_0), a] \,.\, \Psi[b(M_0), M_0]\}^{-1} \,.\, \{1 - W[b(M_0), a]\} . \tag{20b}$$

Under the assumptions

$$H \in C^{(1)}(\Omega_2) ; \quad H'_{M_0} = \frac{\partial H}{\partial M_0} \neq 0 \quad \text{on} \quad \Omega_2 \equiv (M_0^{\min}, M_0^{\max}) \times (a^{\min}, \infty) \in R_2 , \tag{20c}$$

$$H(M_0^{(2)}, a^{(2)}; \omega_0) = 0 \text{ for some } (M_0^{(2)}, a^{(2)}) \in \Omega_2 \text{ and all real } \omega_0$$

we can determinate from the above equations the critical initial relative fuel concentration $M(0) = M_0 = M_0^*(a, \omega_0)$ inducing the desired thermal neutron flux

$$\Phi \equiv \Phi(x; \omega_0) = \Phi_0 \left\{1 + \omega_0 \int_0^x \psi[x; M_0^*(a, \omega_0)]\, dx\right\} \tag{21}$$

in the reactor core. From (21) it follows that choosing $\omega_0 = 0$ we obtain the case of the flattened thermal neutron flux in the reactor core, i.e.

$$\omega_0 = 0 \Rightarrow \Phi = \Phi_0 = \text{const.} > 0 \tag{21a}$$

By substitution of the critical value $M_0(a, \omega_0)$ of the initial relative fuel concentration M_0 into the criticality condition (19) we get the corresponding value of the critical dimension of the reactor core with the desired thermal neutron flux (21) and for the given outer dimension a of the reactor

$$b_{crit} = b_{crit}(a, \omega_0) = b[M_0^*(a, \omega_0); a] . \tag{22}$$

By the preceding reasoning we have proved the validity of the following

Theorem 1. *Let the distribution of the relative fuel concentration $M(x; M_0)$ in the reactor core (in an arbitrary onedimensional geometry) be a solution of the Cauchy's problem* (7a), (8) *and let the corresponding function $\Phi' = (d\Phi/dx) \equiv \equiv \psi[x; M(x; M_0)]$ be the solution of the boundary value problem* (9,) (9b) *for $\Phi'_b = (d\Phi/dx)|_{x=b} = 1$. Let the functions $f(b, M_0; a)$ and $H(M_0, a; \omega_0)$ defined by* *the relation* (18), (20) *resp. fulfill the condition* (18d), (20c) *resp. Let* $a \geqq a^{\min} > b > 0$ [4] *and let a real parameter* ω_0 *exist so that the critical initial relative fuel concentration* $M_0 = M_0^*(a, \omega_0)$ *given by* (20) *lies in the stability interval* $\langle M_0^{\min}, M_0^{\max} \rangle$ [4] *of the solution* $M(x; M_0)$ *of the nonlinear Cauchy problem* (7a), (8), i.e.

$$M_0^*(a, \omega_0) \in \langle M_0^{\min}, M_0^{\max} \rangle . \tag{23}$$

Then the thermal neutron flux $\Phi = \Phi(x; M[x; M_0^*(a, \omega_0)]; \omega_0)$ *in the reactor core induced by the distribution* $M[x; M_0^*(a, \omega_0)]$ *of the relative fuel concentration is given by the relations* (12), (12b). *Especially, for* $\omega_0 = 0$ *we get from* (12) *for* $M = M[x, M_0^*(a, 0)$ *the flattened thermal neutron flux* $\Phi = \Phi_0 = \text{const.}$ *in the reactor core and therefore, (as a consequence of the wellknown theorems of Goertzel and Wilkins), also the minimum of the critical mass:*

$$\int_0^{b[M_0^*(a,0),a]} M[x, M_0^*(a, 0)] \, \mathrm{d}x = \min . \tag{24}$$

Remark 1. From the fact that the condition (12c) is for $\omega_0 = 0$ sufficient for thermal flux flattening it follows that the critical value $M_0^* = M_0^*(a)$ of the initial relative fuel concentration in the papers [1], [4] (which was obtained by substituting $\omega_0 = 0$ into the relations (17) and by eliminating M_0 from them) must be the same as $M_0^*(a, 0)$.

Remark 2. As a natural generalization of the functional (24) for the case of a non-flattened thermal neutron flux Φ induced by a given fuel concentration distribution $M[x; M_0^*(a, \omega_0)$ we have the following "output functional"

$$F(a, \omega_0) \equiv \int_0^{b[M_0^*(a,\omega_0),a]} \Phi(x; M[x; M_0^*(a, \omega_0)]; \omega_0) \, . \, M[x; M_0^*(a, \omega_0)] \, \mathrm{d}x \, , \tag{25}$$

which yields by the mean value theorem and by (24):

$$\begin{aligned} F(a, \omega_0) &= \bar{\Phi} \int_0^{b[M_0^*(a,\omega_0),a]} M[x; M_0^*(a, \omega_0)] \, \mathrm{d}x \geqq \\ &\geqq \bar{\Phi} \int_0^{b[M_0^*(a,0),a]} M[x; M_0^*(a, 0)] \, \mathrm{d}x \end{aligned} \tag{25a}$$

Therefore it may be useful to determine the value ω_0^* of the parameter ω_0 maximizing the value of the functional $F(a, \omega_0)$ for a given value of the parameter a. For ω_0^* we have evidently the condition

$$\left. \frac{\partial F(a, \omega_0)}{\partial \omega_0} \right|_{\omega_0 = \omega_0^*} = 0 \tag{26}$$

under the assumptions

$$\text{(26a)} \qquad \left.\frac{\partial^2 F(a, \omega_0)}{\partial^2 \omega_0}\right|_{\omega_0 = \omega_0^*} < 0 \; ; \quad M_0^*(a, \omega_0^*) \in \langle M_0^{\min}, M_0^{\max}\rangle \, .$$

One can expect that for some a this maximum will be an absolute one. The "output optimization coefficient"

$$\text{(27)} \qquad \eta(a) = \frac{\int_0^{b[M_0^*(a,\omega_0^*),a]} M[x; M_0^*(a, \omega_0^*)] \, \mathrm{d}x}{\int_0^{b[M_0^*(a,0),a]} M[x; M_0^*(a, 0)] \, \mathrm{d}x} \; ; \qquad \lim_{a \to a^{\min}} \eta(a) = 1 \, , \quad [4]$$

does not depend on the mean thermal neutron flux $\bar{\Phi}$ in the reactor core and can be determined (as a function of the outer boundary of the reactor) by solving (e.g. numerically) the equations (26), (20) and the Cauchy problem (7a), (8).

By a generalization of the preceding reasoning we can find also the solution of the inverse problem of the reactor theory: For the given thermal neutron flux $\Phi = \Phi(x)$ the corresponding distribution $M = M[x; M_0; \Phi(x)]$ of the relative fuel concentration in the reactor core is to be determined (in the two-group approximation and in an arbitrary onedimensional geometry [5]) i.e. M inducing the prescribed thermal neutron flux Φ in the reactor core and fulfilling the boundary conditions [1]

$$\text{(28)} \quad M(b; M_0, \Phi) = \tilde{\vartheta}_1(b, a) \equiv \frac{1}{\Phi(b) \, \Sigma_M^a(b)} \left\{ \frac{1}{\mu_2} \, q_R(b) + \operatorname{div} \left(D \operatorname{grad} \Phi\right)\big|_{x=b} \right\} - 1$$

$$\text{(28a)} \qquad \left.\frac{\mathrm{d}M}{\mathrm{d}x}\right|_{x=b} = \tilde{\vartheta}_2(b, a) \equiv \frac{1}{\Phi(b) \, \Sigma_M^a(b)} \left\{ \left.\frac{\tau_R}{\tau} \frac{\mathrm{d}q_R}{\mathrm{d}x}\right|_{x=b} + \right.$$
$$\left. + \frac{\mathrm{d}}{\mathrm{d}x} \left[\operatorname{div} \left(D \operatorname{grad} \Phi\right)\right]\big|_{x=b} - \left[1 + \tilde{\vartheta}_1(b, a) \frac{\mathrm{d}}{\mathrm{d}x} \left(\Phi \Sigma_M^a\right)\right]\bigg|_{x=b} \right\} ,$$

where the outer boundary parameter $a = a^*$ is given and the core boundary parameter b as well as the initial fuel concentration parameter M_0 are to be determined. For the sought function M there follows in this case from the equation (6a) the inhomogeneous quasilinear ordinary differential equation of the second order

$$\text{(29)} \qquad \Delta M + \frac{1}{\tau} \left(\frac{\mathrm{d}\tau}{\mathrm{d}x} + \frac{2\tau}{\Sigma_M^a} \frac{\mathrm{d}\Sigma_M^a}{\mathrm{d}x} + \frac{2\tau}{\Phi} \frac{\mathrm{d}\Phi}{\mathrm{d}x} \right) \frac{\mathrm{d}M}{\mathrm{d}x} - \left\{ \frac{1}{\tau} \left[\left(1 - k(M) \right) - \right. \right.$$
$$\left. \left. - \frac{1}{\Sigma_M^a} \frac{\mathrm{d}\tau}{\mathrm{d}x} \frac{\mathrm{d}\Sigma_M^a}{\mathrm{d}x} - \frac{\tau}{\Sigma_M^a} \Delta \Sigma_M^a \right] - \frac{2}{\Phi \Sigma_M^a} \frac{\mathrm{d}\Sigma_M^a}{\mathrm{d}x} \frac{\mathrm{d}\Phi}{\mathrm{d}x} - \frac{1}{\tau \Phi} \frac{\mathrm{d}\tau}{\mathrm{d}x} \frac{\mathrm{d}\Phi}{\mathrm{d}x} - \frac{1}{\Phi} \Delta \Phi \right\} M =$$
$$= \frac{1}{\Phi \Sigma_M^a} \left\{ D \Delta(\Delta \Phi) + \left(3 \frac{\mathrm{d}D}{\mathrm{d}x} + \frac{D}{\tau} \frac{\mathrm{d}\tau}{\mathrm{d}x} \right) \frac{\mathrm{d}\Delta\Phi}{\mathrm{d}x} + \left(\Delta D + \frac{1}{\tau} \frac{\mathrm{d}\tau}{\mathrm{d}x} \frac{\mathrm{d}D}{\mathrm{d}x} - \right. \right.$$

$$- \frac{D}{\tau} - \Sigma_M^a \Big) \Delta\Phi + \left(2 \frac{d^2 D}{dx^2} + \frac{1}{\tau} \frac{dD}{dx} \frac{d\tau}{dx} \right) \frac{d^2 \Phi}{dx^2} + \left(\frac{d\Delta D}{dx} - \frac{1}{\tau} \Sigma_M^a \frac{d\tau}{dx} - \right.$$

$$\left. - 2 \frac{d\Sigma_M^a}{dx} \right) \frac{d\Phi}{dx} + \frac{1}{\tau} \left(\frac{d\tau}{dx} \frac{d^2 D}{dx^2} \right) \frac{d\Phi}{dx} + \frac{\Sigma_M^a}{\tau} \left(1 - \frac{1}{\Sigma_M^a} \frac{d\tau}{dx} \frac{d\Sigma_M^a}{dx} - \frac{\tau}{\Sigma_M^a} \Delta\Sigma_M^a \right) \Phi \Big\} ,$$

which, under the usual conditions (3), (3a), takes the simplified form

(29a) $$\Delta M + \frac{2}{\Phi} \text{grad } \Phi \, . \, \text{grad } M - \left\{ \frac{1}{\tau} [1 - k(M)] - \frac{\Delta\Phi}{\Phi} \right\} M =$$

$$= \frac{1}{\Phi \Sigma_M^a} \left[D\Delta(\Delta\Phi) - \left(\frac{D}{\tau} + \Sigma_M^a \right) \Delta\Phi + \frac{\Sigma_M^a}{\tau} \Phi \right]$$

and which gives together with the initial conditions (8) the Cauchy problem determining uniquely the sought relative fuel concentration $M = M(x; M_0, \Phi)$ In particular for the flattened thermal neutron flux $\Phi = \Phi_0 = \text{const.}$ we get again the formerly considered Cauchy problem (7a), (8). The two equations (28), (28a) give us the possibility (under the assumption that they are uniquely solvable in the parameters a, b) to determine for the arbitrarily chosen initial relative fuel concentration $M_0 = M(0) \in \langle M_0^{\min}, M_0^{\max} \rangle$, the corresponding value $b(M_0)$ of the critical reactor core boundary parameter b and the corresponding value $a(M_0)$ of the reflector outer boundary parameter a. From the condition

(30) $$a(M_0) = a^*$$

we can then determine (under the assumption that the equation (30) is uniquely solvable in the parameter M_0) the critical value M_0^* of the initial relative fuel concentration M:

(30a) $$M_0^* = M_0^*(a^*) \, ; \quad a(M_0^*) = a^*$$

Solving the Cauchy problem (29), (8) for M and the equations (28), (28a) for a and b with this critical initial relative fuel concentration M_0^* we obtain evidently the sought distribution of fuel $M = M[x; M_0^*, \Phi(x)]$ which induces the prescribed thermal neutron flux $\Phi = \Phi(x)$ in the core of the critical reactor with the core boundary parameter $b^* = b(M_0^*)$ and with the given reflector outer boundary parameter a^*.

Therefore, the following statement is true:

Theorem 2. *Let the Cauchy problem* (29), (8) *have for all values* $M_0 \in \langle M_0^{\min}, M_0^{\max} \rangle$ *of the initial relative fuel concentration* $M_0 = M(0)$ *a unique stable solution* $M = M[x; M_0, \Phi(x)]$ *depending on the given thermal neutron flux* $\Phi = \Phi(x)$. *Let the equations* (28), (28a) *for the fuel distribution* $M = M[x; M_0, \Phi(x)]$ *have unique solutions* $b = b(M_0)$, $a = a(M_0)$ *depending on the parameter* $M_0 \in \langle M_0^{\min}, M_0^{\max} \rangle$ *and let the equation* $a(M_0) = a^*$, *where* a^* *is the prescribed value of the reflector* *outer boundary parameter have the unique solution $M_0^* = M_0^*(a^*)$ such that $a(M_0^*) = a^*$.*

Then the distribution $M[x; M_0^*, \Phi(x)]$ of the relative fuel concentration determined by the Cauchy problem (29), (8) with the initial value $M_0 = M_0^*$ induces in the core of the critical reactor (with the core parameter $b^* = b(M_0^*)$ given by the equations (28), (28a) and with the given value a^* of the reflector outer boundary parameter a) the prescribed thermal neutron flux $\Phi = \Phi(x)$.

Remark 3. From (24) and from the mean value theorem we have for all possible thermal neutron fluxes $\Phi = \Phi(x)$ and the corresponding fuel distributions $M = M[x; M_0^*, \Phi(x)]$ in the critical reactor core with criticality parameter $b^* = b(M_0^*)$ the inequality

$$(31) \qquad F \equiv \int_0^{b*} \Phi(x) \,.\, M[x; M_0^*, \Phi(x)]\,\mathrm{d}x = \bar{\Phi} \int_0^{b*} M[x; M_0^*, \Phi(x)]\,\mathrm{d}x \geqq$$

$$\geqq \bar{\Phi} \int_0^{b[M_0^*(a,0),a]} M[x, M_0^*(a, 0]\,\mathrm{d}x$$

However from (12), (12b), (18c) and (25) it follows due to the "nearly linear" dependence of the output functional $F(a, \omega_0)$ on the parameter ω_0 with the help of the mean value theorem

$$(32)\ F(a, \omega_0) \equiv \Phi_0 \int_0^{b[M_0^*(a,\omega_0),a]} (1 + \omega_0\, \Psi[x; M_0^*(a, \omega_0)] \,.\, M[x, M_0^*(a, \omega_0)]\,\mathrm{d}x =$$

$$= \Phi_0(1 + \omega_0\, \bar{\Psi}[M_0^*(a, \omega_0)]) \int_0^{b[M_0^*(a,\omega_0),a]} M[x; M_0^*(a, \omega_0)]\,\mathrm{d}x =$$

$$= \int_0^{b*} \Phi(x) \,.\, M[x; M_0^*, \Phi(x)]\,\mathrm{d}x \equiv F$$

for all $\Phi(x)$ and the corresponding $M[x; M_0^*, \Phi(x)]$, provided

(32a)

$$\omega_0 = \omega_0^{**} = \frac{1}{\bar{\Psi}[M_0^*(a, \omega_0^{**})} \left\{ \frac{\int_0^{b*} \Phi(x) \,.\, M[x; M_0^*, \Phi(x)]\,\mathrm{d}x}{\Phi_0 \int_0^{b[M_0^*(a,\omega_0^{**}),a]} M[x; M_0^*(a, \omega_0^{**})]\,\mathrm{d}x} - 1 \right\}$$

We see also that if the equation (32a) for ω_0^{**} is solvable (e.g. by iterations) for some $\Phi = \Phi(x)$ and the corresponding $M = M[x; M_0^*, \Phi(x)]$, then for the value ω^* given by the relations (26), (26a) majorizes the output functional $F(a, \omega_0^*)$ the corresponding functional F given by (31) for this possible thermal neutron flux $\Phi = \Phi(x)$ and the corresponding fuel distribution $M = M[x; M_0^*, \Phi(x)]$.

References

[1] *V. Bartošek* and *R. Zezula:* Flat Flux in a Slab Reactor with Natural Uranium, Journal of Nuclear Energy Parts A/B, 1966, vol. 20, pp. 129—134.
[2] *S. Glasstone, M. C. Edlund:* The Elements of Nuclear Reactor Theory. Toronto—New York—London 1952.
[3] *V. Bartošek, R. Zezula:* Flat Flux in a Slab Reactor with Natural Uranium. Report ÚJV 1310 (1965).
[4] *V. Bartošek, R. Zezula:* Stability of flat thermal flux in a slab reactor. Apl. Mat. 13 (1968), 367—375.
[5] *R. Zezula:* Sufficient Conditions for the Flattening of Thermal Neutron Flux and Some Related Problems (in multidimensional geometries). Report ÚJV 1902/67.
[6] *M. Hron, V. Lelek:* Flux Flattening by means of a Non-Uniform Fuel Distribution in a Slab Reactor with Finite Reflector. Report ÚJV 1660 (1966).

Výtah

POSTAČUJÍCÍ PODMÍNKY PRO VYROVNANÝ TOK TEPELNÝCH NEUTRONŮ A NĚKTERÉ PŘÍBUZNÉ PROBLÉMY (V JEDNOROZMĚRNÝCH GEOMETRIÍCH)

ROSTISLAV ZEZULA

Práce se zabývá následujícím problémem teorie jaderných reaktorů: Pro zadaný průběh toku ϕ tepelných neutronů v aktivní zoně reaktoru určit rozložení koncentrace paliva M, které tento tok vytváří. Tento problém je matematicky formulován v t.zv. dvougrupovém difusním přiblížení a pro jednorozměrné geometrie. Je udána postačující podmínka pro existenci jediného řešení tohoto problému, zejména též ve speciálním případě vyrovnaného toku tepelných neutronů. Dále se uvažuje jistá metoda optimalisace celkového výkonu reaktoru.

Hlavní výsledky práce jsou formulovány ve větách 1 a 2.

Author's Address: Dr. *Rostislav Zezula,* CSc., Matematický ústav Karlovy university, Praha 8 Karlín, Sokolovská 83.