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ONEDIMENSIONAL HOMOGENIZED REACTOR WITH NATURAL URANIUM AND WITH FLATTENED SPECIFIC OUTPUT

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Let us consider a onedimensional homogenized reactor with reflector and with natural uranium fueled core, described (in the two-groups diffusion approximation and in the usual denotation [1], [2]) by the equations

(1) $$-D\Delta\Phi + \Sigma_M^a(1 + M)\,\Phi = q$$

(1a) $$-\tau\Delta q \;+ q \qquad = k\Sigma_M^a M\Phi$$

where we suppose (neglecting the moderator-expelling by the fuel)

(2) $$D = \text{const} > 0\,; \quad \tau = \text{const} > 0\,; \quad \Sigma_M^a = \text{const} > 0$$

and where

(2a) $$M = \frac{\Sigma_U^a}{\Sigma_M^a} = \frac{\sigma_U^a}{\Sigma_M^a} N_u > 0$$

is the relative fuel concentration in the reactor core and

(2b) $$k = k(M)$$

is a given function of M. From (2), (2a) there follows that the quantity

(3) $$M\Phi = \frac{\sigma_U^a}{\Sigma_M^a} N_U \Phi$$

is proportional to the specific reactor output and because it is physically interesting to flatten it [6], we assume

(4) $$M\Phi = \text{const} = C_1 > 0$$

and look for such a spatial distribution M of the relative fuel concentration in the reactor core which is necessary for flattening the specific reactor output. To this purpose we write equation (1) in the following (evidently equivalent) form

(5) $$-D\Delta\left(\frac{M\Phi}{M}\right) + \Sigma_M^a\left(\frac{1}{M} + 1\right)M\Phi = q$$

from which we obtain with help of the postulate (4) the following (essentially non-linear) relation between the neutron slowing down density q and the relative fuel concentration M

(5a) $$-C_1 D\Delta\left(\frac{1}{M} + 1\right) + C_1\Sigma_M^a\left(\frac{1}{M} + 1\right) = q$$

But it is evident that equation (5a) is linear in the quantity

(6) $$N = \frac{1}{M} + 1 = N(M) > 1$$

and since we have, with respect to relation (2b)

(6a) $$l(N) = k[(N-1)^{-1}] = k(M), \quad (M = (N-1)^{-1})$$

so equation (1a) must take, for the slowing down density in consequence of relations (6a), (4), the form

(7) $$-\tau\Delta q + q = C_1\Sigma_M^a l(N).$$

Substituting relations (5a), (6) for q into equation (7), we get the following quasi-linear biharmonic equation for $N = N(M)$

(8) $$\Delta(\Delta N) - \frac{\tau\Sigma_M^a + D}{\tau D}\Delta N + \frac{1}{\tau D}f(N)N = 0,$$

$$\left(f(N) = \Sigma_M^a\left[1 - \frac{l(N)}{N}\right]\right)$$

which gives a necessary condition for flattening the specific output of the reactor. It is clear from the symmetry and stability considerations [2] that for equation (8) the following initial conditions must be taken into consideration

(8a) $$N'(0) = N'''(0) = 0; \quad N(0) = N_0 \geqq 1; \quad N''(0) = N_0''$$

where $N_0 \geqq 1$ and N_0'' are real parameters.

The solution of the usual two-group equations for Φ_R, q_R in the reflector has, under the usual homogeneous conditions on the outer boundary a of the reflector

(9) $$\Phi_R(a) = 0; \quad q_R(a) = 0$$

the form

(10) $$q_R(x; a, A) = Ay_1(x; a)$$

(10a) $$\Phi_R(x; a, A, B) = A\mu_1 y_1(x; a) + By_2(x; a)$$

where A, B are free integration constants, the constant μ_1 is a known function of the physical constants of the reflector and the known functions $y_1(x; a)$, $y_2(x; a)$ (linear combinations of the fundamental solutions of the two-group equations for the reflector with coefficients depending on a which fullfil the boundary conditions (9)) depend on the geometry considered.

Let N be an arbitrary solution of the nonlinear biharmonic Cauchy's initial value problem (8), (8a) so that it depends on two real parameters N_0, N_0'' (and on the spatial variable x)

(11) $$N = N(x; N_0, N_0'')$$

and let the values of the real parameters $N_0 \geqq 1$, N_0'' lie in the stability domain $\Omega \subset R_2$ of N. Then there follows from the equivalence of the both equations (1), (5) that a sufficient condition for flattening the specific output in the reactor core shall be

(12) $$\Phi = C_1[N(x; N_0, N_0'') - 1] = \Phi(x; N_0, N_0'')$$

(where C_1 is the normalization constant in (4)), since we know [3] that for the given thermal neutron flux Φ there exists a uniquely determined distribution M of the fuel concentration in the core of the critical reactor inducing Φ which must therefore be identical with

(13) $$M = M(x; N_0, N_0'') = \frac{1}{N(x; N_0, N_0'') - 1}$$

under the condition that the free parameters A, B in (10), (10a), N_0, N_0'' in (8a) and the extrapolated boundaries b and a of the core and the reflector, respectively, will be chosen in accordance with the usual boundary conditions on the boundary b between the core and reflector (insuring reactor criticality):

(14) $$\Phi(b; N_0, N_0'') = \Phi_R(b; a, A, B)\,; \quad D\Phi'(b; N_0, N_0'') = D_R\Phi_R'(b; a, A, B)$$

(14a) $$\frac{1}{\xi\Sigma_s}\, q(b; N_0, N_0'') \doteq \frac{1}{(\xi\Sigma_s)_R}\, q_R(b; a, A)\,; \quad \tau q'(b; N_0, N_0'') = \tau_R q_R'(b; a, A)$$

where the values $\Phi(b; N_0, N_0'')$, $\Phi'(b; N_0, N_0'')$ are given by (12) and the values $q(b; N_0, N_0'')$, $q'(b; N_0, N_0'')$ are given by relations (5a), (6) so that we have

(14b) $$q(b; N_0, N_0'') = C_1[-D\Delta N(x; N_0, N_0'') + \Sigma_M^a N(x; N_0, N_0'')]\big|_{x=b}$$

(14c) $$q'(b; N_0, N_0'') = C_1\left[-D\frac{\mathrm{d}}{\mathrm{d}x}\Delta N(x; N_0, N_0'') + \Sigma_M^a \frac{\mathrm{d}}{\mathrm{d}x} N(x; N_0, N_0'')\right]\bigg|_{x=b}.$$

From the four boundary conditions (14), (14a) we can determine four of the six free parameters N_0, N_0'', A, B, a, b (as functions of the remaining two). Since in (14) the parameters A, B appear linearly we can easily eliminate them. We obtain

$$(15) \qquad A = \frac{C_1}{\mu_1 W_b(y_1, y_2)} \left\{ [N(b; N_0, N_0'') - 1] \, y_2'(b, a) - \right.$$

$$\left. - \frac{D}{D_R} N'(b; N_0, N_0'') \, y_2(b, a) \right\} = A(N_0, N_0'', b, a) \,,$$

$$(15a) \qquad B = \frac{C_1}{\mu_1 W_b(y_1, y_2)} \mu_1 \left\{ y_1(b; a) \frac{D}{D_R} N'(b; N_0, N_0'') - \right.$$

$$\left. - y_1'(b, a) [N(b; N_0, N_0'') - 1] \right\} = B(N_0, N_0'', b, a)$$

where $W_b(y_1, y_2)$ denotes the value of the Wronskian of the known functions $y_1(x; a)$, $y_2(x; a)$ for $x = b$. By substituting (15) and (15a) into (10) and (10a) we get for the slowing down density in the reflector q_R and for the corresponding thermal neutron flux Φ_R the expressions

$$(16) \qquad q_R(x; a, b, N_0, N_0'') = A(N_0, N_0'', b, a) \, y_1(x; a) \,,$$

$$(16a) \qquad \Phi_R(x; a, b, N_0, N_0'') = A(N_0, N_0'', b, a) \, \mu_1 y_1(x; a) + B(N_0, N_0'', b, a) \, y_2(x; a) \,.$$

Using relations (14b), (14c), (16) we can transform the remaining two boundary conditions (14a) as follows

$$(17) \quad H_1(a, b; N_0, N_0'') \equiv \left[\frac{1}{\xi \Sigma_s} q(b; N_0, N_0'') - \frac{1}{(\xi \Sigma_s)_R} q_R(b; a, b, N_0, N_0'') \right] = 0$$

$$(18) \quad H_2(a, b, N_0, N_0'') = \left\{ \tau q'(b; N_0, N_0'') - \tau_R \left[\frac{\mathrm{d}}{\mathrm{d}x} q_R(x; a, b, N_0, N_0'')\big|_{x=b} \right] \right\} = 0 \,.$$

Now, let us assume that the function $H_1(a, b, N_0, N_0'')$ of the variables $a > b > 0$ given on the domain $\Omega_1 \equiv (a_{\min}, \infty) \times (b_{\min}, \infty) \subset R_2$ fulfils the following assumptions:

$$(19) \qquad H_1 \in C^{(1)}(\Omega_1) \,; \quad \frac{\partial}{\partial b} H_1 \neq 0 \quad \text{on} \quad \Omega_1 \,,$$

$$(19a) \quad H_1(a^{(1)}, b^{(1)}; N_0, N_0'') = 0 \text{ for some } (a^{(1)}, b^{(1)}) \in \Omega_1 \text{ and all } N_0 \geqq 1, N_0'' \text{ real} \,,$$

so that there exists (in an interval $(b^{(0)}, b^{(2)})$ containing $b^{(1)}$) the implicit function

$$(20) \qquad a = a(b; N_0, N_0'')$$

for which equation (17) is fulfilled for all $N_0 \geqq 1$, N_0'' real, $b \in (b^{(0)}, b^{(2)})$, i.e.

(20a) $$H_1[a(b; N_0, N_0''), b; N_0, N_0''] = 0 .$$

Substituting relation (20) for a into equation (18) we get the "criticality condition" for b

(21)
$$F(b; N_0, N_0'') \equiv \left\{\tau q'(b; N_0, N_0'') - \tau_R\left[\frac{\mathrm{d}}{\mathrm{d}x}\, q_R(x; a(b; N_0, N_0''), N_0, N_0'')\big|_{x=b}\right]\right\} = 0$$

which under the assumption that there exists on the interval $(b^{(0)}, b^{(2)})$ an inverse function G satisfying the relation

(21a) $$G[F(b; N_0, N_0'')] = b(N_0, N_0'') \text{ for all } b \in (b^{(0)}, b^{(2)}) \text{ and } (N_0, N_0'') \in \Omega_2 \subset \Omega$$

has a unique solution (the so-called "critical core dimension")

(22) $$b^* = b^*(N_0, N_0'') = G(0)$$

satisfying the criticality condition (21) for all $(N_0, N_0'') \in \Omega_2 \subset \Omega$, i.e.

(22a) $$F(b^*(N_0, N_0''); N_0, N_0'') = 0 .$$

For arbitrarily chosen values $(N_0, N_0'') \in \Omega_2 \subset \Omega$ of the both real parameters N_0, N_0'' in the initial conditions (8a) we can therefore evaluate the critical core dimension $b^* = b^*(N_0, N_0'')$ from (22) and then from (20) the corresponding value a^* of the reflector outer boundary

(23) $$a^* = a^*(N_0, N_0'') = a(b^*(N_0, N_0''); N_0, N_0'') .$$

The function $q(x; N_0, N_0'')$, which we need for determining the value a^* by means of relation (17), is obtained by substituting the solution $N(x; N_0, N_0'')$ of the Cauchy's initial value problem (8), (8a) (which evidently forms a twoparametrical family depending on two real parameters $N_0 \geqq 1$, N_0'' to be chosen so that $N(x; N_0, N_0'')$ is a stable solution of the problem (8), (8a) satisfying assumption (21a), (i.e. $(N_0, N_0'') \in \Omega_2 \subset \Omega$)) into relations (5a), (6):

(24) $$q(x; N_0, N_0'') = C_1[-D\Delta N(x; N_0, N_0'') + \Sigma_M^a N(x; N_0, N_0'')] .$$

The relations (11) for N, (12) for Φ, (13) for M, (24) for q, (22) for b^* and (23) for a^* together with relations (16), (16a) for the solution q_R, Φ_R in the reflector give us a twoparametrical family of solutions (M, Φ, q, b^*, a^*) of the problem of flattening the specific output of a critical homogenized onedimensional reactor with natural uranium, depending on N and therefore on two arbitrary real parameters $N_0 \geqq 1$, N_0'', $(N_0, N_0'') \in \Omega_2$ so that we have proved by the preceding reasoning the following.

Theorem 1. *Let* $D > 0$, $\tau > 0$, $\Sigma_M^a > 0$, μ_1 *be given constants.*

Let the real values $N_0 \geqq 1$, N_0'', $(N_0, N_0'') \in \Omega_2 \subset \Omega$ *be chosen so that the function* $N(x; N_0, N_0'')$ *is a stable solution of the Cauchy's initial value problem* (8), (8a) (*in an arbitrary onedimensional geometry*) *and that the function* F *in the critical state condition* (21) *fulfils assumption* (21a).

Let the function $\Phi(x; N_0, N_0'')$, $q(x; N_0, N_0'')$ *be given* (*with help of* $N(x; N_0, N_0'')$) *by relations* (12), (24), *respectively.*

Let the function H_1 *in the boundary condition* (17) *fulfil assumptions* (19), (19a).

Let the value $b^*(N_0, N_0'')$ *of the extrapolated critical core boundary and the corresponding value* $a^*(N_0, N_0'')$ *of the extrapolated reflector outer boundary be given by relations* (22), (23), *respectively.*

Then the distribution $M(x; N_0, N_0'')$ *of the relative fuel concentration in the core, given* (*by means of* $N(x; N_0, N_0'')$) *by relation* (13), *induces in the critical reactor core* (*with extrapolated core boundary* $b^*(N_0, N_0'')$ *and the corresponding reflector outer boundary* $a^*(N_0, N_0'')$) *the thermal neutron flux* $\Phi(x; N_0, N_0'') = C_1[N(x; N_0, N_0'') - 1]$ *giving the flattened specific output* $M\Phi = C_1$ *of the reactor.*

Theorem 1 enables us to look for such a distribution $M(x; N_0, N_0'')$ of the relative fuel concentration for which the total output of the reactor

(25) $$T(N_0, N_0'') = C_1 b^*(N_0, N_0'') = \int_0^{b^*(N_0, N_0'')} M(x; N_0, N_0'')\, \Phi(x; N_0, N_0'')\, dx$$

reaches its local maximum (under the assumption that the function $b^*(N_0, N_0'')$ has such a maximum). This means that we must determine in the twoparametrical families $M(x; N_0, N_0'')$, $\Phi(x; N_0, N_0'')$ the "optimal" values $\tilde{N}_0$, $\tilde{N}_0''$ (lying in the subset $\Omega_2 \subset \Omega$ of the stability domain $\Omega \subset R_2$ of the real parameters $N_0 \geqq 1$, N_0'', in which assumption (21a) holds) for which we have

(25a)
$$T(\tilde{N}_0, \tilde{N}_0'') = \max_{(N_0, N_0'') \in \Omega_2 \subset \Omega} T(N_0, N_0'') = \max_{(N_0, N_0'')} \int_0^{b^*(N_0, N_0'')} M(x; N_0, N_0'')\, \Phi(x; N_0, N_0'')\, dx =$$
$$= C_1 \max_{(N_0, N_0'') \in \Omega_2 \subset \Omega} b^*(N_0, N_0'') .$$

Evidently, for determining the values $\tilde{N}_0$, $\tilde{N}_0''$ we have the following necessary conditions

(26) $$\frac{\partial}{\partial N_0} b^*(N_0, N_0'') = 0 \, ; \quad \frac{\partial}{\partial N_0''} b^*(N_0, N_0'') = 0$$

Let us assume now that both the functions

(26a) $$P_1(N_0, N_0'') = \frac{\partial}{\partial N_0} b^*(N_0, N_0'') \, ; \quad P_2(N_0, N_0'') = \frac{\partial}{\partial N_0''} b^*(N_0, N_0'')$$

fulfil (in the stability domain $\Omega \subset R_2$) the following assumptions

(27) $$P_i \in C^{(1)}(\Omega)\,; \quad \frac{\partial P_i}{\partial N_0''} \neq 0 \quad \text{on} \quad \Omega \subset R_2 \quad (i = 1, 2)$$

(27a) $$P_i(N_0^{(1)}, N_0''^{(1)}) = 0 \quad \text{for some} \quad (N_0^{(1)}, N_0''^{(1)}) \in \Omega \subset R_2 \quad (i = 1, 2)$$

so that there exists an interval I containing $N_0^{(1)}$ in which both the equations (26) have a unique implicit solution

(28) $$N_0'' = Q_1(N_0)\,; \quad N_0'' = Q_2(N_0) \quad (\text{for } N_0 \in I \subset (0, +\infty))\,.$$

By eliminating N_0'' we obtain from (28) the following equation

(28a) $$F(N_0) \equiv Q_1(N_0) - Q_2(N_0) = 0$$

which represents a necessary condition for the determination of the optimal value $\tilde{N}_0$ of the parameter N_0.

Under the assumption that the function $F(N_0)$ has in the interval $I_0 \equiv (1, +\infty)$ the inverse function H so that

(29) $$H[F(N_0)] = N_0$$

for all $N_0 \in I_0$, we can get the (unique) value $\tilde{N}_0$ by solving equation (28a)

(30) $$\tilde{N}_0 = H(0)$$

and then determine the corresponding value N_0'' from relations (28)

(30a) $$\tilde{N}_0'' = Q_1(\tilde{N}_0) = Q_2(\tilde{N}_0)\,.$$

When we assume further that for these values $\tilde{N}_0$, $\tilde{N}_0''$ the well known conditions

(31) $$\frac{\partial^2}{\partial^2 N_0} b^*(N_0, N_0'')\Big|_{N_0=\tilde{N}_0, N_0''=\tilde{N}_0''} < 0\,,$$

$$\det\begin{pmatrix} \dfrac{\partial^2}{\partial^2 N_0''} b^*(N_0, N_0'') & \dfrac{\partial^2}{\partial N_0''\,\partial N_0} b^*(N_0, N_0'') \\ \dfrac{\partial^2}{\partial N_0\,\partial N_0''} b^*(N_0, N_0'') & \dfrac{\partial^2}{\partial^2 N_0} b^*(N_0, N_0'') \end{pmatrix}\Bigg|_{N_0=\tilde{N}_0, N_0''=\tilde{N}_0''} > 0$$

are fulfilled, which are (together with (26)) sufficient for the maximum of the total output functional $T(N_0, N_0'')$ on the stability domain Ω of N, so that

(32) $$T(\tilde{N}_0, \tilde{N}_0'') = \max_{(N_0, N_0'') \in \Omega} T(N_0, N_0'')\,,$$

then the functions (values respectively)

(33) $\quad M(x; \tilde{N}_0, \tilde{N}_0'') ; \quad \Phi(x; \tilde{N}_0, \tilde{N}_0'') ; \quad q(x; \tilde{N}_0, \tilde{N}_0'') ; \quad b^*(\tilde{N}_0, \tilde{N}_0'') ; \quad a^*(\tilde{N}_0, \tilde{N}_0'')$

give the optimized stable solution of the problem of the specific output flattening for which the total reactor output reaches its local maximum in the stability domain Ω of the solutions $N(x; N_0, N_0'')$ of the Cauchy's problem (8), (8a). Thus we have proved the following

Theorem 2. *Let for the functions $M(x; \tilde{N}_0, \tilde{N}_0'')$; $\Phi(x; \tilde{N}_0, \tilde{N}_0'')$; $q(x; \tilde{N}_0, \tilde{N}_0'')$ of the spatial variable x and for the values $b^*(\tilde{N}_0, \tilde{N}_0'')$; $a^*(\tilde{N}_0, \tilde{N}_0'')$ the assumptions of Theorem 1 be fulfilled.*

Let the point $(\tilde{N}_0, \tilde{N}_0'') \in \Omega_2 \subset \Omega$ consisting of the values $\tilde{N}_0$, $\tilde{N}_0''$ of the real parameters $N_0 \geqq 1$, N_0'' lie in the subset $\Omega_2 \subset \Omega$ of the stability domain $\Omega \subset R_2$ of the twoparametrical family of solutions $N(x; N_0, N_0'')$ of the biharmonic Cauchy's initial value problem (8), (8a) *in which assumption* (21a) *holds.*

Let both the functions $P_1(N_0, N_0'') = \partial/\partial N_0 \,.\, b^*(N_0, N_0'')$, $P_2(N_0, N_0'') = \partial/\partial N_0'' \,.$ $.\, b^*(N_0, N_0'')$ fulfil in the stability domain $\Omega \subset R_2$ assumptions (27), (27a).

Let the function $F(N_0)$, defined in (28a) *fulfil assumption* (29) *and let the value $\tilde{N}_0$ of the positive parameter $N_0 \geqq 1$ be determined from relation* (30) *and the corresponding value $\tilde{N}_0''$ of the real parameter N_0'' from relation* (30a).

Let the second derivatives of the function $b^*(N_0, N_0'')$ fulfil (for $N_0 = \tilde{N}_0$, $N_0'' = \tilde{N}_0''$) conditions (31) *for the local maximum of $b^*(N_0, N_0'')$.*

Then the functions $M(x; \tilde{N}_0, \tilde{N}_0'')$, $\Phi(x; \tilde{N}_0, \tilde{N}_0'')$, $q(x; \tilde{N}_0, \tilde{N}_0'')$ give the relative fuel concentration, the thermal neutron flux and the neutron slowing-down density, respectively, in the core (with the extrapolated core boundary $b^*(\tilde{N}_0, \tilde{N}_0'')$) of a critical homogenized onedimensional reactor with flattened specific output (fueled with natural uranium and with the extrapolated reflector outer boundary $a^*(\tilde{N}_0, \tilde{N}_0'')$) for which the total reactor output $T(N_0, N_0'')$ reaches its local maximum in the stability domain $\Omega \subset R_2$ of the twoparametrical family $N(x; N_0, N_0'')$ of solutions of the biharmonic Cauchy's initial value problem (8), (8a).

Remark 1. When the determinant in (31) is less than zero on the whole stability domain Ω of N then it does not exist a stable solution N giving the extremal total reactor output.

It remains to investigate the stability of the quasilinear Cauchy's problem (8), (8a). In order to do this we shall consider this problem in the slab geometry in which this problem takes the simple form

(34)
$$\frac{d^4N}{dx^4} - \frac{\tau\Sigma_M^a + D}{\tau D}\frac{d^2N}{dx^2} + \frac{1}{\tau D} f(N)\, N = 0$$

$$\left(f(N) = \Sigma_M^a \left[1 - \frac{1}{N} k\left(\frac{1}{N-1}\right)\right]\right),$$

$$(34a)\qquad N(0) = N_0\,;\quad N''(0) = N''_0\,;\quad N'(0) = N'''(0) = 0\,.$$

For other usual onedimensional (e.g. cylindrical, spherical) geometries we can get (for $x \neq 0$) problem (8), (8a) in the form analogous to (34), (34a) by the well known transformations [4].

Let us denote

$$(35)\qquad \alpha_1 = \frac{\tau \Sigma_M^a}{\tau D} = \frac{\Sigma_M^a}{D}\,;\quad \alpha_2 = \frac{D}{\tau D} = \frac{1}{\tau}\,.$$

Then we can write equation (34) in the (evidently equivalent) form

$$(36)\qquad \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \alpha_1\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \alpha_2\right) N = g(N)\,N\,;$$

$$\left(g(N) = \left[\alpha_1\alpha_2 - \frac{f(N)}{\tau D}\right] = \frac{\Sigma_M^a}{\tau D}\,\frac{1}{N}\,k\left(\frac{1}{N-1}\right)\right).$$

When we put

$$(37)\qquad \frac{\mathrm{d}^2 N}{\mathrm{d}x^2} - \alpha_2 N = Y\,;\quad N(0) = N_0\,,\quad N'(0) = 0$$

then relations (36), (37) imply

$$(37a)\qquad \frac{\mathrm{d}^2 Y}{\mathrm{d}x^2} - \alpha_1 Y = g(N)\,N\,;\quad Y(0) = Y_0 \text{ real}\,,\quad Y'(0) = 0$$

and therefore, by substituting Y from (37)

$$(38)\qquad \frac{\mathrm{d}^2 Y}{\mathrm{d}x^2} = \alpha_1\,\frac{\mathrm{d}^2 N}{\mathrm{d}x^2} + \frac{f(N)}{\tau D}\,N \equiv h(N, N'')$$

But from relation (38) there follows evidently that the solution $Y = Y(N)$ of the Cauchy's problem (37a) will be stable when the solution N of problem (37) will be stable so that it suffices to investigate the stability of problem (37) for $Y = Y(N)$, i.e.

$$(39)\qquad \frac{\mathrm{d}^2 N}{\mathrm{d}x^2} - \alpha_2 N = Y(N)\,;\quad N(0) = N_0\,,\quad N'(0) = 0\,.$$

For this purpose we put

$$(40)\qquad N' = \frac{\mathrm{d}N}{\mathrm{d}x} = Z$$

so that we get from (39) and (40) the following (evidently autonomous) system of two simultaneous differential equations of the first order

(41)

$$\frac{dN}{dx} = \qquad Z \equiv P(Z)$$

$$\frac{dZ}{dx} = \left[\alpha_2 + \frac{Y(N)}{N}\right] N \equiv Q(N) \doteq Q(N_i) + Q'(N_i)(N - N_i) + O[(N - N_i)^2]$$

$$\left(\text{where } Q'(N) \equiv \frac{dQ}{dN} = \alpha_2 + \frac{dY}{dN}\right).$$

The first (i.e. linear) approximation of the nonlinear system (41) is evidently stationary and the assumptions of the first Liapunoff's method [4] are fulfilled.

Since it follows from equations (41) that we have

(42) $$\frac{dZ}{dx} \equiv \frac{dZ}{dN}\frac{dN}{dx} = \frac{1}{2}\left(2Z\frac{dZ}{dN}\right) \equiv \frac{1}{2}\frac{d}{dN}Z^2 = \alpha_2 N + Y(N)$$

and therefore also

(42a) $$Z \equiv N'(N, N_0) \equiv \frac{dN}{dx} = \pm\sqrt{\left\{2\int_{N_0}^{N}[\alpha_2 N + Y(N)]\,dN\right\}},$$

we can represent the solutions of system (41) in the phase plane (N, N') as a one-parametrical system of phase curves $N'(N; N_0)$ (depending on the real parameter N_0) which are defined by equation (42a) for all N for which the condition

(42b) $$\int_{N_0}^{N}[\alpha_2 N + Y(N)]\,dN \geqq 0$$

holds. The singular points of equation (39) are the points (N, N') in the phase plane with the coordinates [5]

(43) $$N = N_i\,, \quad N' = 0$$

where the values N_i are the real roots of the equation

(43a) $$\frac{1}{\alpha}Q(N_i) \equiv N_i + \frac{1}{\alpha_2}Y(N_i) = 0\,.$$

According to the first Liapunoff's method the character of those singular points $(N_i, 0)$ in the phase plane is determined by the properties of the roots λ_1, λ_2 of the characteristic equation corresponding to the linearized system (41) [5]

$$\det\begin{pmatrix} -\lambda & 1 \\ Q'(N_i) & -\lambda \end{pmatrix} \equiv \lambda^2 - [\alpha_2 + Y'(N_i)] = 0 \tag{44}$$

which are given by the formula

$$\lambda_{1,2}^{(i)} = \pm\sqrt{(\alpha_2 + Y'(N_i))}\,; \quad \left(Y'(N_i) = \frac{\mathrm{d}Y}{\mathrm{d}N}\bigg|_{N=Ni}\right). \tag{45}$$

The two following cases (45a), (45b) may occur (for fixed i):

$$\alpha_2 + Y'(N_i) > 0\,. \tag{45a}$$

In this case the point $(N_i, 0)$ of the phase plane is an unstable singular point of the "saddle" type [5] to which the trivial unstable solution $N = N_i =$ const. corresponds (physically possible only for $b^* = \infty$).

$$\alpha_2 + Y'(N_i) < 0 \tag{45b}$$

In this case the roots λ_1, λ_2 of the characteristic equation (44) are purely imaginary and adjoint, i.e.

$$\lambda_{1,2}^{(i)} = \pm i\sqrt{|\alpha_2 + Y'(N_i)|} \tag{46}$$

so that the point $(N_i, 0)$ is a stable singular point of the phase plane, of the "centre" type, (but it is not asymptotically stable) and to which the trivial stable solution $N = N_i =$ const. corresponds (possible only for $b^* = \infty$ or $b^* = 0$, but physically not realizable [2] with respect to the boundary conditions (14a)).

Let us make now (for physical reasons [2]) the following two assumptions I, II:

I-There exists an interval I_1 of initial values Y_0

$$I_1 = \mathscr{E}\{Y_0 \mid Y_0^{(1)} < Y_0 < Y_0^{(2)}\} \tag{47}$$

such that if $Y_0 \in I_1$ then equation (43a) has only two real roots $N_2 \geqq 1$, $N_1 > N_2$ such that the singular point $(N_2, 0)$ of the phase plane is a stable centre-point and the singular point $(N_1, 0)$ is an unstable saddle-point.

II-There exists a separatrix phase curve

$$N'(N, N_1) = \pm\sqrt{2\int_{N_1}^{N}[\alpha_2 N + Y(N)]\,\mathrm{d}N} \tag{48}$$

which separates the domain of the phase plane containing the centre-point $(N_2, 0)$ and the periodic phase curves $N'(N, N_0)$ corresponding to the stable solutions of problem (37) and has (except for the value N_1) only one further root N_3 given by the relation

$$N'(N_3, N_1) = 0\,. \tag{48a}$$

Under these two assumptions the initial value problem (37) for all initial values N_0 from the interval I_2 where

(49) $$I_2 = \mathscr{E}\{N_0 | N_3 < N_0 < N_1\}$$

and for an arbitrary right hand side $Y(N; Y_0)$ which is the solution of the initial value problem (37a), has a stable periodical solution $N^* = N[x; N_0, Y(N; Y_0)]$ for which equation (37) is fulfilled and for which the relation

(50) $$N_0^{*''} = \alpha_2 N_0^* + Y_0$$

holds. If a solution $N^{**} = N(x; N_0, N_0'')$ of the nonlinear biharmonic Cauchy's problem (34), (34a) will be used for the construction of the right hand side $g(N^{**})\, N^{**}$ in problem (37a), then there follows from (36) that in this case we shall have

(51) $$N[x; N_0, Y(N; Y_0)] \equiv N^* = N^{**} \equiv N(x; N_0, N_0'')$$

and therefore the stability domain $\Omega \subset R_2$ of the twoparametrical family $N(x; N_0, N_0'')$ of solutions of the Cauchy's problem (34), (34a) must be given by the Cartesian product

(52) $$\Omega \equiv I_2 \times I_3$$

where the interval I_3 for the initial value N_0'' is given (with help of the intervals I_1, I_2 defined by equations (47), (49), respectively), by the relation

(52a) $$I_3 = \mathscr{E}\{N_0'' \mid N_0'' = \alpha_2 N_0 + Y_0; N_0 \in I_2; Y_0 \in I_1\}$$

By the preceding reasoning we have thus proved the following:

Theorem 3. *Let $N = N(x; N_0, N_0'')$ be the twoparametrical family of solutions of the nonlinear biharmonic Cauchy's initial value problem* (34), (34a) *for which the assumptions of Theorem 1 are valid. Let $Y = Y(N)$ be the corresponding solution of problem* (37a).

Let equation (43a) *fulfil Assumption I.*

Let the separatrix phase curve given by equation (48) *fulfil Assumption II.*

Then the stability domain $\Omega \subset R_2$ of the twoparametrical family $N(x; N_0, N_0'')$ of solutions of the Cauchy's problem (34), (34a) *is the rectangle given by relations* (52), (52a).

Remark 2. When in (48a) the condition

(53) $$N_3 \geqq 1$$

holds then there may evidently exist two solutions of the problem of the critical reactor with flattened specific output: one for $N_0 \in (N_3, N_2)$, $N_0'' = \alpha_2 N_0 + Y_0$ for which $N(x; N_0, N_0'')$ has in $x = 0$ a local minimum, and the other one for $N_0 \in$

$\in (N_2, N_1)$, $N_0'' = \alpha_2 N_0 + Y_0$, for which $N(x, N_0, N_0'')$ reaches in $x = 0$ its local maximum.

Remark 3. In the same manner as it has been done for the total reactor output, the optimization of other functionals (interesting from the reactor physicist's viewpoint) with respect to the two parameters N_0, N_0'' in the solution $N(x; N_0, N_0'')$ of the problem of the reactor with flattened specific output can be considered. E.g., optimization of the total fuel loading in the reactor core

$$(54) \qquad Q(\tilde{\tilde{N}}_0, \tilde{\tilde{N}}_0) = \max_{(N_0, N_0'') \in \Omega_2 \subset \Omega} Q(N_0, N_0'') = \max_{(N_0, N_0'') \in \Omega_2} \int_0^{b^*(N_0, N_0'')} M(x; N_0, N_0'')\, \mathrm{d}x =$$

$$= \max_{(N_0, N_0'') \in \Omega_2 \subset \Omega} \int_0^{b^*(N_0, N_0'')} [N(x; N_0, N_0'') - 1]^{-1}\, \mathrm{d}x$$

which requires the following two conditions

$$(55) \qquad \frac{\partial}{\partial N_0} Q(N_0, N_0'') = 0\,; \quad \frac{\partial}{\partial N_0''} Q(N_0, N_0'') = 0\,;$$

$$(56) \qquad \left. \frac{\partial^2}{\partial^2 N_0} Q(N_0, N_0'') \right|_{N_0 = \tilde{\tilde{N}}_0, N_0'' = \tilde{\tilde{N}}_0''} < 0\,;$$

$$\det \left. \begin{pmatrix} \dfrac{\partial^2}{\partial^2 N_0} Q(N_0, N_0'') & \dfrac{\partial^2}{\partial N_0\, \partial N_0''} Q(N_0, N_0'') \\ \dfrac{\partial^2}{\partial N_0''\, \partial N_0} Q(N_0, N_0'') & \dfrac{\partial^2}{\partial^2 N_0''} Q(N_0, N_0'') \end{pmatrix} \right|_{N_0 = \tilde{\tilde{N}}_0, N_0'' = \tilde{\tilde{N}}_0''} > 0$$

to be fulfilled. Another example is the optimization of the mean thermal neutron flux in the reactor with flattened specific output

(57)

$$\bar{\Phi}(N_0, N_0'') = \frac{\int_0^{b^*(N_0, N_0'')} \Phi(x; N_0, N_0'')\, \mathrm{d}x}{b^*(N_0, N_0'')} = C_1 \frac{\int_0^{b^*(N_0, N_0'')} [N(x; N_0, N_0'') - 1]\, \mathrm{d}x}{b^*(N_0, N_0'')} =$$

$$= C_1[\bar{N}(N_0, N_0'') - 1]\,, \quad \left(\text{where } \bar{N}(N_0, N_0'') = \frac{\int_0^{b^*(N_0, N_0'')} N(x; N_0, N_0'')\, \mathrm{d}x}{b^*(N_0, N_0'')} \right)$$

whose optimization

$$(58) \qquad \bar{\Phi}(\tilde{\tilde{\tilde{N}}}_0, \tilde{\tilde{\tilde{N}}}_0) = C_1 \max_{(N_0, N_0'') \in \Omega_2 \subset \Omega} [\bar{N}(N_0, N_0'') - 1]$$

must fulfil two conditions analogous to (55), (56).

The relations between these functionals can be studied in the same way.

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Souhrn

JEDNOROZMĚRNÝ HOMOGENIZOVANÝ REAKTOR S PŘÍRODNÍM URANEM A S VYROVNANÝM MĚRNÝM VÝKONEM

ROSTISLAV ZEZULA

Hlavní výsledky práce jsou formulovány ve třech větách.

Ve větě 1 jsou udány postačující podmínky pro existenci resp. konstrukci takového rozložení $M(x; N_0, N_0'') = [N(x; N_0, N_0'') - 1]^{-1}$ relativní koncentrace paliva v jádře kritického reaktoru, které indukuje tok tepelných neutronů $\Phi(x; N_0, N_0'')$ dávající vyrovnaný měrný výkon $M\Phi = \text{const.}$

Ve větě 2 jsou udány postačující podmínky pro existenci resp. konstrukci optimálních hodnot $\tilde{N}_0$, $\tilde{N}_0''$ parametrů N_0, N_0'', pro které celkový výkon reaktoru $T(N_0, N_0'')$ nabývá svého lokálního maxima na oblasti stability Ω řešení $N(x; N_0, N_0'')$ kvasilineárního biharmonického Cauchyova problému pro reaktor s vyrovnaným měrným výkonem.

Ve větě 3 jsou udány podmínky postačující pro existenci obdélníkové oblasti stability Ω řešení $N(x; N_0, N_0'')$.

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