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TO THE INTERPRETATION OF THE OSCULATIONS OF ORBITS AND THE IN-SPACE LAUNCHING POINT OF ARTIFICIAL COSMIC BODIES

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The foremost interest of the present day theoretical astronautics has been concentrated on the problem of launching artificial cosmic bodies into the prescribed trajectory. A number of contributions deals with the determination of the optimal launching, usually with respect to the minimum fuel consumption. Numerous studies of this kind are due to D. F. Lawden. His work [1] contains 40 references to literature, the symposium [2] presentes 394 references! Some general formulations lead to complicated relations [3] whose solution is restricted only to quite special cases. The present paper investigates osculating transfer between two prescribed trajectories without considering the problem of optimization. This point of view allows a final numerical expression of the general case including the special Hohmann transfer [4] with contacts at the apsidal points which was examined in [3].

As the basis for the analysis of the problem expressed by the title some relations from the geometry of conic sections in polar coordinates have served, in particular the condition for osculation of two confocal conic sections and the equations of the common tangents of two nonosculating confocal conic sections. Respective formulae have been obtained by using a method of graphical solution of a certain trigonometric equation[1]). The mathematical proofs of the results presented in this paper are not given for the sake of brevity. The applications of the relations to the motion of rockets presented in the paper leads, on the basis of very traditional knowledge of conic sections, to quite nontraditional results for the kinematics of launching artificial cosmic bodies. Only general conclusions without secondary details are presented in the paper.

[1]) Cf. "Grafické řešení jedné goniometrické rovnice" (Graphical solution of one goniometric equation), Čas. pěst. mat. *80*, (1955), p. 243.

§ 1. CONDITIONS OF THE OSCULATION

The possibility of transferring a rocket from the known path of departure $(i = 1)$ to the transfer orbit $(i = 0)$ osculating the given trajectory of arrival $(i = 2)$, complanar with the path of departure, has been already investigated. For the transfer one change *only* of the velocity modulus of the rocket is assumed so that the path of departure osculates the transfer orbit as well. Let us consider the case of a triplet of conic sections nondegrading to a straight line

$$r = \frac{p_i}{1 + \varepsilon_i \cos(\varphi - \varphi_i)}, \quad i = 0, 1, 2 .$$

The analysis of the requirements of both osculations leads to the conditions

$$(1) \qquad \varepsilon_0^2 + \frac{\varepsilon_i^2 - 1}{p_i^2} p_0^2 - 2\frac{\varepsilon_i}{p_i} p_0 \varepsilon_0 \cos(\varphi_0 - \varphi_i) + \frac{2}{p_i} p_0 = 0, \quad i = 1, 2$$

expressing the couple of relations for the unknown triplet of orbital elements p_0, ε_0, φ_0. Considering the value φ_0 as given, six quantities are known:

$$A_i = \frac{\varepsilon_i^2 - 1}{p_i^2}, \quad B_i = 2\frac{\varepsilon_i}{p_i} \cos(\varphi_0 - \varphi_i), \quad C_i = \frac{2}{p_i}, \quad i = 1, 2$$

and the determination of p_0, ε_0 is transferred to the solution of one arbitrary equation of

$$(2) \qquad \varepsilon_0^2 + A_i p_0^2 - B_i p_0 \varepsilon_0 + C_i p_0 = 1, \quad i = 1, 2$$

together with the equation

$$(3) \qquad (A_1 - A_2) p_0 - (B_1 - B_2) \varepsilon_0 + (C_1 - C_2) = 0 .$$

Numerical evaluation of couples of both possible roots p_{0j}, ε_{0j}, $j = \mathrm{I, II}$ is not difficult. For polar angles φ_{ij} of the contact points of both the transfer orbits $(j = \mathrm{I, II})$ with the trajectory of arrival $(i = 2)$ and with the path of departure $(i = 1)$, a brief analysis gives the unambiguous determination

$$(4) \qquad \operatorname{tg} \frac{\varphi_{ij}}{2} = \frac{p_{0j} \varepsilon_i \sin \varphi_i - p_i \varepsilon_{0j} \sin \varphi_0}{p_i(1 - \varepsilon_{0j} \cos \varphi_0) - p_{0j}(1 - \varepsilon_i \cos \varphi_i)}; \; i = 1, 2, \quad j = \mathrm{I, II} .$$

Both conditions (1) for the osculation, into which quantities p_0, ε_0, φ_0 are substituted, allow to study further aspects of the investigated problem. A natural demand is to determine the contact point situated either on the path of departure (φ_{01}) or on the trajectory of arrival (φ_{02}). According to (4) it signifies that one of the relations

$$\operatorname{tg} \frac{\varphi_{01}}{2} = \frac{p_0 \varepsilon_i \sin \varphi_i - p_i \varepsilon_i \sin \varphi_0}{p_i(1 - \varepsilon_0 \cos \varphi_0) - p_0(1 - \varepsilon_i \cos \varphi_i)} ; \quad i = 1,2$$

be satisfied. Each of these equations allow to determine, together with the couple (1), the triplet of orbital parameters p_0, ε_0, φ_0. To calculate such a solution may require an iterative procedure, e.g. a successive selection of the angle φ_0. Thus the problem is reduced to (2), (3).

§ 2. STARTING VELOCITY

The point of view used until now was exclusively geometrical. Let us consider the problem kinematically. By the localization of the orbital point r_0, by the tangential angle ϑ_0 and by the value of the corresponding orbital velocity v_0 the transfer orbit is uniquely determined. If we consider r_0, ϑ_0 as given, all the orbital elements can be expressed in dependence on v_0. The result of the corresponding analysis can be expressed by the following Theorem:

Let be

$$r = \frac{p_i}{1 + \varepsilon_i \cos (\varphi - \varphi_i)}$$

for $i = 1, 2$ the equation of the conic section – nondegrading into a straight line – representing the path of departure and the trajectory of arrival, respectively, and for $i = 0$ the transfer orbit osculating these both conic sections. Let us denote by r_0, Φ_0, ϑ_0, v_0 the quantities related to the contact point of the path of departure and of the transfer orbit (the starting point, the in-space launching point): the radius vector, the polar angle, the tangential angle, and the starting velocity, respectively. Then

$$v_0^2 = v_{0\text{II}}^2 \frac{\{p_2 - r_0[1 + \varepsilon_2 \cos (\varphi_0 - \Phi_0)]\} \, p_2}{p_2^2 + (\varepsilon_2 - 1)(r_0 \sin \vartheta_0)^2 - 2 p_2 \varepsilon_2 r_0 \sin (\vartheta_0 + \Phi_0 - \varphi_2)} \tag{5}$$

where $v_{0\text{II}}$ is the value of the second cosmical velocity at the starting point.

If the starting point is inside the trajectory of arrival, there always exists a finite real velocity v_0. If the starting point is outside the trajectory of arrival, there exists a finite real value only if the tangent t of the transfer orbit at the starting point has no common point with the trajectory of arrival. If t is at the same time the tangent of the trajectory of arrival, then $v_0 \to \infty$ and the transfer orbit degrades into a straight line "passed through" at infinitely high speed. If t intersects the trajectory of arrival really in two different points then no real v_0 exists. If the starting point is at the same time a point of the trajectory of arrival, then if it is the osculation *point of the path of departure with the trajectory of arrival as well, the conic section for* $i = 0$ *expresses formally the transfer orbit for any* v_0. *If it is not the osculation point, then* $v_0 = 0$ *and the transfer orbit degrades into a straight line of the free fall to the central body; the concept of osculation loses its sense.*

When deriving the relation (5) it is necessary to consider the singular case of a parabola, for which some general expressions become undefinite. The investigation can be made either by a limit process or by taking other expressions, e.g. instead of $p = \mu(1 - \varepsilon^2)/(2\mu/r - v^2)$ to put $p = (r\, v\sin \vartheta)^2/\mu$. The zero value of the denominator in (5) expresses geometrically the condition that the common tangent of the transfer orbit and of the path of departure is that of the trajectory of arrival. Physically this is the case of a formal straight line transfer orbit. Both statements are evident.

§ 3. STARTING SECTIONS

Up to now we have investigated the problem of existence of the transfer orbit corresponding to a completely definite point of the path of departure. Let us enlarge the problem to the whole course of the path of departure with the aim to look for the possibilities of launching the rocket from the path of departure to the transfer orbit by a convenient change of the velocity modulus without the completely definite localization of the starting point. As it can be expected, the analysis shows that such launching is possible at any point of the path of departure if the latter lies as a whole either inside or outside the trajectory of arrival. In boundary cases when the path of departure itself osculates the trajectory of arrival (inside or outside), it itself fuses with the transfer orbit. On the other hand, in the case of the path of departure intersecting the trajectory of arrival, the transition can be realized from all points of the path situated inside the trajectory of arrival (by decreasing the velocity), but from the points situated outside, the transition (by increasing the velocity) is possible only from the section existing between the contact points of the common tangents of the path of departure and the trajectory of arrival.

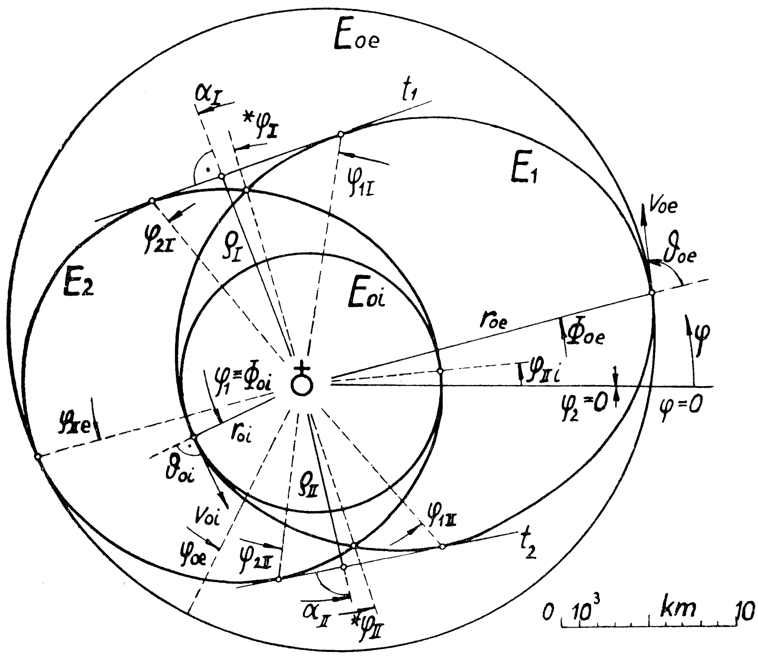
In case of a hyperbolic path such a section need not exist. The analysis of calculating this problem is relatively labourious.

The contents of this § correspond to the concluding part of the preceding §.

§ 4. SYNTHETIC VIEW

The formulation using the concept of the starting velocity was a kinematic one. Regardless of the physically mechanical conception, the whole problem can be formulated purely geometrically:

Six values r_0, Φ_0, ϑ_0, p_1, ε_1, φ_1 determine the tangent t_0 of the searched transfer orbit with the contact point M_0 and the form (a_2, b_2) and the localization of the trajectory of arrival. From this stand-point we try to determine the ellipse[2]) (of the transfer orbit) touching the given ellipse[2]) (the trajectory of arrival), having one common focus F with it and touching the given tangent t_0 at the given point M_0. The concrete solution is evident: Let us denote the second focus of the trajectory of arrival (of the transfer orbit) by $F''(F')$. The bearer of the focus F' is the straight line f passing through the point M_0 symmetrically to the tangent t_0. Let us indicate a point M' on the line f so that $M_0M' = 2a_2 - r_0$ is (respecting the sign of the orientation) plotted from t_0 to the half-plane which contains F. The searched focus F' has then the same distance from the both points F'', M'. Thus the transfer orbit is practically determined. The contact point is situated on the straight line joining the uncommon foci of the transfer orbit and of the trajectory of arrival.



Launching of a rocket from the path of departure E_1 into the trajectory of arrival E_2 through two transfer orbits E_{0e}, E_{0i} having external (e) and internal (i) osculations recpectively.

As a numerical example we choose the path of departure E_1

$$a_1 = 14\,000 \text{ km}, \quad e_1 = 7000 \text{ km}, \quad \varphi_1 = 205°00'00''$$

[2]) Generally a conic section; to give a more concrete idea the word "ellipse" is used.

and the trajectory of arrival E_2

$$a_2 = 12\,000 \text{ km}, \quad e_2 = 4000 \text{ km}, \quad \varphi_2 = 0°00'00''$$

so that

$$b_1 = 12\,124{,}4 \text{ km}, \quad p_1 = 10\,500{,}0 \text{ km}, \quad \varepsilon_1 = 0{,}50000,$$
$$b_2 = 11\,313{,}7 \text{ km}, \quad p_2 = 10\,666{,}7 \text{ km}, \quad \varepsilon_2 = 0{,}33333$$

and the polar angles of their intersections

$$^*\varphi_{\mathrm{I}} = 106°14'28'', \quad ^*\varphi_{\mathrm{II}} = 286°14'26''.$$

The connecting line of centres of both conic sections has the length 10 758,6 km and the direction 15°57′38″ and therefore the normal straight lines of both common tangents $t_{1,2}$ have the length

$$\varrho_{\mathrm{I}} = 12\,758{,}9 \text{ km}, \quad \varrho_{\mathrm{II}} = 10565{,}7 \text{ km}$$

their polar angles being

$$\alpha_{\mathrm{I}} = 109°55'39'', \quad \alpha_{\mathrm{II}} = 281°09'41''$$

and thus the polar angles of the contact points

$$\varphi_{1\mathrm{I}} = 80°02'21'', \quad \varphi_{2\mathrm{I}} = 128°11'28'',$$
$$\varphi_{1\mathrm{II}} = 310°12'50'', \quad \varphi_{2\mathrm{II}} = 262°04'04''.$$

The starting section for the external osculation is then limited on the path of departure by the angles

(*) $$(-49°47'10'', 80°02'21'')$$

and enables launching into the trajectory of arrival into the section

$$(128°11'28'', 262°04'04'').$$

The transfer with the internal osculation can be reached from the path of departure only from the locations in

$$(106°14'28'', 286°14'26'')$$

with possible contact with the trajectory of arrival in

$$(73°45'34'', 106°14'28'').$$

For computing the transfer orbit with the external osculation we choose the centre of the interval (*)

$$\Phi_0 = 15°07'35''$$

with the corresponding launching orbit E_{0e}:

$$r_{0e} = 20\,693{,}8 \text{ km}\,, \quad \vartheta_{0e} = 80°24'33''\,, \quad v_{0\text{II}e} = 6204{,}4 \text{ m/s}$$

so that

$$v_{0e} = 4133{,}6 \text{ m/s}$$

$$a_{0e} = 18\,607{,}4 \text{ km}\,, \quad b_{0e} = 18\,231{,}6 \text{ km}$$

$$e_{0e} = 3\,720{,}5 \text{ km}\,, \quad p_{0e} = 17\,863{,}6 \text{ km}$$

$$\varepsilon_{0e} = 0{,}19994\,, \quad \varphi_{0e} = 241°58'18''$$

with the osculation point on the trajectory of arrival at

$$\varphi_{\text{II}e} = 194°32'26''\,.$$

For the realization of the internal osculation we choose the perigee of the path of departure ($\Phi_{0i} = \varphi_1$)

$$r_{0i} = e_1 = 7000 \text{ km}\,, \quad \vartheta_{0i} = 90°00'00''\,, \quad v_{0\text{II}} = 10\,667{,}8 \text{ m/s}$$

whose apsidal straight line then coincides with the apsidal line of the transfer orbit. Using the value of the starting velocity

$$v_{0i} = 7801{,}6 \text{ m/s}$$

we obtain a nearly circular transfer orbit E_{0i}

$$a_{0i} = 7523{,}3 \text{ km}\,, \quad b_{0i} = 7507{,}1 \text{ km}\,. \quad p_{0i} = 7491{,}1 \text{ km}\,,$$

$$\varepsilon_{01} = 0{,}065498\,, \quad e_{0i} = 492{,}76 \text{ km}$$

with the contact at the point

$$\varphi_{\text{II}i} = 5°21'42''\,.$$

§ 5. COSMICAL RENDEZ-VOUS

The time of duration of the flight of a rocket on the transient orbit can be expressed by using Kepler's equation. The knowledge of positions in dependence on time on the path of departure imposes the question how to realize the cosmical rendez-vous of a rocket describing a conic section – which has been spoken of as the path of departure – with a rocket on a trajectory previously called the trajectory of arrival. A very simple case is that one when the conic section of departure as well as that of arrival reduce to circles.

If we denote the polar angles of the osculation points by φ_{01}, φ_{02}; $\varphi_{02} = \varphi_{01} + \pi$ and the times for each revolution in the circular paths by T_1, T_2, respectively, and the localization of both rockets before the starting moments by corresponding angles ${}_{0i}\varphi$ at times ${}_{0i}t$, $i = 1, 2$, then for the moment t_{02} of the start at the transfer orbit we have

$$t_{02} = \frac{({}_{02}\varphi - {}_{12}\varphi)/2\pi + [(2m + 1)\,\tau_0 - {}_{02}t]/T_2 + {}_{01}t/T_1 + n - \frac{1}{2}}{1/T_1 - 1/T_2};$$

$$n = 0, \pm 1, \pm 2, \ldots, \quad m = 0, 1, 2, \ldots.$$

The quantity τ_0 is the known half-time of the assumable total circulation on the transfer orbit and the presence of a nonnegative integer m shows the possibility of realizing the rendez-vous not only after passing the half of the transfer orbit $(m = 0)$, but also after passing it m-times. The parameter n takes into account the periodicity of passing the trajectory of arrival.

The uniform passing of the departure and arrival conic sections allows the immediate construction of a graphical flight schedule for both rockets with the instructive interpretation of integer parameters m, n. The immediate generalization of this idea to the case of noncircular conic sections leads to a numerically laborious solution.

Literatura

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[2] *G. Leitmann* (editor): Optimization Techniques with Applications to the Aerospace Systems, New York, London, Academic Press 1962.

[3] *D. F. Lawden:* Chapter XI in [2].

[4] *W. Hohmann:* Die Erreichbarkeit der Himmelskörper, München, Oldenbourg 1925.

Souhrn

K INTERPRETACI OSKULAČNÍCH ORBITŮ A NAVÁDĚCÍCH BODŮ UMĚLÝCH KOSMICKÝCH TĚLES

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Jde o převedení rakety z předepsané původní dráhy na přechodový orbit oskulující jinou danou komplanární cílovou raketovou trajektorii. Navedení do přechodového orbitu se uskutečňuje jedinou změnou modulu rychlosti rakety beze změny letového směru, takže i původní dráha má s přechodovým orbitem oskulační styk. Požadavek oskulace dvou kuželoseček vede na numericky pracné řešení algebraicko-goniometrické soustavy. Předpis polohy apsidální přímky přechodového orbitu je zjednodušením na soustavu závislých algebraických rovnic (2) (3) s možností explicitního

vyjádření polárních úhlů oskulačních bodů (4). Vyjadřuje se naváděcí rychlost (5) v závislosti na místě navedení a vymezují se úseky původní dráhy, z nichž je dvoj-oskulační navedení možné (§ 3).

K úloze oskulace se přihlíží rovněž ryze geometricky, bez kinematického hlediska (§ 4). Závěrečný § 5 vyšetřuje kosmické rendez-vous, případ, kdy raketa letící v přechodové dráze prochází oskulačním bodem s cílovou trajektorií tak, že zde právě zastihne raketu z této trajektorie.

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