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ON THE EXISTENCE AND UNIQUENESS OF SOLUTION AND SOME VARIATIONAL PRINCIPLES IN LINEAR THEORIES OF ELASTICITY WITH COUPLE-STRESSES

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I. COSSERAT CONTINUUM

INTRODUCTION

In the last few years a number of theories appeared taking into account the microstructure of materials. One of the simplest and most elaborated is the theory of Cosserat continuum. In this continuum each mass point has six degrees of freedom: three components of the displacement vector and three components of the microrotation vector. For this case a number of simple examples of practical importance was solved, but due to complicated equations to be solved, usually the only way to succeed was to find the stress functions. More complicated problems, however, are to be solved approximately. Variational methods rank among very important approximate methods. The existence, uniqueness and continuous dependence of the solutions upon the given data, and the estimates of errors of some approximate variational solutions are discussed in this part of our paper for the case of Cosserat continuum.

We restrict ourselves only to the static case of bounded bodies, the material being generally anisotropic and inhomogeneous. We define a certain weak solution of a boundary-value problem. On the base of inequalities of KORN's type, following J. NEČAS, I. HLAVÁČEK in [2] we prove the existence, uniqueness and continuous dependence of the weak solution upon the given data. The functionals of potential and complementary energy are defined and the existence of the minima of these functionals in a certain class of functions is proved. These minima are attained by the weak solution and the corresponding stresses, respectively. An estimate of errors of the approximate solution obtained from variational principles is suggested.

Finally, some analogies of the principles of HU-WASHIZU and of HELLINGER-REISSNER are given for Cosserat bodies.

In the second part of our paper we shall deal with some other theories of elasticity with couple-stresses, namely with Mindlin's theory of microstructure and with the first strain -- gradient theory.

1. BOUNDARY-VALUE PROBLEMS FOR ELLIPTIC SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

In this section some definitions and theorems are presented which are taken from [2]. The proofs of these theorems can be found in [1] a [2].

Ω will denote a bounded region with Lipschitz boundary[1]) in the three-dimensional Euclidean space with Cartesian coordinates $X \equiv (x_1, x_2, x_3)$. $L_2(\Omega)$ denotes the space of real functions square-integrable in Ω in the Lebesgue sense. $W_2^{(k)}(\Omega)$ denotes the subspace of $L_2(\Omega)$ of functions whose derivatives up to the order k in the sense of distributions are in $L_2(\Omega)$.

Let us introduce the scalar product on $W_2^{(k)}(\Omega)$ by

$$(v, u)_k = \sum_{|\alpha| \leqq k} \int_\Omega D^\alpha v \; D^\alpha u \, \mathrm{d}X$$

where

$$D^\alpha \equiv \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \, \partial x_2^{\alpha_2} \, \partial x_3^{\alpha_3}} \, .$$

W denotes the Cartesian product $\prod_{s=1}^{m} W_2^{(\varkappa_s)}(\Omega)$, $s = 1, 2, \ldots, m$ where m, $\varkappa_s$ are positive integers. Let a bilinear form $A(\boldsymbol{v}, \boldsymbol{u})$ be given on $W \times W$ in the form

$$A(\boldsymbol{v}, \boldsymbol{u}) = \int_\Omega \sum_{r,s=1}^{m} \sum_{\substack{|i| \leqq \varkappa_r \\ |j| \leqq \varkappa_s}} a_{ij}^{rs} D^i v_r D^j u_s \, \mathrm{d}X$$

where $a_{ij}^{rs}(X)$ are real bounded and measurable functions on Ω. Furthermore, let $\mathscr{D}(\Omega)$ be the space of real functions with compact support in Ω, which are differentiable any times. Let $\mathring{W}_2^{(k)}(\Omega)$ denote the closure of $\mathscr{D}(\Omega)$ in $W_2^{(k)}(\Omega)$, $\mathring{W} = \prod_{s=1}^{m} \mathring{W}_2^{(\varkappa_s)}(\Omega)$.

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[1]) We call the boundary Γ *Lipschitz* if

a) to each point $X \in \Gamma$ such open sphere S_X with the centre X exists, that the intersection $S_X \cap \Gamma$ may be described by means of a Lipschitz function and

b) $S_X \cap \Gamma$ divides S_X into external and internal parts with respect to Ω.

Let V be a closed subspace of W such that $\overset{\circ}{W} \subset V \subset W$. Let us define functionals

$$f(v) = \int_{\Omega} \sum_{s=1}^{m} f_s v_s \, dX, \quad f_s \in L_2(\Omega), \quad v \in W,$$

$$g(v) = \int_{\Gamma'} \sum_{s=1}^{m} g_s v_s \, d\Gamma + \int_{B} \sum_{s=1}^{m} G_s v_s \, dB, \quad g_s \in L_2(\Gamma'), \quad G_s \in L_2(B), \quad v \in W$$

and if $\varkappa_s < 2$, then $G_s \equiv 0$.

Here $L_2(\Gamma')$ and $L_2(B)$ denote the spaces of real functions square-integrable on $\Gamma' \subset \Gamma$ and $B \subset \Gamma$, respectively. B is a one-dimensional set (e.g. consisting of curves) of a finite measure. The theorems of embedding imply that the functionals $f(v)$, $g(v)$ are continuous on W.

Definition 1.1. Let $\bar{u} \in W$. We say that $u \in W$ is a weak solution of the boundary-value problem if

$$u - \bar{u} \in V$$

and for each $v \in V$

(1.1) $$A(v, u) = f(v) + g(v).$$

Let the operators $N_l v$ $(l = 1, 2, \ldots, h)$ mapping W into $L_2(\Omega)$ be given in the form

$$N_l v = \sum_{s=1}^{m} \sum_{|\alpha| \leq \varkappa_s} n_{ls\alpha} D^\alpha v_s$$

where $n_{ls\alpha}(X)$ are bounded and measurable on Ω.

Definition 1.2. We say that the *operators $N_l v$ form a coercive system on W* if for each $v \in W$

(1.2) $$\sum_{l=1}^{h} |N_l v|_{L_2}^2 + \sum_{s=1}^{m} |v_s|_{L_2}^2 \geqq c_1 |v|_W^2, \quad c_1 > 0$$

holds where c_1 does not depend on v, $| \ |_{L_2}$ and $| \ |_W$ denote the usual norms in $L_2(\Omega)$ and W, respectively.

Theorem 1.1. *Let $n_{ls\alpha}$ be constants for $|\alpha| = \varkappa_s$. Then the system $N_l v$ is coercive on W if and only if the rank of the matrix*

(1.3) $$N_{ls}\xi \equiv \sum_{|\alpha| = \varkappa_s} n_{ls\alpha} \xi_\alpha$$

equals m for each $\xi \neq 0$, $\xi \in C_3$ where C_3 denotes the complex three-dimensional space and $\xi_\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \xi_3^{\alpha_3}$

Let us suppose that for each $\boldsymbol{v} \in W$

$$(1.4) \qquad A(\boldsymbol{v}, \boldsymbol{v}) \geqslant c_2 \sum_{l=1}^{h} |N_l \boldsymbol{v}|_{L_2}^2 , \quad c_2 > 0$$

and c_2 does not depend on $\boldsymbol{v}$.

Let us denote

$$\mathscr{P} = \{\boldsymbol{v} \in V, \sum_{l=1}^{h} |N_l \boldsymbol{v}|_{L_2}^2 = 0\} .$$

Let $V/\mathscr{P}$ be the factor-space of classes $\tilde{\boldsymbol{v}} = \{\boldsymbol{v} + \boldsymbol{p}, \boldsymbol{v} \in V, \boldsymbol{p} \in \mathscr{P}\}$ with the norm

$$|\tilde{\boldsymbol{v}}|_{V/\mathscr{P}} = \inf_{\boldsymbol{p} \in \mathscr{P}} |\boldsymbol{v} + \boldsymbol{p}|_W .$$

Theorem 1.2. *Let*

$$(1.5) \qquad A(\boldsymbol{v}, \boldsymbol{u}) \equiv [\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{u}}]$$

define a bilinear form for each $\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{u}} \in W/\mathscr{P}$, $\boldsymbol{u} \in \tilde{\boldsymbol{u}}$, $\boldsymbol{v} \in \tilde{\boldsymbol{v}}$. *Let* (1.2), (1.4) *hold. Then the necessary and sufficient condition for the existence of a weak solution to the boundary-value problem is*

$$(1.6) \qquad \boldsymbol{p} \in \mathscr{P} \Rightarrow f(\boldsymbol{p}) + g(\boldsymbol{p}) = 0 .$$

The solution is determined except for an element $\boldsymbol{p} \in \mathscr{P}$. *Furthermore, for the weak solution* $\boldsymbol{u} \in W$ *the estimate*

$$(1.7) \qquad |\boldsymbol{u}|_{W/\mathscr{P}} \leqslant c_3 \left[|\bar{\boldsymbol{u}}|_W + \left(\sum_{s=1}^{m} |f_s|_{L_2(\Omega)}^2 \right)^{1/2} + \left(\sum_{s=1}^{m} |g_s|_{L_2(\Gamma_f)}^2 \right)^{1/2} + \right.$$
$$\left. + \left(\sum_{s=1}^{m} |G_s|_{L_2(B)}^2 \right)^{1/2} \right] , \quad c_3 > 0$$

holds. Further, for each $\tilde{\boldsymbol{v}} \in W/\mathscr{P}$ *there is*

$$(1.8) \qquad A(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{v}}) \geqslant c_4 |\tilde{\boldsymbol{v}}|_{W/\mathscr{P}}^2 , \quad c_4 > 0 ,$$

which we call the inequality of Korn's type.

Theorem 1.3. *Let* (1.2), (1.4) *and* (1.6) *hold. Let* $p_i(\boldsymbol{v})$, $i = 1, 2, \ldots, k$, *be linearly independent functionals on* $\mathscr{P}$, *i.e. if for each* $\boldsymbol{p} \in \mathscr{P}$

$$\sum_{i=1}^{k} \alpha_i \, p_i(\boldsymbol{p}) = 0 , \quad \textit{then} \quad \sum_{i=1}^{k} \alpha_i^2 = 0 ,$$

α_i *being real numbers. Furthermore, let*

$$\boldsymbol{p} \in \mathscr{P} , \quad \sum_{i=1}^{k} p_i^2(\boldsymbol{p}) = 0 \Rightarrow \boldsymbol{p} = \boldsymbol{0} .$$

Let us define

$$V_p = \{v \in V, \sum_{i=1}^{k} p_i^2(v) = 0\}.$$

Then there exists one and only one weak solution $\boldsymbol{u}$ *of the problem such that*

$$\boldsymbol{u} - \bar{\boldsymbol{u}} \in V_p .$$

There holds

$$|\boldsymbol{u}|_W \leqq c_5 \left[|\bar{\boldsymbol{u}}|_W + \left(\sum_{s=1}^{m} |f_s|^2_{L_2(\Omega)} \right)^{1/2} + \left(\sum_{s=1}^{m} |g_s|^2_{L_2(\Gamma')} \right)^{1/2} + \right.$$
$$\left. + \left(\sum_{s=1}^{m} |G_s|^2_{L_2(B)} \right)^{1/2} \right], \quad c_5 > 0 . \tag{1.9}$$

Furthermore, for $v \in V_p$ *an inequality of Korn's type*

$$A(v, v) \geqq c_6 |v|^2_W , \quad c_6 > 0 \tag{1.10}$$

holds.

Let the suppositions of Theorem 1.2 and (1.6) hold. Furthermore, let

$$A(v, u) = A(u, v) , \quad v, u \in W . \tag{1.11}$$

Define the quadratic functional on V by

$$\Phi(v) = A(v, v) - 2\{f(v) + g(v) - A(v, \bar{u})\} \tag{1.12}$$

Let $u \in W$ be such that

$$u - \bar{u} \in V .$$

Then from (1.12) there follows

$$\Phi(u - \bar{u}) = A(u, u) - 2f(u) - 2g(u) + \Phi_1(\bar{u}) \tag{1.13}$$

with

$$\Phi_1(\bar{u}) = -A(\bar{u}, \bar{u}) + 2f(\bar{u}) + 2g(\bar{u}) .$$

If $\mathring{u} \in W$ is a weak solution to the boundary-value problem, then

$$\mathring{u} - \bar{u} \equiv w \in V$$

and (1.1) can be written in the form

$$A(v, w) = f(v) + g(v) - A(v, \bar{u}) , \quad v \in V .$$

Then (1.12) and (1.11) yield

$$\Phi(v) = A(v, v) - 2A(v, w) = A(v - w, v - w) - A(w, w) . \tag{1.14}$$

(1.14) and (1.8) imply that $\Phi(\boldsymbol{v})$ attains its minimum on V if and only if

$$\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{p} = \mathring{\boldsymbol{u}} - \bar{\boldsymbol{u}} + \boldsymbol{p}\,, \quad \boldsymbol{p} \in \mathscr{P}\,.$$

As $\Phi_1(\bar{\boldsymbol{u}})$ does not depend on $\boldsymbol{u}$, *the functional* $\mathscr{L}(\boldsymbol{u})$ *defined by*

$$\mathscr{L}(\boldsymbol{u}) = \tfrac{1}{2}A(\boldsymbol{u}, \boldsymbol{u}) - f(\boldsymbol{u}) - g(\boldsymbol{u}) \tag{1.15}$$

attains its minimum on the set

$$\boldsymbol{u} = \bar{\boldsymbol{u}} \oplus V$$

if and only if

$$\boldsymbol{u} = \mathring{\boldsymbol{u}} + \boldsymbol{p}\,, \quad \boldsymbol{p} \in \mathscr{P}\,, \quad i.e. \quad \tilde{\boldsymbol{u}} = \tilde{\mathring{\boldsymbol{u}}}\,.$$

In order to avoid the use of the classes $\tilde{\boldsymbol{u}}$, $\tilde{\boldsymbol{v}}$ we can make use of subspaces V_p as introduced in Theorem 1.3. Let the suppositions of Theorem 1.3 and (1.11) hold.

Let us define the quadratic functional on V_p by (1.12). In the same way as above and using (1.10), we obtain that *the functional*

$$\mathscr{L}(\boldsymbol{u}) = \tfrac{1}{2}A(\boldsymbol{u}, \boldsymbol{u}) - f(\boldsymbol{u}) - g(\boldsymbol{u})$$

attains its minimum on the set

$$\bar{\boldsymbol{u}} \oplus V_p$$

if and only if

$$\boldsymbol{u} = \mathring{\boldsymbol{u}}$$

where $\mathring{\boldsymbol{u}}$ *is the (unique) weak solution in* $\bar{\boldsymbol{u}} \oplus V_p$.

2. THE BASIC EQUATIONS OF COSSERAT CONTINUUM

For the equations of equilibrium and the geometrical equations see for example [5] or [6], for the constitutive equations of anisotropic bodies see [7].

We write the equations of static equilibrium in the form

$$\tau_{ji,j} + X_i = 0\,, \tag{2.1}$$

$$m_{ji,j} + \varepsilon_{ijk}\tau_{jk} + Y_i = 0 \tag{2.2}$$

and the geometrical equations in the form

$$\gamma_{ij} = u_{j,i} - \varepsilon_{ijk}\varphi_k\,, \tag{2.3}$$

$$\varkappa_{ij} = \varphi_{j,i}\,. \tag{2.4}$$

Here the indices i, j, k etc. take the values 1, 2, 3 and summing is assumed over the couples of the same indices. The comma denotes partial differentiation, as usual. ε_{ijk} stands for the usual alternator, X_i and Y_i represent the volume density of the body forces and the body couples, respectively. τ_{ij}, m_{ij}, u_i, φ_k in this order, designate the asymmetrical stress tensor, the couple-stress tensor, the displacement vector and the vector of micro-rotation, φ_i are independent of u_i. γ_{ij} and $\varkappa_{ij}$ are called the strain tensor and the curvature-twist tensor, respectively.

The consistitutive equations for the anisotropic inhomogeneous material are written in the form

$$\tau_{ij} = E_{ijkl}\gamma_{kl} + K_{ijkl}\varkappa_{kl}\,, \tag{2.5}$$

$$m_{ij} = K_{klij}\gamma_{kl} + M_{ijkl}\varkappa_{kl}\,, \tag{2.6}$$

with

$$E_{ijkl} = E_{klij}\,, \quad M_{ijkl} = M_{klij}\,, \tag{2.7}$$

E_{ijkl}, K_{ijkl}, M_{ijkl} being bounded and measurable functions of X defined on $\bar{\Omega} = \Omega \cap \Gamma$.

The energy of deformation per unit volume $\mathscr{A}$ is given by the quadratic form

$$2\mathscr{A}(\gamma_{ij}, \varkappa_{ij}) = E_{ijkl}\gamma_{ij}\gamma_{kl} + 2K_{ijkl}\gamma_{ij}\varkappa_{kl} + M_{ijkl}\varkappa_{ij}\varkappa_{kl}\,. \tag{2.8}$$

We suppose that the form (2.8) *is uniformly positive definite*, i.e. there exists $c > 0$ such that for each $X \in \bar{\Omega}$

$$\mathscr{A}(\gamma_{ij}, \varkappa_{ij}) \geqq c \sum_{i,j=1}^{3} (\gamma_{ij}^2 + \varkappa_{ij}^2)\,. \tag{2.9}$$

The matrix of the system of equations (2.5), (2.6)

$$\begin{Vmatrix} (E_{ijkl}), & (K_{ijkl}) \\ (K_{klij}), & (M_{ijkl}) \end{Vmatrix} \tag{2.10}$$

is symmetric and represents at the same time the matrix of the quadratic form (2.8). As the determinant of (2.10) by virtue of (2.9) is greater than a certain number $c' > 0$ for each $X \in \bar{\Omega}$, (c' does not depend on X), there exists the inverse matrix to (2.10) and it is symmetric, as well. Consequently, we can solve equations (2.5), (2.6) for γ_{ij}, $\varkappa_{ij}$ in the form

$$\gamma_{ij} = P_{ijkl}\tau_{kl} + Q_{ijkl}m_{kl}\,, \tag{2.11}$$

$$\varkappa_{ij} = Q_{klij}\tau_{kl} + S_{ijkl}m_{kl}\,, \tag{2.12}$$

where

$$P_{ijkl} = P_{klij}\,, \quad S_{ijkl} = S_{klij}\,. \tag{2.13}$$

P_{ijkl}, Q_{ijkl}, S_{ijkl} are bounded and measurable functions of X on $\bar{\Omega}$. Using (2.5), (2.6), (2.11), (2.12), we can rewrite (2.8) as follows:

$$(2.14) \qquad 2\mathscr{A}(\gamma_{ij}, \varkappa_{ij}) = 2\overline{\mathscr{A}}(\tau_{ij}, m_{ij}) = P_{ijkl}\tau_{ij}\tau_{kl} + 2Q_{ijkl}\tau_{ij}m_{kl} + S_{ijkl}m_{ij}m_{kl} .$$

Let $\Gamma_{u^n}, \Gamma_{u^t}, \Gamma_{T^n}, \Gamma_{T^t}, \Gamma_{\varphi^n}, \Gamma_{\varphi^t}, \Gamma_{M^n}, \Gamma_{M^t} \subset \Gamma$ be either open in Γ or empty sets such that the following mutually disjoint decompositions hold (equalities being valid except for sets of the zero surface measure)

$$\Gamma = \Gamma_{u^n} \cup \Gamma_{T^n} = \Gamma_{u^t} \cup \Gamma_{T^t} = \Gamma_{\varphi^n} \cup \Gamma_{M^n} = \Gamma_{\varphi^t} \cup \Gamma_{M^t} .$$

Let the functions

$$(2.15a) \qquad \bar{u}_i \in W_2^{(1)}(\Omega) , \quad \bar{\varphi}_i \in W_2^{(1)}(\Omega) , \quad i = 1, 2, 3 ,$$

$$(2.15b) \quad \bar{T}^n \in L_2(\Gamma_{T^n}) , \quad \bar{T}_i^t \in L_2(\Gamma_{T^t}) , \quad \bar{M}^n \in L_2(\Gamma_{M^n}) , \quad \bar{M}_i^t \in L_2(\Gamma_{M^t}) , \quad i = 1, 2, 3$$

be given and let the following boundary conditions be prescribed:

$$(2.16a) \qquad \bar{u}^n = u^n \quad \text{on} \quad \Gamma_{u^n}$$

$$(2.16b) \qquad \bar{u}_i^t = u_i^t \quad \text{on} \quad \Gamma_{u^t}$$

$$(2.16c) \qquad \bar{\varphi}^n = \varphi^n \quad \text{on} \quad \Gamma_{\varphi^n}$$

$$(2.16d) \qquad \bar{\varphi}_i^t = \varphi_i^t \quad \text{on} \quad \Gamma_{\varphi^t}$$

$$(2.17a) \qquad \bar{T}^n = T^n \quad \text{on} \quad \Gamma_{T^n}$$

$$(2.17b) \qquad \bar{T}_i^t = T_i^t \quad \text{on} \quad \Gamma_{T^t}$$

$$(2.17c) \qquad \bar{M}^n = M^n \quad \text{on} \quad \Gamma_{M^n}$$

$$(2.17d) \qquad \bar{M}_i^t = M_i^t \quad \text{on} \quad \Gamma_{M^t} .$$

Here n_i denotes the unit outward normal to Γ, the indices n or t denote the normal or tangential components of a vector into the direction of the outward normal n_i or of the tangential plane to Γ, respectively, i.e.

$$\bar{u}^n = \bar{u}_j n_j , \quad \bar{u}_i^t = \bar{u}_i - \bar{u}_j n_j n_i , \quad u^n = u_j n_j , \quad u_i^t = u_i - u_j n_j n_i ,$$
$$T^n = \tau_{jk} n_j n_k , \quad T_i^t = \tau_{ji} n_j - \tau_{jk} n_j n_k n_i$$

and similarly for φ_i, m_{jk}.

Now we deduce the principle of virtual work: Let X_i, $Y_i \in L_2(\Omega)$ and (2.15a), (2.15b) hold. We say that $\{u_i, \varphi_i\}$ *is a kinematically admissible displacement and micro-rotational field*, if u_i, $\varphi_i \in W_2^{(1)}(\Omega)$ meet the kinematic boundary conditions (2.16a)–(2.16d) in the sense of traces. We say that $\{\tau_{ij}, m_{ij}\}$ *is a statically admissible stress field*, if τ_{ij}, $m_{ij} \in W_2^{(1)}(\Omega)$ meet (2.1), (2.2) in the sense of $L_2(\Omega)$ and the boundary conditions (2.17a)–(2.17d) in the sense of $L_2(\Gamma)$. If $\{\tau_{ij}, m_{ij}\}$ and $\{u_i, \varphi_i\}$ are a statically admissible stress field and a kinematically admissible displacement and micro-rotational field, respectively, then using (2.1), (2.2) and the divergence theorem, we obtain

$$(2.18) \qquad \int_\Omega [\tau_{ij}(u_{j,i} - \varepsilon_{ijk}\varphi_k) + m_{ij}\varphi_{j,i}]\,\mathrm{d}X = \int_\Omega (X_i u_i + Y_i\varphi_i)\,\mathrm{d}X +$$

$$\int_{\Gamma_{Tn}} \overline{T}^n u^n \,\mathrm{d}\Gamma + \int_{\Gamma_{Tt}} \overline{T}_i^t u_i^t \,\mathrm{d}\Gamma + \int_{\Gamma_{Mn}} \overline{M}^n \varphi^n \,\mathrm{d}\Gamma + \int_{\Gamma_{Mt}} \overline{M}_i^t \varphi_i^t \,\mathrm{d}\Gamma +$$

$$\int_{\Gamma_{un}} T^n \bar{u}^n \,\mathrm{d}\Gamma + \int_{\Gamma_{ut}} T_i^t \bar{u}_i^t \,\mathrm{d}\Gamma + \int_{\Gamma_{\varphi n}} M^n \bar{\varphi}^n \,\mathrm{d}\Gamma + \int_{\Gamma_{\varphi t}} M_i^t \bar{\varphi}_i^t \,\mathrm{d}\Gamma \,.$$

Equation (2.18) which holds for arbitrary, in general mutually independent statically and kinematically admissible fields of stress and displacement and micro-rotation, respectively, expresses the p r i n c i p l e o f v i r t u a l w o r k f o r C o s s e r a t c o n t i n u u m.[2])

3. THE EXISTENCE AND UNIQUENESS OF THE WEAK SOLUTION OF THE BOUNDARY-VALUE PROBLEMS FOR COSSERAT CONTINUUM

Now let us make the following choice of the quantities introduced in Section 1: Let $m = 6$, $\varkappa_s = 1$, $s = 1, 2, \ldots, 6$. The components of the displacement vector are denoted by u_i or v_i, the components of the micro-rotation vector by φ_i or ψ_i. We denote

$$\{u_1, u_2, u_3, \varphi_1, \varphi_2, \varphi_3\} \equiv \{u_i, \varphi_i\} \equiv \boldsymbol{u}\,,$$
$$\{v_1, v_2, v_3, \psi_1, \psi_2, \psi_3\} \equiv \{v_i, \psi_i\} \equiv \boldsymbol{v}$$

so that $\boldsymbol{u}$ or $\boldsymbol{v}$ represents a displacement and micro-rotation field. W is defined as the space of elements

$$\boldsymbol{u} = \{u_i, \varphi_i\}\,, \quad u_i, \varphi_i \in W_2^{(1)}(\Omega)$$

[2]) In case that the elastic coefficients $E_{ijkl}(X)$, $K_{ijkl}(X)$, $M_{ijkl}(X)$ are piecewise continuous with jump discontinuities on a finite number of surfaces (as it is the case for example with layered bodies), the real τ_{ij}, m_{ij} do not belong to $W_2^{(1)}(\Omega)$ for all $i, j = 1, 2, 3$. In order to derive the principle of virtual work, we define the statically admissible stress field as follows: Let $\bar{\Omega}$ can be divided into a finite number of subregions Ω_S so that

$$\bar{\Omega} = \bigcup_S \bar{\Omega}_S\,, \quad \Omega_i \cap \Omega_j = \emptyset \quad \text{for} \quad i \neq j\,.$$

Let E_{ijkl}, K_{ijkl}, M_{ijkl} be continuous in every Ω_S. Let τ_{ij}, $m_{ij} \in W_2^{(1)}(\Omega_S)$ satisfy equations (2.1), (2.2) in every Ω_S in the sense of $L_2(\Omega_S)$ and let on all surfaces of discontinuities

$$(\tau_{ij}n_i)' = -(\tau_{ij}n_i)''\,, \quad (m_{ij}n_i)' = -(m_{ij}n_i)''$$

hold for the limits from one and the other side of these surfaces. Then using the divergence theorem for every Ω_S we obtain again (2.18) because the corresponding surface integrals cancel out.

with the norm

$$(3.1) \qquad |\boldsymbol{u}|_W^2 = \sum_{i=1}^{3} \left(|u_i|^2_{W_2^{(1)}(\Omega)} + |\varphi_i|^2_{W_2^{(1)}(\Omega)}\right).$$

V will be the subspace of W of all elements $\boldsymbol{v} \in W$ which meet the homogeneous boundary conditions (2.16a)–(2.16d), i.e. for

$$\bar{u}^n = \bar{u}_i^t = \bar{\varphi}^n = \bar{\varphi}_i^t = 0$$

(in the sense of traces). The bilinear form $A(\boldsymbol{v}, \boldsymbol{u})$ on $W \times W$ is defined by

$$(3.2) \qquad A(\boldsymbol{v}, \boldsymbol{u}) = \int_\Omega [E_{ijkl}\, \gamma_{ij}(\boldsymbol{v})\, \gamma_{kl}(\boldsymbol{u}) + K_{ijkl}\{\gamma_{ij}(\boldsymbol{u})\, \varkappa_{kl}(\boldsymbol{v}) + \gamma_{ij}(\boldsymbol{v})\, \varkappa_{kl}(\boldsymbol{u})\} + \\ + M_{ijkl}\, \varkappa_{ij}(\boldsymbol{v})\, \varkappa_{kl}(\boldsymbol{u})]\, \mathrm{d}X$$

where

$$\gamma_{ij}(\boldsymbol{v}) = v_{j,i} - \varepsilon_{ijk}\psi_k\,, \quad \varkappa_{ij}(\boldsymbol{v}) = \psi_{j,i}\,,$$
$$\gamma_{ij}(\boldsymbol{u}) = u_{j,i} - \varepsilon_{ijk}\varphi_k\,, \quad \varkappa_{ij}(\boldsymbol{u}) = \varphi_{j,i}\,,$$
$$\boldsymbol{u} = \{u_i, \varphi_i\} \in W\,, \quad \boldsymbol{v} = \{v_i, \psi_i\} \in W\,.$$

E_{ijkl}, K_{ijkl}, M_{ijkl} are bounded measurable functions in $\bar{\Omega}$ which meet (2.7). Obviously

$$A(\boldsymbol{u}, \boldsymbol{v}) = A(\boldsymbol{v}, \boldsymbol{u})\,, \quad A(\boldsymbol{u}, \boldsymbol{u}) = 2\int_\Omega \mathscr{A}(\gamma_{ij}(\boldsymbol{u}), \varkappa_{ij}(\boldsymbol{u}))\, \mathrm{d}X\,.$$

Let us define the functionals

$$(3.3) \qquad f(\boldsymbol{v}) = \int_\Omega (X_i v_i + Y_i \psi_i)\, \mathrm{d}X\,, \quad \boldsymbol{v} \in W\,, \quad X_i, Y_i \in L_2(\Omega)\,,$$

$$(3.4) \qquad g(\boldsymbol{v}) = \int_{\Gamma_{Tn}} \bar{T}^n v^n\, \mathrm{d}\Gamma + \int_{\Gamma_{Tt}} \bar{T}_i^t v_i^t\, \mathrm{d}\Gamma + \int_{\Gamma_{Mn}} \bar{M}^n v^n\, \mathrm{d}\Gamma + \int_{\Gamma_{Mt}} \bar{M}_i^t v_i^t\, \mathrm{d}\Gamma\,, \quad \boldsymbol{v} \in W$$

where all the quantities in (3.4) are defined in (2.15), (2.16a)–(2.17d).

Now let us define the weak solution in the sense of Definition 1.1.

Definition 3.1. We say that $\boldsymbol{u} \in W$ *is a weak solution of the boundary-value problem* if

$$\boldsymbol{u} - \bar{\boldsymbol{u}} \in V \quad \text{where} \quad \bar{\boldsymbol{u}} = \{\bar{u}_i, \bar{\varphi}_i\}$$

and if

$$(3.5) \qquad \int_\Omega [E_{ijkl}\, \gamma_{ij}(\boldsymbol{v})\, \gamma_{kl}(\boldsymbol{u}) + K_{ijkl}\{\gamma_{ij}(\boldsymbol{u})\, \varkappa_{kl}(\boldsymbol{v}) + \gamma_{ij}(\boldsymbol{v})\, \varkappa_{kl}(\boldsymbol{u})\} + \\ + M_{ijkl}\, \varkappa_{ij}(\boldsymbol{v})\, \varkappa_{kl}(\boldsymbol{u})]\, \mathrm{d}X = \int_\Omega (X_i v_i + Y_i \psi_i)\, \mathrm{d}X +$$

$$+ \int_{\Gamma_{Tn}} \bar{T}^n v^n \,\mathrm{d}\Gamma + \int_{\Gamma_{Tt}} \bar{T}_i^t v_i^t \,\mathrm{d}\Gamma + \int_{\Gamma_{Mn}} \bar{M}^n v^n \,\mathrm{d}\Gamma + \int_{\Gamma_{Mt}} \bar{M}_i^t v_i^t \,\mathrm{d}\Gamma$$

holds for each $\boldsymbol{v} = \{v_i, \psi_i\} \in V$.

Note the connection between this definition of the weak solution, the principle of virtual work and the principle of virtual displacement and micro-rotation. Let $\boldsymbol{u}$ be a kinematically admissible displacement and micro-rotation field such that, using (2.3)–(2.6), we obtain a statically admissible stress field $\{\tau_{ij}(\boldsymbol{u}), m_{ij}(\boldsymbol{u})\}$. Applying the principle of virtual work to these $\{\tau_{ij}(\boldsymbol{u}), m_{ij}(\boldsymbol{u})\}$ and first to the field $\boldsymbol{u}$ and second to a kinematically admissible displacement and micro-rotation field $\boldsymbol{u} + \boldsymbol{v}$, $\boldsymbol{v} \in V$ and then subtracting, we derive just (3.5), i.e. $\boldsymbol{u}$ is the weak solution in the sense of Definition 3.1. On the contrary, if $\boldsymbol{u}$ is a weak solution then $\tau_{ij}(\boldsymbol{u}), m_{ij}(\boldsymbol{u}) \in \in L_2(\Omega)$, but they need not belong to $W_2^{(1)}(\Omega)$. That is why we call this solution weak. The definition of the weak solution expresses the principle of virtual displacement and microrotation, interpreting $\boldsymbol{v}$ as the variations $\delta\boldsymbol{u}$ and $\boldsymbol{u} + \boldsymbol{v}$, $\boldsymbol{v} \in V$ as "virtual" (i.e. kinematically admissible) displacement and microrotation fields.

Now it remains to make a choice of operators $N_l\boldsymbol{v}$. We choose

$$\begin{aligned}
(3.6) \qquad & N_1\boldsymbol{v} = \gamma_{11}(\boldsymbol{v}), \quad N_2\boldsymbol{v} = \gamma_{12}(\boldsymbol{v}), \quad N_3\boldsymbol{v} = \gamma_{13}(\boldsymbol{v}),\\
& N_4\boldsymbol{v} = \varkappa_{11}(\boldsymbol{v}), \quad N_5\boldsymbol{v} = \varkappa_{12}(\boldsymbol{v}), \quad N_6\boldsymbol{v} = \varkappa_{13}(\boldsymbol{v}),\\
& N_7\boldsymbol{v} = \gamma_{21}(\boldsymbol{v}), \quad N_8\boldsymbol{v} = \gamma_{22}(\boldsymbol{v}), \quad N_9\boldsymbol{v} = \gamma_{23}(\boldsymbol{v}),\\
& N_{10}\boldsymbol{v} = \varkappa_{21}(\boldsymbol{v}), \quad N_{11}\boldsymbol{v} = \varkappa_{22}(\boldsymbol{v}), \quad N_{12}\boldsymbol{v} = \varkappa_{23}(\boldsymbol{v}),\\
& N_{13}\boldsymbol{v} = \gamma_{31}(\boldsymbol{v}), \quad N_{14}\boldsymbol{v} = \gamma_{32}(\boldsymbol{v}), \quad N_{15}\boldsymbol{v} = \gamma_{33}(\boldsymbol{v}),\\
& N_{16}\boldsymbol{v} = \varkappa_{31}(\boldsymbol{v}), \quad N_{17}\boldsymbol{v} = \varkappa_{32}(\boldsymbol{v}), \quad N_{18}\boldsymbol{v} = \varkappa_{33}(\boldsymbol{v}).
\end{aligned}$$

Then

$$\int_{\Omega} \sum_{i,j=1}^{3} [\gamma_{ij}^2(\boldsymbol{v}) + \varkappa_{ij}^2(\boldsymbol{v})] \,\mathrm{d}X = \sum_{l=1}^{18} |N_l\boldsymbol{v}|_{L_2}^2$$

and (2.9), (3.2) imply (1.4). It follows from Theorem 1.1 that the *system of operators* (3.6) *is coercive on* $\boldsymbol{W}$, because the matrix (1.3) is composed of three diagonal matrices $\xi_i\boldsymbol{E}$, $\boldsymbol{E}$ being the unit matrix, so that the rank of (1.3) is 6 for each vector $\xi \in C_3$, $\xi \neq 0$.

According to the definition of $\mathscr{P}$, for each $\boldsymbol{v} \in \mathscr{P}$ there holds

$$(3.7) \qquad \gamma_{ij}(\boldsymbol{v}) = \varkappa_{ij}(\boldsymbol{v}) = 0$$

almost everywhere in Ω.

Hence (3.2) defines a bilinear form $[\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{u}}]$ on $W/\mathscr{P} \times W/\mathscr{P}$, all the suppositions of Theorem 1.2 are satisfied and (1.6) is the necessary and sufficient condition for the existence of the weak solution as defined in Definition 3.1. Similarly to the case of the classical elasticity (cf. Lemma II.1 in [2]), we can prove that

$$\mathscr{P} = \{v \in V, v_k = a_k + \varepsilon_{klm} b_l x_m, \psi_k = b_k\}, \quad a_k = \text{const.}, \quad b_k = \text{const.} \tag{3.8}$$

The elements of $\mathscr{P}$ are displacements and micro-rotations of the rigid body, which satisfy the homogeneous kinematic boundary conditions (2.16a)–(2.16d).

Let us investigate some important cases of boundary-value problems.

Theorem 3.1. *Let $\mathscr{P} = \{0\}$. There exists one and only one weak solution $\boldsymbol{u} \in W$ and the estimate*

$$|\boldsymbol{u}|_W \leqq c\,[(\sum_{i=1}^{3} \{|\bar{u}_i|^2_{W_2^{(1)}(\Omega)} + |\bar{\varphi}_i|^2_{W_2^{(1)}(\Omega)}\})^{1/2} + (\sum_{i=1}^{3} \{|X_i|^2_{L_2(\Omega)} + |Y_i|^2_{L_2(\Omega)}\})^{1/2} +$$
$$+ (|\overline{T}^n|^2_{L_2(\Gamma_{T^n})} + \sum_{i=1}^{3} |\overline{T}^t_i|^2_{L_2(\Gamma_{T^t})} + |\overline{M}^n|^2_{L_2(\Gamma_{M^n})} + \sum_{i=1}^{3} |\overline{M}^t_i|^2_{L_2(\Gamma_{M^t})})^{1/2}]$$

holds for it.

The proof follows immediately from Theorem 1.2, from the coerciveness of operators (3.6) and the validity of (1.4). The following theorem gives some sufficient conditions for $\mathscr{P} = \{0\}$.

Theorem 3.2. *Let one of the following conditions* (a)–(f) *be satisfied:*

(a) *$\Gamma_{u^n} \cap \Gamma_{u^t}$ is a non-empty set open in Γ;*

(b) *Γ_{u^n} contains such an open part of a smooth surface F_0 which is neither surface of rotation nor cylindrical nor helical*[3]) *and also any open part of F_0 is neither rotational nor cylindrical nor helical;*

(c) *Γ_{u^n} is a non-empty set open in Γ satisfying any condition from Theorems II.5–II.10 in* [2] *that is sufficient for the implication* (31) *in* [2];

(d) *Γ_{u^n} is a non-empty set open in Γ and not consisting only of cylindrical surfaces parallel with one straight line and moreover, $\Gamma_{\varphi^t} \cap \Gamma_{\varphi^n}$ is a non-empty set open in Γ;*

(e) *Γ_{u^t} is empty and Γ_{u^n} contains*

(1) *an open part of a surface of rotation with the axis x_3 (excluding planes, cylinders and spheres) or*

(2) *an open part of a circular cylinder parallel to the x_3-axis and an open part of a plane normal to the x_3-axis or an open part of a sphere with the centre on the x_3-axis, or*

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[3]) Here the surfaces of rotation are meant in the general sense including planes (orthogonal to the axis of rotation), circular cylinders and spheres; the cylindrical surfaces include planes. The helical surfaces are those that can be described in cylindrical coordinates (ϱ, φ, x_3) by $x_3 = \varrho \cos \varphi$, $x_2 = \varrho \sin \varphi$, $x_3 = f(\varrho) + h\varphi$ where $h \neq 0$ is a constant, $f(\varrho)$ is a continuously differentiable function for $0 \leqq \varrho < \infty$.

(3) *two open parts of spheres with different centres on the x_3-axis or*

(4) *an open part of a sphere with the centre on the x_3-axis and an open part of a plane normal to the x_3-axis or*

(5) *an open part of a helical surface with the axis in x_3.*

Furthermore,

(α) *$\Gamma_{\varphi t}$ is empty and $\Gamma_{\varphi n}$ contains an open part of a surface which is not cylindrical parallel to the x_3-axis or*

(β) *$\Gamma_{\varphi n}$ is empty and $\Gamma_{\varphi t}$ does not consist only of parts of planes normal to the x_3-axis;*

(f) *Γ_{ut} is empty, Γ_{un} contains an open part of a sphere with the centre in the origin. Furthermore, either*

(α) *$\Gamma_{\varphi t}$ is empty, $\Gamma_{\varphi n}$ does not consist only of parts of cylindrical surfaces parallel with one straight line or*

(β) *$\Gamma_{\varphi n}$ is empty, $\Gamma_{\varphi t}$ does not consist of parts of mutually parallel planes only.*

Then $\mathscr{P} = \{0\}$.

The proof of sufficiency of conditions (a), (b), (c) can be found in [2]. It remains to prove the sufficiency of (d), (e), (f). Let condition (d) hold. Then according to (3.8) $\psi_k = b_k = 0$ $(k = 1, 2, 3)$ on $\Gamma_{\varphi n} \cap \Gamma_{\varphi t}$. $v_k n_k = 0$ on Γ_{un} yields $a_k n_k = 0$ on Γ_{un}. As $n_k(X)$ are not complanar for each $X \in \Gamma_{un}$, we obtain $a_k = 0$ $(k = 1, 2, 3)$. Let condition (e) hold. Then according to (3.8) and [2] (see Lemmas II.7, II.9, Theorems II.8, II.9) there holds either

$$v_1 = -b_3 x_1\,, \quad v_2 = b_3 x_2\,, \quad v_3 = 0\,, \qquad \psi_1 = \psi_2 = 0\,, \quad \psi_3 = b_3$$

(for cases (1)–(4))

or

$$v_1 = -b_3 x_1\,, \quad v_2 = b_3 x_2\,, \quad v_3 = b_3 h\,, \quad \psi_1 = \psi_2 = 0\,, \quad \psi_3 = b_3$$

(for case (5)).

If (α) holds then $b_3 n_3 \neq 0$ on $\Gamma_{\varphi n}$ and consequently $b_3 = 0$, as there exists a part of $\Gamma_{\varphi n}$ with $n_3 \neq 0$. If (β) holds then

$$\varphi_3^t = b_3(1 - n_3^2) = 0$$

and there exists a part of $\Gamma_{\varphi t}$ with $n_3 \neq 1$ which implies $b_3 = 0$. Let condition (f) hold. Then Theorem II.9 in [2] yields

$$v_i = \varepsilon_{ijk} b_j x_k\,, \quad \psi_i = b_i\,.$$

Let us suppose $\boldsymbol{b} \neq 0$. We put the x_3-axis parallel to the vector $\boldsymbol{b}$ and repeat the proof of the case (e). This completes the proof of the theorem.

Now let us investigate the inverse case, namely when tractions are given all over Γ, i.e.

$$\Gamma_{u^n} = \Gamma_{u^t} = \Gamma_{\varphi^n} = \Gamma_{\varphi^t} = \emptyset\,,$$
$$\Gamma = \Gamma_{T^n} = \Gamma_{T^t} = \Gamma_{M^n} = \Gamma_{M^t}$$

(the second row being valid except for sets of surface measure zero).
In this case $V = W$ and $\mathscr{P}$ is given precisely by (3.8), a_k, b_k being arbitrary constants. Then the necessary and sufficient condition (1.6) for the existence of a weak solution takes the form of the following system

$$\text{(3.9)} \qquad \int_\Omega X_i \,\mathrm{d}X + \int_\Gamma \bar{T}_i \,\mathrm{d}\Gamma = 0\,, \quad i = 1, 2, 3\,,$$
$$\int_\Omega (\varepsilon_{ijk} x_j X_k + Y_i)\,\mathrm{d}X + \int_\Gamma (\varepsilon_{ijk} x_j \bar{T}_k + \bar{M}_i)\,\mathrm{d}\Gamma = 0\,,$$

which expresses the total equilibrium for external forces and couples.

To get the unique solution, we use functionals p_i introduced in Theorem 1.3. We show here some possible sets of $p_i(\boldsymbol{v})$, $\boldsymbol{v} = \{v_i, \psi_i\}$:

$$\text{(3.10)} \qquad \begin{cases} p_i(\boldsymbol{v}) = \displaystyle\int_{\Omega^*} v_i \,\mathrm{d}X\,, & i = 1, 2, 3\,, \\ p_j(\boldsymbol{v}) = \displaystyle\int_{\Omega^*} \varepsilon_{(j-3)kl} v_{l,k} \,\mathrm{d}X\,, & j = 4, 5, 6\,; \end{cases}$$

$$\text{(3.11)} \qquad \begin{cases} p_i(\boldsymbol{v}) = \displaystyle\int_M v_i \,\mathrm{d}M\,, & i = 1, 2, 3\,, \\ p_j(\boldsymbol{v}) = \displaystyle\int_M \varepsilon_{(j-3)kl} x_k v_l \,\mathrm{d}M\,, & j = 4, 5, 6\,; \end{cases}$$

$$\text{(3.12)} \qquad \begin{cases} p_i(\boldsymbol{v}) = \displaystyle\int_M v_i \,\mathrm{d}M\,, & i = 1, 2, 3\,, \\ p_j(\boldsymbol{v}) = \displaystyle\int_M \psi_{j-3} \,\mathrm{d}M\,, & j = 4, 5, 6\,, \end{cases}$$

where

$$M = \Omega^* \quad \text{or} \quad \Gamma^*\,.$$

Here $\Omega^* \subset \Omega$ is an arbitrary set of positive volume measure, $\Gamma^* \subset \bar{\Omega}$ is a non-empty sum of a finite number of Lipschitz surfaces.[4] In particular, $\Omega^* = \Omega$, $\Gamma^* \subset \Gamma$ may be chosen.

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[4]) Lipschitz surface is described in a Cartesian coordinate system by

$$\xi_3 = \chi(\xi_1, \xi_2)\,, \quad (\xi_1, \xi_2) \in \bar{g}$$

where χ is a Lipschitz function on $\bar{g}$ and $\bar{g}$ is a closed region.

Theorem 3.2. *Let (except for sets of surface measure zero)*

$$\Gamma = \Gamma_{T^n} = \Gamma_{T^t} = \Gamma_{M^n} = \Gamma_{M^t}$$

and let (3.9) *hold. Define the subspace* $V_p \subset W$ *as the set*

$$V_p = \{v \in W, p_l(v) = 0, l = 1, 2, \ldots, 6\},$$

where $p_l(v)$ *are chosen as in* (3.10) *or* (3.11) *or* (3.12). *Then there exists one and only one weak solution* $u_p \in V_p$ *and the inequality*

$$|u_p|_W \leqq c[(\sum_{i=1}^{3} \{|X_i|^2_{L_2(\Omega)} + |Y_i|^2_{L_2(\Omega)}\})^{1/2} + (\sum_{i=1}^{3} \{|\overline{T}_i|^2_{L_2(\Gamma)} + |\overline{M}_i|^2_{L_2(\Gamma)}\})^{1/2}], \quad c > 0$$

holds for it.

To prove this theorem it is sufficient to verify that the functionals (3.10)–(3.12) have the properties required in Theorem 1.3. The properties of functionals (3.10), (3.11) were proved in [2], Section 2. Here we restrict ourselves to the choice (3.12). $p_j(p) = 0$, $p \in \mathscr{P}$, $j = 4, 5, 6$ yields $\psi_k = b_k = 0$, $k = 1, 2, 3$. Further, $p_i(p) = 0$, $p \in \mathscr{P}$, $i = 1, 2, 3$ yields $v_k = a_k = 0$, $k = 1, 2, 3$ and therefore $p = 0$. $p_i(v)$ are linear and continuous on W due to the continuity of the embedding of $W_2^{(1)}\Omega)$ into $L_2(M)$. It remains to prove the linear independence of $p_i(v)$ on $\mathscr{P}$. Let

$$(3.13) \qquad 0 = \sum_{j=1}^{6} \alpha_j\, p_j(p) = \int_M [\sum_{i=1}^{3} \alpha_i(a_i + \varepsilon_{ijk} b_j x_k) + \sum_{j=4}^{6} \alpha_j b_{j-3}]\, dM\,.$$

We could easily verify that the form of the functionals $p_i(v)$ and of the elements $p \in \mathscr{P}$ is invariant under the orthogonal transformation of Cartesian coordinates (see [2]). Let us translate the origin of coordinates into the centroid of M. Then choosing $a_i = \alpha_i$, $i = 1, 2, 3$, $b_{j-3} = \alpha_j$, $j = 4, 5, 6$, by virtue of (3.13)

$$\sum_{i=1}^{6} \alpha_i^2 = 0$$

and the proof is complete.

Finally let us consider briefly the boundary-value problems with $1 \leqq p_N \leqq 5$ where p_N denotes the number of indeterminate coefficients in the elements $p \in \mathscr{P}$ (i.e. the number of "degrees of freedom of the rigid body"). A group of such problems is discussed in [2], Theorem II.13 in dependence on the shape of $\Gamma_{u^t} \cap \Gamma_{u^n}$. The corresponding conditions of equilibrium may be derived again by substituting $p \in \mathscr{P}$ into (1.6). These conditions are necessary and sufficient for the existence of a class of solutions. As in the preceding case we can choose a subspace V_p with the unique solution by means of appropriate functionals $p_i(v)$, e.g. according to (37)–(39), Lemma II.11 and Theorem II.14 of [2], or using also functionals for ψ similarly to (3.12).

4. THE PRINCIPLE OF MINIMUM POTENTIAL ENERGY AND THE PRINCIPLE OF MINIMUM COMPLEMENTARY ENERGY

Let $A(\boldsymbol{v}, \boldsymbol{u})$, $f(\boldsymbol{v})$ and $g(\boldsymbol{v})$ be defined by (3.2)–(3.4), $\bar{\boldsymbol{u}} \in W$ and V defined in Section 3. $\frac{1}{2}A(\boldsymbol{u}, \boldsymbol{u})$ represents the elastic energy, $f(\boldsymbol{u}) + g(\boldsymbol{u})$ the work of the external forces and couples. As $A(\boldsymbol{v}, \boldsymbol{u})$ is symmetrical and the suppositions of Theorem 1.2 and (1.6) hold, hence following the procedure in Section 1 we can define $\mathscr{L}(\boldsymbol{u})$ as in (1.15) and establish the principle of minimum potential energy:

The quadratic functional of total potential energy $\mathscr{L}(\boldsymbol{u})$ defined for $\boldsymbol{u} = \{u_i, \varphi_i\} \in W$ by

$$\mathscr{L}(\boldsymbol{u}) = \int_\Omega [\tfrac{1}{2}E_{ijkl}\,\gamma_{ij}(\boldsymbol{u})\,\gamma_{kl}(\boldsymbol{u}) + K_{ijkl}\,\gamma_{ij}(\boldsymbol{u})\,\varkappa_{kl}(\boldsymbol{u}) + \tfrac{1}{2}M_{ijkl}\,\varkappa_{ij}(\boldsymbol{u})\,\varkappa_{kl}(\boldsymbol{u})]\,\mathrm{d}X - \int_\Omega (X_i u_i + Y_i\varphi_i)\,\mathrm{d}X - \int_{\Gamma_{T^u}} \overline{T}^n u^n\,\mathrm{d}\Gamma - \int_{\Gamma_{T^t}} \overline{T}{}_i^t u_i^t\,\mathrm{d}\Gamma - \int_{\Gamma_{M^n}} \overline{M}^n \varphi^n\,\mathrm{d}\Gamma - \int_{\Gamma_{M^t}} \overline{M}{}_i^t \varphi_i^t\,\mathrm{d}\Gamma \tag{4.1}$$

where

$$\gamma_{ij}(\boldsymbol{u}) = u_{j,i} - \varepsilon_{ijk}\varphi_k\,, \quad \varkappa_{ij}(\boldsymbol{u}) = \varphi_{j,i}\,,$$

attains the minimum on the set

$$\bar{\boldsymbol{u}} \oplus V$$

if and only if

$$\boldsymbol{u} = \mathring{\boldsymbol{u}} + \boldsymbol{p}$$

where $\mathring{\boldsymbol{u}}$ is a weak solution and $\boldsymbol{p} \in \mathscr{P}$ is defined in (3.8).

Following the end of Section 1, we can formulate the principle of minimum potential energy using subspaces V_p as defined in Section 3:

The quadratic functional (4.1) *attains the minimum on the set*

$$\bar{\boldsymbol{u}} \oplus V_p$$

if and only if

$$\boldsymbol{u} = \mathring{\boldsymbol{u}}$$

where $\mathring{\boldsymbol{u}}$ is the weak solution being unique in V_p.

Using the method of orthogonal projections in Hilbert space, we deduce the principle of minimum complementary energy.

Let $\mathscr{T}$ be a Banach space of stress fields $\boldsymbol{T}$

$$\boldsymbol{T} = \{\tau_{ij}, m_{ij}\}\,, \quad \tau_{ij}, m_{ij} \in L_2(\Omega)$$

with the norm

$$|\boldsymbol{T}|^2_{\mathscr{T}} = \sum_{i,j=1}^{3} \left(|\tau_{ij}|^2_{L_2(\Omega)} + |m_{ij}|^2_{L_2(\Omega)}\right). \tag{4.2}$$

Let us introduce for $\boldsymbol{T}', \boldsymbol{T}'' \in \mathscr{T}$ a bilinear form

$$(\boldsymbol{T}', \boldsymbol{T}'') = \int_\Omega \left[P_{ijkl}\tau'_{ij}\tau''_{kl} + Q_{ijkl}(\tau'_{ij}m''_{kl} + \tau''_{ij}m'_{kl}) + S_{ijkl}m'_{ij}m''_{kl}\right] \mathrm{d}X\,. \tag{4.3}$$

We show that (4.3) defines a scalar product in $\mathscr{T}$. By virtue of (2.13), (4.3) is symmetrical. $\mathscr{A}(\gamma_{ij}, \varkappa_{ij})$ being positive definite (see (2.9)), with help of (2.14) we have

$$(\boldsymbol{T}, \boldsymbol{T}) = 2\int_\Omega \overline{\mathscr{A}}(\tau_{ij}, m_{ij})\,\mathrm{d}X \geqq c\int_\Omega \sum_{i,j=1}^{3} (\gamma_{ij}^2 + \varkappa_{ij}^2)\,\mathrm{d}X\,, \quad c > 0\,. \tag{4.4}$$

Using (2.5), (2.6) and the boundedness of E_{ijkl}, K_{ijkl}, M_{ijkl}, we easily prove that

$$\sum_{i,j=1}^{3} (\tau_{ij}^2 + m_{ij}^2) \leqq c_1 \sum_{i,j=1}^{3} (\gamma_{ij}^2 + \varkappa_{ij}^2)\,, \quad c_1 > 0\,. \tag{4.5}$$

Joining (4.4) and (4.5) together and using (4.2), we get

$$(\boldsymbol{T}, \boldsymbol{T}) \geqq \frac{c}{c_1}|\boldsymbol{T}|^2_{\mathscr{T}}\,.$$

We could easily verify also the inverse inequality

$$(\boldsymbol{T}, \boldsymbol{T}) \leqq c_2|\boldsymbol{T}|^2_{\mathscr{T}}\,, \quad c_2 > 0\,.$$

Let us write

$$(\boldsymbol{T}, \boldsymbol{T})^{1/2} \equiv |\boldsymbol{T}|_{\mathscr{H}}\,. \tag{4.6}$$

Thus we have created a Hilbert space of stress fields $\boldsymbol{T}$ with the scalar product (4.3) and the associated norm (4.6). The norms (4.2) and (4.6) are equivalent.

Denote by $\mathscr{H}_1 \subset \mathscr{H}$ the subset of all $\boldsymbol{T} \in \mathscr{H}$ to which $\boldsymbol{u} = \{u_i, \varphi_i\} \in V$ exists such that using (2.3), (2.4), equations (2.5), (2.6) hold. (i.e. $\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{u})$). Second, denote by $\mathscr{H}_2 \subset \mathscr{H}$ the subset of $\boldsymbol{T} \in \mathscr{H}$ such that for each $\boldsymbol{v} = \{v_i, \psi_i\} \in V$

$$\int_\Omega \left[\tau_{ij}(v_{j,i} - \varepsilon_{ijk}\psi_k) + m_{ij}\psi_{j,i}\right] \mathrm{d}X = 0$$

holds. Let $T' \in \mathscr{H}_1$, $T'' \in \mathscr{H}_2$. (4.3) yields

$$(T', T'') = \int_\Omega (\tau''_{ij}\gamma'_{ij} + m''_{ij}\varkappa'_{ij})\, dX$$

and there exists $u' = \{u'_i, \varphi'_i\} \in V$ such that

$$\gamma'_{ij} = u'_{j,i} - \varepsilon_{ijk}\varphi'_k, \quad \varkappa'_{ij} = \varphi'_{j,i}.$$

As $T'' \in \mathscr{H}_2$, we have

$$(T', T'') = \int_\Omega [\tau''_{ij}(u'_{j,i} - \varepsilon_{ijk}\varphi'_k) + m''_{ij}\varphi'_{j,i}]\, dX = 0,$$

therefore $\mathscr{H}_1$ and $\mathscr{H}_2$ are orthogonal.

Let $T \in \mathscr{H}$ be an arbitrary stress field satisfying the equations of equilibrium (2.1), (2.2) and the statical boundary conditions on Γ_{Tn}, Γ_{Tt}, Γ_{Mn}, Γ_{Mt} in the weak sense, i.e. let (cf. (2.18) or (3.5))

$$\int_\Omega [\tau_{ij}(v_{j,i} - \varepsilon_{ijk}\psi_k) + m_{ij}\psi_{j,i}]\, dX = \int_\Omega (X_i v_i + Y_i\psi_i)\, dX + \tag{4.7}$$

$$+ \int_{\Gamma_{Tn}} \bar{T}^n v^n\, d\Gamma + \int_{\Gamma_{Tt}} \bar{T}^t_i v^t_i\, d\Gamma + \int_{\Gamma_{Mn}} \bar{M}^n \psi^n\, d\Gamma + \int_{\Gamma_{Mt}} \bar{M}^t_i \psi^t_i\, d\Gamma$$

hold for each $v = \{v_i, \psi_i\} \in V$. Denote $\mathring{T} = \{\mathring{\tau}_{ij}, \mathring{m}_{ij}\} \equiv T(\mathring{u})$, i.e. $\mathring{T}$ corresponds to the weak solution $\mathring{u}$, by means of (2.3)–(2.6). From (3.7) there follows

$$T(\mathring{u} + p) = T(\mathring{u}), \quad p \in \mathscr{P}.$$

If we write

$$\mathring{u} = \bar{u} + w$$

we have

$$\mathring{T} = T(\bar{u}) + T(w) \quad \text{and} \quad T(w) \in \mathscr{H}_1.$$

By virtue of (3.5) $\mathring{T}$ meets (4.7) nad consequently $T - \mathring{T} \in \mathscr{H}_2$. Because of the orthogonality of $\mathscr{H}_1$ and $\mathscr{H}_2$ we have

$$|T - T(\bar{u})|^2_{\mathscr{H}} = |(T - \mathring{T}) + T(w)|^2_{\mathscr{H}} = |T - \mathring{T}|^2_{\mathscr{H}} + |T(w)|^2_{\mathscr{H}}. \tag{4.8}$$

It is obvious that $|T - T(\bar{u})|^2_{\mathscr{H}}$ attains its minimum on the set of $T \in \mathscr{H}$ which satisfy (4.7), if and only if $T = \mathring{T}$. The same holds for the functional

$$\mathscr{S}(T) = \tfrac{1}{2}\{|T - T(\bar{u})|^2_{\mathscr{H}} - |T(\bar{u})|^2_{\mathscr{H}}\} = \tfrac{1}{2}(T, T) - (T, T(\bar{u})).$$

Hence we can formulate the principle of minimum complementary energy: *The quadratic functional*

$$(4.9) \qquad \mathscr{S}(T) = \int_\Omega [\tfrac{1}{2}P_{ijkl}\tau_{ij}\tau_{kl} + Q_{ijkl}\tau_{ij}m_{kl} + \tfrac{1}{2}S_{ijkl}m_{ij}m_{kl}]\,\mathrm{d}X - \\ - \int_\Omega [\tau_{ij}(\bar{u}_{j,i} - \varepsilon_{ijk}\bar{\varphi}_k) + m_{ij}\bar{\varphi}_{j,i}]\,\mathrm{d}X$$

attains the minimum on the set of $T \in \mathscr{T}$ *which satisfy the equations of equilibrium* (2.1), (2.2) *and the statical boundary conditions in the sense of* (4.7), *if and only if*

$$|T - \mathring{T}|_{\mathscr{T}} = 0$$

where $\mathring{T} = T(\mathring{u})$, $\mathring{u}$ *being the weak solution.*

If moreover the weak solution $\mathring{u}$ is such that $T(\mathring{u})$ meets the equations of equilibrium (2.1), (2.2) in the sense of $L_2(\Omega)$ [5]) and the boundary conditions (2.17a) to (2.17d) in the sense of traces and therefore $T(\mathring{u})$ is a statically admissible stress field, then we can take for T the statically admissible stress fields and apply the principle of virtual work (2.18) to the fields T and $\bar{u}$:

$$\int_\Omega [\tau_{ij}(\bar{u}_{j,i} - \varepsilon_{ijk}\bar{\varphi}_k) + m_{ij}\bar{\varphi}_{j,i}]\,\mathrm{d}X = \int_\Omega (X_i\bar{u}_i + Y_i\bar{\varphi}_i)\,\mathrm{d}X + \\ + \int_{\Gamma_{un}} T^n\bar{u}^n\,\mathrm{d}\Gamma + \int_{\Gamma_{ut}} T_i^t\bar{u}_i^t\,\mathrm{d}\Gamma + \int_{\Gamma_{\varphi n}} M^n\bar{\varphi}^n\,\mathrm{d}\Gamma + \int_{\Gamma_{\varphi t}} M_i^t\bar{\varphi}_i^t\,\mathrm{d}\Gamma + \\ + \int_{\Gamma_{Tn}} \bar{T}^n\bar{u}^n\,\mathrm{d}\Gamma + \int_{\Gamma_{Tt}} \bar{T}_i^t\bar{u}_i^t\,\mathrm{d}\Gamma + \int_{\Gamma_{Mn}} \bar{M}^n\bar{\varphi}^n\,\mathrm{d}\Gamma + \int_{\Gamma_{Mt}} \bar{M}_i^t\bar{\varphi}_i^t\,\mathrm{d}\Gamma\,.$$

If we omit the integrals not depending on T, we can formulate the principle of minimum complementary energy in the common form: *The quadratic functional (complementary energy)*

$$(4.10) \qquad \mathscr{S}(T) = \int_\Omega [\tfrac{1}{2}P_{ijkl}\tau_{ij}\tau_{kl} + Q_{ijkl}\tau_{ij}m_{kl} + \tfrac{1}{2}S_{ijkl}m_{ij}m_{kl}]\,\mathrm{d}X - \\ - \int_{\Gamma_{un}} T^n\bar{u}^n\,\mathrm{d}\Gamma - \int_{\Gamma_{ut}} T_i^t\bar{u}^t\,\mathrm{d}\Gamma - \int_{\Gamma_{\varphi n}} M^n\bar{\varphi}^n\,\mathrm{d}\Gamma - \int_{\Gamma_{\varphi t}} M_i^t\bar{\varphi}_i^t\,\mathrm{d}\Gamma$$

attains the minimum on the set of statically admissible stress fields $T \in \mathscr{T}$, *if and only if* $|T - \mathring{T}|_{\mathscr{T}} = 0$.

Remark. If T is a statically admissible stress field, then T meets (4.7), because each $v \in V$ is kinematically admissible displacement and micro-rotation field for $\bar{u}_i = \bar{\varphi}_i = 0$ and (4.7) follows from the principle of virtual work (2.18).

[5]) Possibly in the sense of $L_2(\Omega_S)$ — see the footnote[2]).

In this section we shall establish variational principles for Cosserat continuum which correspond to the principles of Hu-Washizu and of Reissner-Hellinger in classical elasticity.

We use the same approach, based on Lagrange multipliers method, as in [3] for elastic bodies. For the sake of simplicity let us consider only the particular case when

$$\Gamma_u = \Gamma_{u^n} = \Gamma_{u^t} = \Gamma_{\varphi^n} = \Gamma_{\varphi^t}\,, \quad \Gamma_T = \Gamma_{T^n} = \Gamma_{T^t} = \Gamma_{M^n} = \Gamma_{M^t}$$

and $\bar{u}_i$, $\bar{\varphi}_i \in W_2^{(1)}(\Omega)$, $\bar{T}_i$, $\bar{M}_i \in L_2(\Gamma_T)$. Then the boundary conditions are

$$(5.1) \qquad \bar{u}_i = u_i\,, \quad \bar{\varphi}_i = \varphi_i \quad \text{on} \quad \Gamma_u$$

$$(5.2) \qquad \bar{T}_i = \tau_{ji} n_j\,, \quad \bar{M}_i = m_{ji} n_j \quad \text{on} \quad \Gamma_T\,.$$

Let us add conditions (2.3), (2.4), (5.1) to the functional (4.1) by means of coefficients λ_{ij}, μ_{ij}, ξ_i, η_i. The new functional has the form

$$\mathscr{I}(u_i, \varphi_i, \gamma_{ij}, \varkappa_{ij}, \lambda_{ij}, \mu_{ij}, \xi_i, \eta_i) = \int_\Omega [\mathscr{A}(\gamma_{ij}, \varkappa_{ij}) - X_i u_i - Y_i \varphi_i]\, \mathrm{d}X +$$
$$+ \int_\Omega [\lambda_{ij}(-\gamma_{ij} + u_{j,i} - \varepsilon_{ijk}\varphi_k) + \mu_{ij}(-\varkappa_{ij} + \varphi_{j,i})]\, \mathrm{d}X -$$
$$- \int_{\Gamma_T} (\bar{T}_i u_i + \bar{M}_i u_i)\, \mathrm{d}\Gamma + \int_{\Gamma_u} [\xi_i(u_i - \bar{u}_i) + \eta_i(\varphi_i - \bar{\varphi}_i)]\, \mathrm{d}\Gamma$$

where $\mathscr{A}(\gamma_{ij}, \varkappa_{ij})$ is defined in (2.8) and all variable functions are mutually independent. From the necessary conditions for $\delta\mathscr{I} = 0$, it is obvious that λ_{ij}, μ_{ij} have the sense of τ_{ij}, m_{ij}, respectively and

$$\xi_i = -\lambda_{ji} n_j\,, \quad \eta_i = -\mu_{ij} n_j \quad \text{on} \quad \Gamma_u\,.$$

We can establish a variational principle which is a counterpart of Hu-Washizu principle in classical elasticity, as follows:

The condition

$$\delta\mathscr{I}(u_i, \varphi_i, \gamma_{ij}, \varkappa_{ij}, \tau_{ij}, m_{ij}) = 0$$

where

$$(5.3) \qquad \mathscr{I}(u_i, \varphi_i, \gamma_{ij}, \varkappa_{ij}, \tau_{ij}, m_{ij}) = \int_\Omega [\mathscr{A}(\gamma_{ij}, \varkappa_{ij}) + \tau_{ij}(-\gamma_{ij} + u_{j,i} - \varepsilon_{ijk}\varphi_k) +$$
$$+ m_{ij}(\varphi_{j,i} - \varkappa_{ij}) - (X_i u_i + Y_i \varphi_i)]\, \mathrm{d}X -$$
$$- \int_{\Gamma_T} (\bar{T}_i u_i + \bar{M}_i \varphi_i)\, \mathrm{d}\Gamma + \int_{\Gamma_u} [\tau_{ij}(\bar{u}_j - u_j) + m_{ij}(\bar{\varphi}_j - \varphi_j)]\, n_i\, \mathrm{d}\Gamma$$

yields the following Euler's conditions in Ω and natural boundary conditions respectively: the equations of equilibrium (2.1), (2.2), the geometrical equations (2.3), (2.4), the constitutive equations (2.5), (2.6); the boundary conditions (5.1) on Γ_u and the boundary conditions (5.2) on Γ_T.

Similarly we can derive another variational principle (a counterpart of Reissner-Hellinger principle) when adding conditions (2.1), (2.2) and (5.2) to the functional $\mathscr{S}(\boldsymbol{T})$ of complementary energy (4.10). The principle reads like this:

The condition

$$\delta\mathscr{R}(\tau_{ij}, m_{ij}, u_i, \varphi_i) = 0\,,$$

(5.4)
$$\mathscr{R}(\tau_{ij}, m_{ij}, u_i, \varphi_i) = \int_\Omega [\overline{\mathscr{A}}(\tau_{ij}, m_{ij}) - (\tau_{ij}\gamma_{ij} + m_{ij}\varphi_{j,i}) + (X_i u_i + Y_i\varphi_i)]\,\mathrm{d}X + \int_{\Gamma_u} [(u_j - \bar{u}_j)\,\tau_{ij} + (\varphi_j - \bar{\varphi}_j)\,m_{ij}]\,n_i\,\mathrm{d}\Gamma + \int_{\Gamma_T} (\bar{T}_i u_i + \bar{M}_i\varphi_i)\,\mathrm{d}\Gamma$$

where $\overline{\mathscr{A}}(\tau_{ij}, m_{ij})$ and γ_{ij} are defined by (2.14) and (2.3), respectively, yields the following Euler's conditions in Ω and natural boundary conditions: the equations of equilibrium (2.1), (2.2) and the equations

$$u_{j,i} - \varepsilon_{ijk}\varphi_k = P_{ijkl}\tau_{kl} + Q_{ijkl}m_{kl}\,,$$
$$\varphi_{j,i} = Q_{klij}\tau_{kl} + S_{ijkl}m_{kl} \quad \text{in} \quad \Omega\,,$$

the boundary conditions (5.1) on Γ_u and the boundary conditions (5.2) on Γ_T.

We could also deduce the latter principle (5.4) from the former (5.3) when supposing that the constitutive equations (2.5), (2.6) are satisfied a priori.

From these two principles we could deduce a group of variational theorems (as it was done for the classical elasticity in [3]) when choosing various Euler's conditions as additional conditions. In this way we could have also deduced the principle of minimum potential energy and that of the minimum complementary energy.

6. ESTIMATES OF ERRORS OF THE APPROXIMATE SOLUTIONS OBTAINED FROM THE PRINCIPLES OF MINIMUM POTENTIAL AND MINIMUM COMPLEMENTARY ENERGY

Let

$${}^n\boldsymbol{u} = \bar{\boldsymbol{u}} + {}^n\boldsymbol{w}$$

where ${}^n\boldsymbol{u} = \{{}^n u_i, {}^n\varphi_i\}$ is an approximate solution obtained on the base of the principle of minimum potential energy, i.e. the $\boldsymbol{n}$-th term of a sequence minimizing the functional $\mathscr{L}(\boldsymbol{u})$ in (4.1). Let ${}^m\boldsymbol{T} = \{{}^m\tau_{ij}, {}^m m_{ij}\}$ be an approximate solution obtained on the base of the principle of minimum complementary energy, i.e. the m-th term of a sequence minimizing the functional $\tilde{\mathscr{S}}(T)$ in (4.9) or $\mathscr{S}(T)$ in (4.10), respectively. ${}^mT$ meets (4.7) or it is a statically admissible stress field, respectively. (1.14) yields

$$A({}^n u - \mathring{u}, {}^n u - \mathring{u}) = \Phi({}^n w) + A(w, w) \tag{6.1}$$

where $\mathring{u} = \bar{u} + w$ is the weak solution. From (2.5), (2.6) or (2.11), (2.12), respectively, (3.2), (4.3) and (4.8) we obtain

$$A(w, w) = |T(w)|^2_{\mathscr{H}} \leqq |{}^m T - T(\bar{u})|^2_{\mathscr{H}}\,. \tag{6.2}$$

(1.13) and (1.15) yield

$$\Phi({}^n w) = 2\mathscr{L}({}^n u) + \Phi_1(\bar{u})\,. \tag{6.3}$$

The definition of $\tilde{\mathscr{S}}(T)$ implies

$$|{}^m T - T(\bar{u})|^2_{\mathscr{H}} = 2\tilde{\mathscr{S}}(T) + |T(\bar{u})|^2_{\mathscr{H}}\,. \tag{6.4}$$

Combining (6.1)–(6.4) we obtain the estimate

$$A({}^n u - \mathring{u}, {}^n u - \mathring{u}) \leqq 2[\mathscr{L}({}^n u) + \tilde{\mathscr{S}}({}^m T) + f(\bar{u}) + g(\bar{u})]$$

where f, g are defined by (3.3), (3.4).

If moreover ${}^m T$ is a statically admissible stress field, then there holds (see the relation between (4.9) and (4.10))

$$A({}^n u - \mathring{u}, {}^n u - \mathring{u}) \leqq 2[\mathscr{L}({}^n u) + \mathscr{S}({}^m T)]\,.$$

Next let us estimate $|{}^m T - \mathring{T}|_{\mathscr{H}}$. From (4.8), (1.8), (6.1), (6.2) we obtain

$$|{}^m T - \mathring{T}|^2_{\mathscr{H}} = |{}^m T - T(\bar{u})|^2_{\mathscr{H}} - |T(w)|^2_{\mathscr{H}} \leqq \Phi({}^n w) + |{}^m T - T(\bar{u})|^2_{\mathscr{H}}$$

and again

$$|{}^m T - \mathring{T}|^2_{\mathscr{H}} \leqq 2[\mathscr{L}({}^n u) + \tilde{\mathscr{S}}({}^m T) + f(\bar{u}) + g(\bar{u})]\,.$$

If ${}^m T$ is an admissible stress field, there is

$$|{}^m T - \mathring{T}|^2_{\mathscr{H}} \leqq 2[\mathscr{L}({}^n u) + \mathscr{S}({}^m T)]\,.$$

7. ISOTROPIC MATERIAL

For the case of the isotropic material, the form (2.8) is invariant with respect to all orthogonal transformations of coordinates. Then E_{ijkl}, M_{ijkl} and K_{ijkl} are isotropic tensors and furthermore,

$$K_{ijkl} = 0 \tag{7.1}$$

because $\varkappa_{ij}$ is an axial tensor as the gradient of micro-rotations. We can write

$$(7.2)\qquad \begin{aligned} E_{ijkl} &= E_1\,\delta_{ik}\delta_{jl} + E_2\,\delta_{il}\,\delta_{jk} + E_3\,\delta_{ij}\,\delta_{kl}\,, \\ M_{ijkl} &= M_1\,\delta_{ik}\,\delta_{jl} + M_2\,\delta_{il}\delta_{jk} + M_3\,\delta_{ij}\,\delta_{kl} \end{aligned}$$

where E_i, M_i are constants for homogeneous bodies. As the matrix (2.10) is positive definite, all the principal minors of the matrix (2.10) are positive. This fact combined with (7.1), (7.2) yields inequalities

$$(7.3)\qquad \begin{aligned} &E_1 + E_2 > 0\,, && M_1 + M_2 > 0 \\ &E_1 - E_2 > 0\,, && M_1 - M_2 > 0 \\ &E_1 + E_2 + 3E_3 > 0\,, && M_1 + M_2 + 3M_3 > 0\,. \end{aligned}$$

(7.3) are the necessary and sufficient conditions for the form (2.8) to be positive definite in the isotropic case.

Instead of M_i, E_i other constants may be introduced, which are commonly used in the literature (for example see [5]):

$$(7.4)\qquad \begin{aligned} &\lambda = 2E_3\,, \quad k = 2(E_1 - E_2)\,, && 2\mu + k = 2(E_1 + E_2)\,, \\ &\alpha = 2M_3\,, \quad \beta = 2M_2\,, && \gamma = 2M_1\,. \end{aligned}$$

Here λ, μ represent the Lamé's constants of classical elasticity. Then the quadratic form (2.8) may be rewritten as follows

$$(7.5)\qquad \begin{aligned} \mathscr{A}(\gamma_{ij}, \varkappa_{ij}) &= \tilde{\mathscr{A}}(\varepsilon_{ij}, r_i, \varphi_i) = \tfrac{1}{2}[\lambda\varepsilon_{kk}\varepsilon_{ll} + (2\mu + k)\,\varepsilon_{kl}\varepsilon_{kl}] + \\ &+ k(r_k - \varphi_k)\,(r_k - \varphi_k) + \tfrac{1}{2}[\alpha\varphi_{k,k}\varphi_{l,l} + \beta\varphi_{k,l}\varphi_{l,k} + \gamma\varphi_{k,l}\varphi_{k,l}] \end{aligned}$$

where

$$(7.6)\qquad \varepsilon_{kl} = \tfrac{1}{2}(u_{k,l} + u_{l,k})\,, \quad r_k = \tfrac{1}{2}\varepsilon_{klm}u_{m,l}\,.$$

According to (7.3), (7.4), the necessary and sufficient conditions for the form (7.5) to be positive definite with respect to γ_{ij}, $\varkappa_{ij}$ are the following inequalities

$$(7.7)\qquad \begin{aligned} &3\lambda + 2\mu + k > 0\,, \quad k > 0\,, && 2\mu + k > 0\,, \\ &\gamma + \beta > 0\,, \quad \gamma - \beta > 0\,, && 3\alpha + \beta + \gamma > 0\,. \end{aligned}$$

Using (2.3), (2.4), (7.6) we write the constitutive equations of an isotropic continuum in the form

$$\begin{aligned} \tau_{ij} &= \lambda\varepsilon_{kk}\delta_{ij} + (2\mu + k)\,\varepsilon_{ij} + k\,\varepsilon_{ijk}(r_k - \varphi_k)\,, \\ m_{ij} &= \alpha\varphi_{kk}\delta_{ij} + \beta\varphi_{i,j} + \gamma\varphi_{j,i}\,. \end{aligned}$$

If λ, μ, k, α, β, γ are e.g. piecewise constant functions satisfying the inequalities (7.7) in $\bar{\Omega}$, then all the results of the preceding sections hold again.

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Výtah

EXISTENCE A JEDNOZNAČNOST ŘEŠENÍ A NĚKTERÉ VARIAČNÍ PRINCIPY V LINEÁRNÍCH TEORIÍCH PRUŽNOSTI S MOMENTOVÝMI NAPĚTÍMI

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Část 1: COSSERATOVO KONTINUUM

V první části práce se definuje zobecněné řešení okrajových úloh pro Cosseratovo prostředí. Jsou dokázány existence, jednoznačnost a spojitá závislost slabého řešení na daných zatíženích pro statický případ omezených, anisotropních, nehomogenních těles. Formulují se princip minima potenciální energie a minima doplňkové energie a jiné zobecněné variační principy.

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