Hana Kamasová; Antonín Šimek

Inversion of quasi-triangular matrices

Aplikace matematiky, Vol. 15 (1970), No. 2, 146–148

Persistent URL: `http://dml.cz/dmlcz/103278`

INVERSION OF QUASI-TRIANGULAR MATRICES

Hana Kamasová, Antonín Šimek

(Received April 1, 1969)

1. INTRODUCTION

In the paper [4] the following results were proved:

Let $\boldsymbol{A}$ be a square matrix of order $n \geqq r$ over a field of characteristic zero, divided into blocks $\alpha_{i,k}$ of the type $(n_i \times n_k)$,

$$\sum_{i=1}^{r} n_i = \sum_{k=1}^{r} n_k = n .$$

Let further be

$$(1,1) \qquad \begin{aligned} \boldsymbol{A}_1 &= \alpha_{1,1} \\ \boldsymbol{A}_2 &= \begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{bmatrix} \\ &\vdots \\ \boldsymbol{A}_{r-1} &= \begin{bmatrix} \alpha_{1,1} & \dots & \alpha_{1,r-1} \\ \vdots & & \\ \alpha_{r-1,1} & \dots & \alpha_{r-1,r-1} \end{bmatrix} \\ \boldsymbol{A}_r &= \boldsymbol{A} . \end{aligned}$$

Let us define matrices $\boldsymbol{Z}_{i,k}^{(p)}$ for $i, k, p = 1, \dots, r$ in the following way:

$$(1,2) \qquad \begin{aligned} &1. \ \boldsymbol{Z}_{i,k}^{(1)} = \alpha_{i,k} \\ &2. \ \boldsymbol{Z}_{i,k}^{(p)} = \boldsymbol{Z}_{i,k}^{(p-1)} - \boldsymbol{Z}_{i,p-1}^{(p-1)} \boldsymbol{Z}_{p-1}^{-1} \boldsymbol{Z}_{p-1,k}^{(p-1)} \end{aligned}$$

for $p = 2, \dots, r$, where $\boldsymbol{Z}_{p,p}^{(p)} = \boldsymbol{Z}_p$.
For matrices $\boldsymbol{Z}_p$ we have

Theorem 1,1. *The matrices $\boldsymbol{A}_p$ are regular iff $\boldsymbol{Z}_p$ are regular for $p = 1, 2, \dots, r$.* (For the proof, see [4].)

Let us introduce $V_{i,k}^{(p)}$ as the set of subsequences of the sequence $(i, i+1, \dots \dots, p-1, p, p-1, \dots, k+1, k)$ which have the following properties:

1. The first element is i, the last is k.
2. Each two neighboring elements in any of the subsequences are different.

Theorem 1,2. *Let the matrices* (1,1) *be regular; let* $\mathbf{A}^{-1} = [\beta_{i,k}]$ *be a partitioned matrix conformal to* $\mathbf{A}$. *Then we have*

$$\beta_{i,k} = \sum(-1)^{1+m(j_1,\dots,j_s)} \mathbf{Z}_{j_1}^{-1}\mathbf{Z}_{j_1,j_2}^{(q_1)}\mathbf{Z}_{j_2}^{-1} \dots \mathbf{Z}_{j_{s-1},j_s}^{(q_{s-1})}\mathbf{Z}_{j_s}^{-1},$$

$$(i = j_1, j_2, \dots, j_s = k) \in V_{i,k}^{(r)},$$

$$q_t = \min(j_t, j_{t+1}), \quad t = 1, \dots, s-1$$

where $m(j_1, \dots, j_s)$ *is the number of the elements of the sequence* $(i = j_1, \dots, j_s = k)$. (For the proof, see [4].)

In this paper, formulas for the blocks of the inverse matrix to a quasi-triangular matrix $\mathbf{A} = [\alpha_{i,k}]$ where $\alpha_{i,k} = 0$ for $i > k$, are showed. M. FIEDLER achieved in [2] the same results.

2. RESULTS

Theorem 2,1. *Let the matrices* (1,1) *be regular; let* $\alpha_{i,k} = 0$ *for* $i > k$. *Then we have*

$$\mathbf{Z}_{i,k}^{(p)} = \alpha_{i,k} \text{ for } i \geqq p,\ k \geqq p,\ p \geqq 2\,. \tag{2,1}$$

Proof. By induction

1. $p = 2$

$$\mathbf{Z}_{i,k}^{(2)} = \mathbf{Z}_{i,k}^{(1)} - \mathbf{Z}_{i,1}^{(1)}\mathbf{Z}_1^{-1}\mathbf{Z}_{1k}^{(1)}\,.$$

By the assumption it is $i \geqq 2$, thus $\mathbf{Z}_{i,1}^{(1)} = \alpha_{i,1} = 0$ and $\mathbf{Z}_{i,k}^{(2)} = \mathbf{Z}_{i,k}^{(1)} = \alpha_{i,k}$.

2. Let us suppose that (2,1) is true for p.

Then $\mathbf{Z}_{i,k}^{(p+1)} = \mathbf{Z}_{i,k}^{(p)} - \mathbf{Z}_{i,p}^{(p)}\mathbf{Z}_p^{-1}\mathbf{Z}_{p,k}^{(p)}$.

By the assumption it is $i \geqq p + 1$, thus $\mathbf{Z}_{i,p}^{(p)} = \alpha_{i,p} = 0$ and $\mathbf{Z}_{i,k}^{(p+1)} = \mathbf{Z}_{i,k}^{(p)} = \alpha_{i,k}$.

Note. By lemma 2,1 in [4] we have $\mathbf{Z}_{i,k}^{(p)} = 0$ for $i < p$, $\mathbf{Z}_{i,k}^{(p)} = 0$ for $k < p$.

Theorem 2,2. *The matrices* $\mathbf{A}_p$ *are regular iff* $\alpha_{p,p}$ *are regular for* $p = 1, 2, \dots, r$.

Proof. The assertion is a corollary of Theorems 1,1 and 2,1.

Theorem 2,3. *Let the matrices* (1,1) *be regular, let* $\alpha_{i,k} = 0$ *for* $i > k$. *Let* $\mathbf{A}^{-1} = [\beta_{i,k}]$ *be a partitioned matrix conformal to* $\mathbf{A}$. *Then we have*

$$\beta_{i,k} = \sum(-1)^{1+m} \alpha_{j_1,j_1}^{-1}\alpha_{j_1,j_2}\alpha_{j_2,j_2}^{-1} \dots \alpha_{j_{s-1},j_s}\alpha_{j_s,j_s}^{-1} \tag{2,2}$$

$$(i = j_1, j_2, \dots, j_s = k) \in V_{i,k}^{(k)}$$

for $i \leqq k$ *where* m *is the number of terms of the sequence* $(i = j_1, j_2, \ldots, j_s = k)$

$$\beta_{i,k} = 0 \; for \; i > k \, . \tag{2,3}$$

Proof. We prove the assertion (2,2). Let $i \leqq k$. By Theorem 1,2 it is

$$\beta_{i,k} = \sum (-1)^{m+1} \mathbf{Z}_{j_1}^{-1} \ldots \mathbf{Z}_{j_s}^{-1} \, .$$

From Theorem 2,1 we get

$$\beta_{i,k} = \sum_{V^{(r)}{}_{i,k}} (-1)^{m+1} \alpha_{j_1,j_1}^{-1} \alpha_{j_1,j_2} \alpha_{j_2,j_2}^{-1} \ldots \alpha_{j_s,j_s}^{-1} \, . \tag{2,4}$$

From the definition of $V_{i,k}^{(p)}$ it is obvious that $V_{i,k}^{(k)} \subset V_{i,k}^{(r)}$ for $r \geqq k$. Since $\alpha_{j_t,j_{t+1}} = 0$ for $j_t > j_{t+1}$, the only non-zero terms of the sum (2,4) correspond to the increasing sequences. Thus, it is sufficient to summarize over all sequences from $V_{i,k}^{(k)}$.

To prove (2,3) it is sufficient to note that no sequence from $V_{i,k}^{(r)}$ is increasing; thus $\beta_{i,k} = 0$.

References

[1] *W. J. Duncan:* Reciprocation of Triply Partitioned Matrices, Journal of the Royal Aeronautical Society, Vol. 60., Feb. 1956, 131—132.

[2] *M. Fiedler:* On inverting partitioned matrices. Czechoslovak Mathematical Journal, 13 (88), 4, 1963, 574—586.

[3] Ф. Р. Гантмахер: Теория матриц, Москва, 1966.

[4] *H. Kamasová, A. Šimek:* Metoda inverze matice rozdělené na bloky, Aplikace matematiky, sv. 14 (1969), č. 2, 105—114.

[5] *E. Kosko:* Matrix Inversion by Partitioning, The Aeronautical Quarterly, Vol. VIII., 1957, 157—184.

[6] *E. Kosko:* Reciprocation of Triply Partitioned Matrices, Journal of the Royal Aeronautical Society, Vol. 60., July 1956, 490—491.

Souhrn

INVERZE KVAZITRIANGULARNÍCH MATIC

HANA KAMASOVÁ, ANTONÍN ŠIMEK

V článku je uvedena metoda inverze kvazitriangulárních matic, rozdělených na $r \times r$ bloků.

Věta. *Nechť matice* (1,1) *jsou regulární, nechť* $\alpha_{i,k} = 0$ *pro* $i > k$. *Nechť* $\mathbf{A}^{-1} = [\beta_{i,k}]$ *je bloková matice konformní s* $\mathbf{A}$. *Potom platí* (2,2), (2,3).

Author's address: Hana Kamasová, Ing. *Antonín Šimek:* Vysoká škola chemicko-technologická, katedra matematiky, Technická 1905, Praha 6.