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Aplikace matematiky, Vol. 16 (1971), No. 1, 64–80

Persistent URL: http://dml.cz/dmlcz/103326

ON THE EXISTENCE AND UNIQUENESS OF SOLUTION OF THE CAUCHY PROBLEM FOR LINEAR INTEGRO-DIFFERENTIAL EQUATIONS WITH OPERATOR COEFFICIENTS

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(Received May 7, 1970)

INTRODUCTION

In the theory of neutron fields some problems arise, which may be described by means of integro-differential equations with initial conditions. The aim of the present paper is to state a class of problems, covering the above-mentioned physical example, and to prove the existence, uniqueness and continuous dependence of their solutions on the given data. A variational approach (see [1]) is used to establish the definition of a generalized (weak) solution of the Cauchy problem, which is an extension of the concept of generalized solutions in case of differential equations (compare [2]). Following the work of Lions [2], the proof of existence is based on a special projection theorem and that of uniqueness on a method, originated from Ладыженская [3].

1. NOTATION. STATEMENT OF THE PROBLEM

Let a bounded interval $I = \langle 0, T \rangle$ and a basic Hilbert space H be given, with the scalar product (u, v) and the associated norm $|u| = (u, u)^{1/2}$.

V will denote a Hilbert space with the scalar product $((u, v))$ and the norm $\|u\| = ((u, u))^{1/2}$.

$L_2(I, H)$ and $L_2(I, V)$ will denote the spaces of functions $u(t)$, mapping the interval I into H and V, respectively, and such that

$$\int_0^T |u(t)|^2 \, dt < \infty \quad \text{and} \quad \int_0^T \|u(t)\|^2 \, dt < \infty ,$$

respectively.

$\mathscr{L}(X, Y)$ denotes the space of linear continuous mappings of a Hilbert space X into a Hilbert space Y.

Let use denote $u'(t) = \mathrm{d}u/\mathrm{d}t$, $u''(t) = \mathrm{d}^2u/\mathrm{d}t^2$. Suppose that

(1) $$V \subset H\,, \quad \|u\| \geqq c|u| \quad \text{for every} \quad u \in V \quad \text{and}$$

(2) $$(u \in V,\ |u| = 0) \Rightarrow \|u\| = 0\,.$$

Let three real bilinear forms a_0, a_1, a_2, be given, depending on parameters t and t, τ, respectively, such that

(3) $a_0(t; u, v)$ and $a_1(t; u, v)$ are continuous on $V \times V$ for $t \in I$, $a_2(t, \tau; u, v)$ is continuous on $H \times H$ for $t, \tau \in I$.

(4) Moreover, let the operators $B(t) \in \mathscr{L}(V, H)$ and $C(t) \in \mathscr{L}(H, H)$ for $t \in I$ be given and let there exist positive constants $\alpha, \lambda, c_1, c_2, c_3$ such that

(5) $$a_0(t; u, v) = a_0(t, v, u)\,,$$

(6) $$a_0(t; v, v) + \lambda|v|^2 \geqq \alpha\|v\|^2$$

holds for every $u, v \in V$ and $t \in I$,

(7) $a_0(t; u, v) \in C^{(1)}(I)$, (continuously differentiable function of $t \in I$) for any fixed $u, v \in V$,

(8) $$a_1(t; u, v) \in C^{(1)}(I)\,,$$
$$|a_1(t; u, v)| \leqq c_1\|u\| \,.\, |v| \quad \text{for} \quad t \in I,\ u, v \in V\,,$$

(9) $$(v, B(t)\, v) \leqq c_2|v|^2\,,$$

(10) $$(f, B(t)\, v) \in C^{(1)}(I) \quad \text{for every} \quad f \in H,\ v \in V,\ t \in I\,,$$

(11) $$(C(t)\, u, v) = (u, C(t)\, v)\,,$$
$$(C(t)\, v, v) \geqq c_3|v|^2\,, \quad \text{for} \quad t \in I,\ u, v \in H\,,$$

(12) $$(C(t)\, u, v) \in C^{(1)}(I) \quad \text{for every} \quad u, v \in H\,,$$

(13) $$\left|\int_0^t a_2(t, \tau; u(\tau), v)\, \mathrm{d}\tau\right| \leqq c_4|v| \left(\int_0^t |u(\tau)|^2\, \mathrm{d}\tau\right)^{1/2}$$

for every $v \in H$, $u \in L_2(I, H)$ and almost all $t \in I$.

Furthermore, let

(14) $$f \in L_2(I, H), \quad v_0 \in H$$

be given.

Definition 1. (Weak solution of the Cauchy problem.)

$D(I)$ will denote the linear manifold of functions $\varphi \in L_2(I, V)$, for which $\varphi' \in L_2(I, H)$ and $\varphi(T) = \theta$.

We say that a function u is a solution of the Cauchy problem $\mathscr{P}(\theta, v_0, f)$, if $u \in L_2(I, V)$, $u' \in L_2(I, H)$, $u(0) = \theta$ and

$$(15) \qquad \int_0^T \Big\{ - (C(t)\, u'(t), \varphi'(t)) + (u'(t), B(t)\, \varphi(t)) + a_0(t; u(t), \varphi(t)) + \\ + a_1(t; u(t), \varphi(t)) + \int_0^t a_2(t, \tau; u(\tau), \varphi(t))\, d\tau \Big\}\, dt = \\ = \int_0^T (f(t), \varphi(t))\, dt + (C(0)\, v_0, \varphi(0))$$

holds for every $\varphi \in D(I)$.

Remark 1. The problem $\mathscr{P}(\theta, v_0, f)$ may be formally interpreted by an integro-differential equation

$$(16) \quad Lu \equiv \frac{d}{dt}(C(t)\, u'(t)) + B^*(t)\, u'(t) + A(t)\, u(t) + \int_0^t K(t, \tau)\, u(\tau)\, d\tau = f(t)$$

with initial conditions $u(0) = \theta$, $u'(0) = v_0$, if $u'' \in L_2(I, H)$, $A(t)$ means the (unbounded) operator, to which the bilinear form $a_0 + a_1$ is associated, $B(t) \in \mathscr{L}(H, H)$, $B^*(t)$ is the operator adjoint of $B(t)$ and $K(t, \tau) \in \mathscr{L}(H, H)$,

$$\int_0^T \|K(t, \tau)\|^2\, d\tau < \infty \quad \text{for almost all} \quad t \in I\,.$$

Remark 2. Let us suppose the "convolution symmetry" of operators occuring in (16), i.e., let

$$\int_0^T \Big(A(t)\, u(t) + \int_0^t K(t, \tau)\, u(\tau)\, d\tau\,, \quad v(T - t)\Big)\, dt = \\ = \int_0^T \Big(A(t)\, v(t) + \int_0^t K(t, \tau)\, v(\tau)\, d\tau, \; u(T - t)\Big)\, dt\,,$$

$$\int_0^T (B^*(t)\, u(t), v(T - t))\, dt = \int_0^t (B^*(t)\, v(t), u(T - t))\, dt\,,$$

$$\frac{d}{dt} \int_0^t (B^*(\tau)\, u(\tau), v(t - \tau))\, d\tau\Big|_{t=T} = \frac{d}{dt} \int_0^t (B^*(\tau)\, v(\tau), u(t - \tau))\, d\tau\Big|_{t=T}\,,$$

$$\int_0^T (C(t)\, u(t), v(T - t))\, dt = \int_0^T (C(t)\, v(t), u(T - t))\, dt\,.$$

Then (15) means the condition of the stationary value of the following functional (see [1])

$$\mathscr{F}(u) = \int_0^T (L\, u(t) - 2f(t), \; u(T - t))\, dt + (u'(0) - 2v_0, C(0)\, u(T))\,,$$

if we set $\delta u(T - t) = \varphi(t)$. Hence the Definition 1 extends the variational formulation of the problem to the equations with non-symmetric operators.

Example. In the theory of neutron fields the following integro-differential equation occurs for the unknown function $u(x, t)$ on $Q = (0, l) \times (0, T)$

$$c \frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + b \frac{\partial u}{\partial t} + au - \frac{\partial^2 u}{\partial x^2} + \sum_{j=1}^{m} \alpha_j \int_0^t e^{-\lambda_j(t-\tau)} u(\tau) \,\mathrm{d}\tau = \sum_{j=1}^{m} \beta_j e^{-\lambda_j t},$$

where $a, b, c, \alpha_j, \beta_j, \lambda_j$ $(j = 1, 2, \ldots, m)$ are given constants. Let us choose $H = L_2(0, l)$, $\mathring{W}_2^{(1)}(0, l) \subset V \subset W_2^{(1)}(0, l)$ in accordance with the kind of the boundary conditions and

$$a_0(t; u, v) = \int_0^l \left(auv + \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \right) \mathrm{d}x,$$

$$a_1(t; u, v) = 0,$$

$$a_2(t, \tau; u(\tau), v) = \sum_{j=1}^{m} \alpha_j e^{-\lambda_j(t-\tau)} \int_0^l u(\tau)\, v \,\mathrm{d}x.$$

Then all the suppositions (1) till (14) can be easily verified. The problem is even "potential" in the sense of Remark 2, because all the operators are "symmetrical in convolutions".

2. EXISTENCE AND UNIQUENESS THEOREM FOR THE PROBLEM $\mathscr{P}(0, v_0, f)$

The main object of the present section is the

Theorem 1. *Assume that (1) till (14) hold. Then there exists one and only one solution u of the problem $\mathscr{P}(0, v_0, f)$ and*

$$(17) \qquad \int_0^T (\|u(t)\|^2 + |u'(t)|^2) \,\mathrm{d}t \leqq \left\{ \int_0^T |f(t)|^2 \,\mathrm{d}t + |v_0|^2 \right\}.$$

First we shall prove some auxiliary lemmas.

Lemma 1. *Let H_1 and H_2 be two Hilbert spaces with the scalar products $(u, v)_1$ and $(u, v)_2$, respectively. Suppose that $a(t; u, v)$ is a bilinear form continuous on $H_1 \times H_2$ for every $t \in I$ and $a(t; u, v)$ is bounded on I for any fixed $u \in H_1$ and $v \in H_2$.*

Then there exists a constant M, independent of t, u, v, such that

$$|a(t, u, v)| \leqq M |u|_1 \, |v|_2$$

holds for every $t \in I_1$, $u \in H_1$ and $v \in H_2$.

Proof. For any $t \in I$ we may write

$$a(t; u, v) = (A(t)\, u, v)_2 ,$$

where $A(t) \in \mathscr{L}(H_1, H_2)$. Then the norms $\|A(t)\|$ are bounded on I. Indeed, if the contrary is true, there exists a sequence $\{t_n\} \subset I$, such that

$$\|A(t_n)\| \geqq n .$$

Using the well-known approach (see e.g. the proof of Banach's theorem on the boundedness of norms of linear continuous operators [4] § 19), we can prove the existence of an element $\bar{u} \in H_1$ and of a subsequence $\{t_k\} \subset \{t_n\}$, such that

$$|A(t_k)\, \bar{u}|_2 \geqq k .$$

Let us set $A(t_k)\, \bar{u} = w_k$. The supposition of our Lemma implies

$$|(w_k, v)_2| = |(A(t_k)\, \bar{u}, v)_2| < M(v)$$

for every $v \in H_2$, $k = 1, 2, \ldots$ Then also the norms of the functionals $(w_k, v)_2$, i.e. $|w_k|_2$, are bounded (see e.g. [5] § 23), which is a contradiction. Finally, it suffices to write

$$|a(t; u, v)| = |(A(t)\, u, v)_2| \leqq \|A(t)\|\, |u|_1\, |v|_2$$

and to restrict the norms of $A(t)$ by a number M.

Lemma 2. *The problem $\mathscr{P}(\theta, v_0, f)$ is equivalent to the problem $\mathscr{P}(\theta, v_0, e^{-kt}f)$ with the bilinear forms $\tilde{a}_i$ $(i = 0, 1, 2)$ and the operators $\tilde{B}(t)$, $C(t)$, satisfying (3) till (13) and such that*

(18) $$\tilde{a}_0(t; v, v) \geqq \alpha_1 \|v\|^2 ,$$

(19) $$\tilde{a}_0(t; v, v) + 2\tilde{a}_1(t; v, v) \geqq \alpha_1 \|v\|^2$$

holds for every $t \in I$, $v \in V$, with $\alpha_1 > 0$ independent of t, v.

Proof. Let us introduce a transformation

(20) $$u(t) = w(t)\, e^{kt} , \quad (k \text{ real constant}) .$$

The corresponding transformation of the trial functions in Definition 1 is

(21) $$\varphi(t) = \psi(t)\, e^{-kt}$$

according to Remark 2. Indeed,

$$\varphi(t) = \delta u(T - t) = e^{k(T-t)} \delta w(T - t) = e^{kT} \delta w(T - t)\, e^{-kt} .$$

Inserting (20) and (21) into (15), the result can be arranged in the original form, if we set

$$(22)\qquad \begin{aligned} \tilde{a}_0(t; w, \psi) &= a_0(t; w, \psi) + k^2(Cw, \psi)\,, \\ \tilde{a}_1(t; w, \psi) &= a_1(t; w, \psi) + k(w, B\psi) + k(C'w, \psi)\,, \\ \tilde{a}_2(t, \tau; w(\tau), \psi) &= e^{k(\tau - t)}\, a_2(t, \tau; w(\tau), \psi)\,, \\ \tilde{B}(t) &= B(t) + 2k\, C(t)\,, \quad \tilde{f}(t) = e^{-kt} f(t)\,, \end{aligned}$$

where $C' = d\, C(t)/dt$. We can verify without difficulties that the new bilinear forms, operator $\tilde{B}(t)$ and $\tilde{f}$ satisfy (3) till (14).

Using (7), (9), (12) and Lemma 1 (with $H_1 = H_2 = H, M = c'$), we derive

$$(23)\qquad \tilde{a}_1(t; v, v) \leqq c_1 \|v\|\, |v| + k\, c_2 |v|^2 + k\, c' |v|^2\,.$$

Let $0 < \alpha_1 < \alpha$. Then according to (22), (6), (11) and (23)

$$\begin{aligned} &\tilde{a}_0(t; v, v) + 2\tilde{a}_1(t; v, v) - \alpha_1 \|v\|^2 \geqq \\ &\geqq (\alpha - \alpha_1) \|v\|^2 - \lambda |v|^2 + k^2 c_3 |v|^2 - 2c_1 \|v\|\, |v| - 2k(c_2 + c')\, |v|^2 = \\ &= (\alpha - \alpha_1) \|v\|^2 - 2c_1 \|v\|\, |v| + (k^2 c_3 - 2kc_2 - 2kc' - \lambda)\, |v|^2 \geqq 0 \end{aligned}$$

holds for sufficiently great k^2. If

$$k^2 c_3 - \lambda \geqq 0$$

at the same time, we obtain (18) and (19).

Convention. *Henceforth we shall use the inequalities* (18) *and* (19) *instead of* (6), *which is justified by Lemma* 2.

The proof of existence of Theorem 1 will be based on the following projection

Theorem 2. (Lions [2], chpt. 3.) *Let F denote a Hilbert space with the norm $|u|_F$ and Φ its linear subset with the norm $|\varphi|_\Phi$. (Φ need not to be complete with this norm.) Let a constant $c > 0$ exists such that*

$$(24)\qquad |\varphi|_F \leqq c |\varphi|_\Phi \quad \textit{for every} \quad \varphi \in \Phi\,.$$

Let $E(u, \varphi)$ be a bilinear form on $F \times \Phi$ such that

(25) *$E(u, \varphi)$ represents a continuous functional on F for every $\varphi \in \Phi$ and a constant $\alpha > 0$ exists such that*

$$(26)\qquad |E(\varphi, \varphi)| \geqq \alpha |\varphi|_\Phi^2 \quad \textit{for every} \quad \varphi \in \Phi\,.$$

Moreover, let $L(\varphi)$ be a linear continuous functional on Φ. Then there exists $u \in F$ such that

$$E(u, \varphi) = L(\varphi) \quad \textit{for every} \quad \varphi \in \Phi \quad \textit{and} \tag{27}$$

$$|u|_F \leqq c\, \alpha^{-1} \|L\| , \tag{28}$$

where

$$\|L\| = \sup |L(\varphi)| \quad \textit{on the set} \quad \varphi \in \Phi , \quad |\varphi|_\Phi = 1 .$$

Lemma 3. *Let φ satisfy the following conditions*

$$\varphi \in L_2(I, V) , \quad \varphi' \in L_2(I, V) , \quad \varphi'' \in L_2(I, H) , \tag{29}$$

$$\varphi(0) = \theta , \quad \varphi'(T) = \theta . \tag{30}$$

Then the inequality

$$\int_0^T \Big\{ a_0(t; e^{-\gamma t}\varphi, e^{-\gamma t}\varphi') + a_1(t; e^{-\gamma t}\varphi, e^{-\gamma t}\varphi') + \tag{31}$$

$$+ \int_0^t a_2(t, \tau; e^{-\gamma t}\, \varphi(\tau), e^{-\gamma t}\, \varphi'(t))\, d\tau + (e^{-\gamma t}\varphi', B(t)\, e^{-\gamma t}\varphi') -$$

$$- (C(t)\, \varphi', (e^{-2\gamma t}\varphi')') \Big\}\, dt \geqq c_6 |\varphi|_\Phi^2 ,$$

where

$$|\varphi|_\Phi^2 = \int_0^T \{\|e^{-\gamma t}\varphi\|^2 + |e^{-\gamma t}\varphi'|^2\}\, dt + |\varphi'(0)|^2 , \tag{32}$$

holds for sufficiently great $\gamma > 0$.

Proof. We shall estimate separate terms gradually. Using (5), (7), (30), (18) and the inequality

$$|a_0'(t; \varphi, \varphi)| \leqq c_5 \|\varphi\|^2 ,$$

which is a consequence of (3), (7) and Lemma 1, we may write

$$2\int_0^T a_0(t; e^{-\gamma t}\varphi, e^{-\gamma t}\varphi')\, dt = \int_0^T e^{-2\gamma t} \left\{ \frac{d}{dt}\, a_0(t; \varphi, \varphi) - a_0'(t; \varphi, \varphi) \right\} dt =$$

$$= \int_0^T e^{-2\gamma t}\{2\gamma a_0(t; \varphi, \varphi) - a_0'\, (t; \varphi, \varphi)\}\, dt + e^{-2\gamma T} a_0(T; \varphi(T), \varphi(T) \geqq$$

$$\geqq \int_0^t e^{-2\gamma t}(2\gamma\alpha_1 \|\varphi(t)\|^2 - c_5 \|\varphi(t)\|^2)\, dt .$$

Next we derive on the base of (13)

$$(33) \qquad 2\left|\int_0^T \mathrm{d}t \int_0^t a_2(t, \tau; e^{-\gamma t}\,\varphi(\tau), e^{-\gamma t}\,\varphi'(t))\,\mathrm{d}\tau\right| \leqq$$

$$\leqq c_4 \int_0^T 2|e^{-\gamma t}\,\varphi'(t)| \left(\int_0^t |e^{-\gamma t}\,\varphi(\tau)|^2\,\mathrm{d}\tau\right)^{1/2} \mathrm{d}t \leqq$$

$$\leqq c_4 \int_0^T |e^{-\gamma t}\,\varphi'(t)|^2\,\mathrm{d}t + c_4 \int_0^T \mathrm{d}t \int_0^t |e^{-\gamma t}\,\varphi(\tau)|^2\,\mathrm{d}\tau\,.$$

In the last integral it is possible to change the order of integration to obtain

$$(34) \qquad \int_0^T \mathrm{d}\tau \int_\tau^T e^{-2\gamma t}|\varphi(\tau)|^2\,\mathrm{d}t = \frac{1}{2\gamma}\int_0^T (e^{-2\gamma\tau} - e^{-2\gamma T})|\,\varphi(\tau)|^2\,\mathrm{d}\tau \leqq$$

$$\leqq \frac{1}{2\gamma}\int_0^T |e^{-\gamma t}\,\varphi(t)|^2\,\mathrm{d}t\,.$$

Using also (8) and (1), we may write

$$2\left|\int_0^T \left\{a_1(t; e^{-\gamma t}\varphi, e^{-\gamma t}\varphi') + \int_0^t a_2(t, \tau; e^{-\gamma t}\,\varphi(\tau), e^{-\gamma t}\,\varphi'(t))\,\mathrm{d}\tau\right\} \mathrm{d}t\right| \leqq$$

$$\leqq \int_0^T \{2c_1\|e^{-\gamma t}\varphi\|\;|e^{-\gamma t}\varphi'| + c_4|e^{-\gamma t}\,\varphi'(t)|^2 + \tfrac{1}{2}c_4\gamma^{-1}c^{-2}\|e^{-\gamma t}\,\varphi(t)\|^2\}\,\mathrm{d}t\,.$$

According to (9), (30), (11) and (12), we have

$$2\left|\int_0^T (e^{-\gamma t}\varphi', B(t)\,e^{-\gamma t}\varphi')\,\mathrm{d}t\right| \leqq 2c_2 \int_0^T |e^{-\gamma t}\,\varphi'(t)|^2\,\mathrm{d}t$$

and

$$-2\int_0^T (C(t)\,\varphi', (e^{-2\gamma t}\varphi')')\,\mathrm{d}t = -\int_0^T \{(C(t)\,\varphi', (e^{-2\gamma t}\varphi')') +$$

$$+ ((e^{-2\gamma t}\varphi')', C(t)\,\varphi')\}\,\mathrm{d}t = (C(0)\,\varphi'(0), \varphi'(0)) + 2\gamma \int_0^T e^{-2\gamma t}(C(t)\,\varphi', \varphi')\,\mathrm{d}t +$$

$$+ \int_0^T e^{-2\gamma t}(C'(t)\,\varphi', \varphi')\,\mathrm{d}t \geqq c_3|\varphi'(0)|^2 + \int_0^T (2\gamma c_3 - c')\,|e^{-\gamma t}\,\varphi'(t)|^2\,\mathrm{d}t\,.$$

Altogether, the left-hand side of (31) is greater or equal to the expression

$$\int_0^T \{(\gamma\alpha_1 - \tfrac{1}{2}c_5 - \tfrac{1}{4}c_4c^{-2}\gamma^{-1})\,\|e^{-\gamma t}\,\varphi(t)\|^2 - c_1\|e^{-\gamma t}\,\varphi(t)\|\;|e^{-\gamma t}\,\varphi'(t)| +$$

$$+ (\gamma c_3 - \tfrac{1}{2}c' - c_2 - \tfrac{1}{2}c_4)\,|e^{-\gamma t}\,\varphi'(t)|^2\}\,\mathrm{d}t + \tfrac{1}{2}c_3|\,\varphi'(0)|^2\,.$$

Substituting for the product in the second term the sum of quadrates, we obtain the assertion to be proved.

Proof of Theorem 1. *Existence.* Let us choose in Theorem 2

$$F : \{u \in L_2(I, V),\ u' \in L_2(I, H),\ u(0) = \theta\}$$

with the norm

$$|u|_F^2 = \int_0^T (\|e^{-\gamma t} u(t)\|^2 + |e^{-\gamma t} u'(t)|^2)\, dt$$

and let Φ be the set of functions, satisfying (29), (30) and possessing the norm (32). Obviously (24) holds with $c = 1$. Let us define for $u \in F$, $\varphi \in \Phi$

$$E(u, \varphi) = \int_0^T \{a_0(t; e^{-\gamma t} u(t), e^{-\gamma t} \varphi'(t)) + a_1(t; e^{-\gamma t} u(t), e^{-\gamma t} \varphi'(t)) +$$
$$+ \int_0^t a_2(t, \tau; e^{-\gamma t} u(\tau), e^{-\gamma t} \varphi'(t))\, d\tau + (e^{-\gamma t} u'(t), B(t)\, e^{-\gamma t} \varphi'(t)) -$$
$$- (C(t)\, u'(t), (e^{-2\gamma t} \varphi'(t))')\, dt\,.$$

We shall show, that the functional $E(u, \varphi)$ is continuous on F for every $\varphi \in \Phi$. Indeed, using (3), (7), Lemma 1, (1), (8), (13), further (4), (10), (12) and again Lemma 1, we obtain

$$|E(u, \varphi)| \leqq \int_0^T \Big\{M\|e^{-\gamma t}\varphi'\|\, \|e^{-\gamma t}u\| + c_1 c^{-1}\|e^{-\gamma t}\varphi'\|\, \|e^{-\gamma t}u\| +$$
$$+ c_4|e^{-\gamma t}\varphi'| \left(\int_0^t |e^{-\gamma t} u(\tau)|^2\, d\tau\right)^{1/2} + c_7|e^{-\gamma t}u'|\, \|e^{-\gamma t}\varphi'\| +$$
$$+ 2\gamma c_8|e^{-\gamma t}u'|\, |e^{-\gamma t}\varphi'| + c_8|e^{-\gamma t}u'|\, |e^{-\gamma t}\varphi''|\Big\}\, dt \leqq$$
$$\leqq \int_0^T \left(\|e^{-\gamma t}u\|^2 + |e^{-\gamma t}u'|^2 + \int_0^t |e^{-\gamma t} u(\tau)|^2\, d\tau\right)^{1/2} G(\varphi, \varphi', \varphi'')\, dt\,,$$

where $G(\varphi, \varphi', \varphi'')$ is a square-integrable function on I. Applying the Cauchy-Buniakovski inequality, we derive

$$|E(u, \varphi)| \leqq \left(\int_0^T (\|e^{-\gamma t}u\|^2 + |e^{-\gamma t}u'|^2 +\right.$$
$$\left. + \int_0^t |e^{-\gamma t} u(\tau)\ ^2\, d\tau)\, dt\right)^{1/2} \left(\int_0^T G^2(\varphi, \varphi', \varphi'')\, dt\right)^{1/2}.$$

Fubini's theorem yields, like in (34), that

$$\int_0^T dt \int_0^t |e^{-\gamma t} u(\tau)|^2\, d\tau \leqq \tfrac{1}{2}\gamma^{-1} \int_0^T |e^{-\gamma t} u(t)|^2\, dt \leqq \tfrac{1}{2}\gamma^{-1} c^{-2} \int_0^T \|e^{-\gamma t} u(t)\|^2\, dt\,,$$

consequently

$$|E(u, \varphi)| \leqq |u|_F (1 + \tfrac{1}{2}\gamma^{-1}c^{-2})^{1/2} \left(\int_0^T G^2(\varphi, \varphi', \varphi'')\, dt\right)^{1/2}.$$

According to Lemma 3, we can choose a positive γ such that (26) will be satisfied. Furthermore, let us set

$$L(\varphi) = \int_0^T (e^{-\gamma t} f(t), e^{-\gamma t} \varphi'(t))\, dt + (C(0)\, v_0, \varphi'(0)).$$

$L(\varphi)$ is continuous on Φ, because

$$(34') \quad |L(\varphi)| \leqq \left(\int_0^T |e^{-\gamma t} f(t)|^2\, dt\right)^{1/2} \left(\int_0^T |e^{-\gamma t} \varphi'(t)|^2\, dt\right)^{1/2} + \|C(0)\|\, |v_0|\, |\varphi'(0)| \leqq$$

$$\leqq |\varphi|_\Phi \left(\int_0^T |e^{-\gamma t} f(t)|^2\, dt + \|C(0)\|^2\, |v_0|^2\right)^{1/2}.$$

Theorem 2 yields the existence of an element $u \in F$, which satisfies (27) and (28). We shall prove, that this element represents a solution of the problem $\mathscr{P}(\theta, v_0, f)$ according to Definition 1. It suffices to verify (15). Let us consider an arbitrary $\varphi \in D(I)$ and set

$$(35) \qquad \varphi_0(t) = \int_0^t e^{2\gamma\tau} \varphi(\tau)\, d\tau \quad \text{for} \quad t \in I,$$

consequently

$$\varphi(t) = e^{-2\gamma t} \varphi_0'(t), \quad \varphi_0(0) = \varphi_0'(T) = \theta, \quad \varphi_0 \in \Phi.$$

Then

$$E(u, \varphi_0) = L(\varphi_0)$$

follows from (27) and inserting (35) into this equation, we obtain

$$\int_0^T \Big\{ a_0(t; e^{-\gamma t} u(t), e^{\gamma t} \varphi(t)) + a_1(t; e^{-\gamma t} u(t), e^{\gamma t} \varphi(t)) +$$

$$+ \int_0^t a_2(t, \tau; e^{-\gamma \tau} u(\tau), e^{\gamma t} \varphi(t))\, d\tau + (e^{-\gamma t} u'(t), B(t)\, e^{\gamma t} \varphi(t)) -$$

$$- (C(t)\, u'(t), \varphi'(t)) \Big\}\, dt = \int_0^T (e^{-\gamma t} f(t), e^{\gamma t} \varphi(t))\, dt + (C(0)\, v_0, \varphi(0)).$$

The exponential functions may be cancelled out and we conclude that (15) holds. Thus the proof of existence is complete.

Uniqueness. It suffices to prove, that the problem $\mathscr{P}(\theta, \theta, \theta)$ has only trivial solution. Let u be a solution of $\mathscr{P}(\theta, \theta, \theta)$. Choose $0 < s < T$ and define

$$\varphi(t) = -\int_t^s u(\tau)\,\mathrm{d}\tau \quad \text{for} \quad t \leqq s\,, \tag{36}$$
$$\varphi(t) = \theta \quad \text{for} \quad t \geqq s\,.$$

We can easily verify, that $\varphi \in D(I)$. Inserting $f = \theta$, $v_0 = \theta$ and (36) into (15), we obtain

$$2\int_0^s \Big\{a_0(t;\varphi'(t),\varphi(t)) + a_1(t;\varphi'(t),\varphi(t)) + \int_0^t a_2(t,\tau;u(\tau),\varphi(t))\,\mathrm{d}\tau + $$
$$+ (u'(t), B(t)\,\varphi(t)) - (C(t)\,u'(t), u(t))\Big\}\,\mathrm{d}t = 0\,.$$

Making use of (5), (7) and (11), we may write

$$\int_0^s \Big\{\frac{\mathrm{d}}{\mathrm{d}t}\,a_0(t;\varphi(t),\varphi(t)) - a_0'(t;\varphi(t),\varphi(t)) + 2a_1(t;\varphi'(t),\varphi(t)) + $$
$$+ 2(u'(t), B(t)\,\varphi(t)) - \frac{\mathrm{d}}{\mathrm{d}t}(C(t)\,u(t), u(t)) + (C'(t)\,u(t), u(t)) + $$
$$+ 2\int_0^t a_2(t,\tau;u(\tau),\varphi(t))\,\mathrm{d}\tau\Big\}\,\mathrm{d}t = 0$$

and therefore

$$a_0(0;\varphi(0),\varphi(0)) + (C(s)\,u(s), u(s)) = \tag{37}$$
$$= \int_0^s \Big\{2a_1(t;\varphi',\varphi) - a_0'(t;\varphi,\varphi) + 2(u'(t), B(t)\,\varphi(t)) + $$
$$+ (C'(t)\,u(t), u(t)) + 2\int_0^t a_2(t,\tau;u(\tau),\varphi(t))\,\mathrm{d}\tau\Big\}\,\mathrm{d}t\,.$$

Using the identities

$$\int_0^s a_1(t;\varphi',\varphi)\,\mathrm{d}t = -a_1(0;\varphi(0),\varphi(0)) - \int_0^s \{a_1(t;\varphi,\varphi') + a_1'(t;\varphi,\varphi)\}\,\mathrm{d}t\,,$$
$$\int_0^s (u', B(t)\,\varphi)\,\mathrm{d}t = -\int_0^s (u, B(t)\,u)\,\mathrm{d}t - \int_0^s (u, B'(t)\,\varphi)\,\mathrm{d}t\,,$$

the relation (37) may be rewritten as follows

$$a_0(0;\varphi(0),\varphi(0)) + 2a_1(0;\varphi(0),\varphi(0)) + (C(s)\,u(s), u(s)) = \tag{38}$$
$$= \int_0^s \Big\{-2a_1(t;\varphi,\varphi') - 2a_1'(t;\varphi,\varphi) - a_0'(t;\varphi,\varphi) - 2(u, B(t)\,u) - $$
$$- 2(u, B'(t)\,\varphi) + (C'(t)\,u, u) + 2\int_0^t a_2(t,\tau;u(\tau),\varphi(t))\,\mathrm{d}\tau\Big\}\,\mathrm{d}t\,.$$

The left-hand side of (38) can be estimated from below by means of Lemma 2, (19) and (11), the right-hand side from above by means of (8), (7), (3), (9), (4), (10), (12), Lemma 1 and the inequality

$$\left|\int_0^s 2\,\mathrm{d}t \int_0^t a_2(t, \tau; u(\tau), \varphi(t))\,\mathrm{d}\tau\right| \leqq c_4 \int_0^s 2|\varphi(t)| \left(\int_0^t |u(\tau)|^2\,\mathrm{d}\tau\right)^{1/2} \mathrm{d}t \leqq$$

$$\leqq c_4 \int_0^s \left\{|\varphi(t)|^2 + \int_0^s |u(\tau)|^2\,\mathrm{d}\tau\right\} \mathrm{d}t = c_4 \int_0^s (|\varphi(t)|^2 + s|u(t)|^2)\,\mathrm{d}t\,.$$

Thus we obtain

$$\alpha_1\|\varphi(0)\|^2 + c_3|u(s)|^2 \leqq$$

$$\leqq \int_0^s \{2c_1\|\varphi\|\,|u| + 2c_1'\|\varphi\|^2 + c_5\|\varphi\|^2 + 2c_2|u|^2 + 2c_2'|u|\,\|\varphi\| + c'|u|^2 +$$

$$+ c_4|\varphi|^2 + c_4\,T|u|^2\}\,\mathrm{d}t\,,$$

which yields

$$\|\varphi(0)\|^2 + |u(s)|^2 \leqq c_9 \int_0^s (\|\varphi(t)\|^2 + |u(t)|^2)\,\mathrm{d}t\,.$$

Let us introduce

$$v(t) = \int_0^t u(\tau)\,\mathrm{d}\tau\,,$$

so that $\varphi(t) = v(t) - v(s)$. Then

$$\|v(s)\|^2 + |u(s)|^2 \leqq c_9 \int_0^s \{2\|v(t)\|^2 + |u(t)|^2\}\,\mathrm{d}t + 2c_9 s\|v(s)\|^2\,. \tag{39}$$

In case that $1 - 2c_9 T > 0$, we have

$$\|v(s)\|^2 + |u(s)|^2 \leqq k \int_0^s \{\|v(t)\|^2 + |u(t)|^2\}\,\mathrm{d}t \tag{40}$$

for all $s \in (0, T)$, (k = const.). We shall need the following

Lemma 4. *Let $\omega(s) \in L_2(0, s_0)$ be a real function such that*

$$\omega^2(s) \leqq k \int_0^s \omega^2(t)\,\mathrm{d}t \tag{41}$$

holds (almost everywhere) in $(0, s_0)$. Then $\omega(s) = 0$ almost everywhere.

Proof. Let us set

$$\int_0^{s_0} \omega^2(t)\,\mathrm{d}t = v\,.$$

From (41) it follows gradually

$$\omega^2(s) \leqq kv\,,$$

$$\omega^2(s) \leqq k\int_0^s kv\,\mathrm{d}t = k^2 vs\,,$$

in general

$$\omega^2(s) \leqq vk^{n+1}\,\frac{s^n}{n!}\,,\quad n = 0, 1, 2, \ldots$$

Consequently

$$\omega^2(s) \leqq vk^{n+1}\,\frac{s_0^n}{n!}\,,$$

which converges to zero for $n \to \infty$, hence $\omega(s) = 0$.

By virtue of Lemma 4 (for $s_0 = T$) and (40), in case that $1 - 2c_9 T > 0$, we have

$$\int_0^T |u(t)|^2\,\mathrm{d}t = 0$$

and using also (2), we obtain

$$\int_0^T \|u(t)\|^2\,\mathrm{d}t = 0\,. \tag{42}$$

Next suppose that $1 - 2c_9 T \leqq 0$. Then there exists $0 < s_0 < T$ such that $1 - 2c_9 s_0 = \frac{1}{2}$, Lemma 4 yields that $|u(t)| = 0$ almost everywhere in $(0, s_0)$. The function $u(t)$ is equal, however, to a continuous mapping of I into H almost everywhere in I (a consequence of Definition 1). Therefore we may set $|u(t)| = 0$ for all $t \in \langle 0, s_0\rangle$. The above-described procedure, starting from definition (36), can be now repeated on the interval $\langle s_0, T\rangle$ or $\langle s_0, 2s_0\rangle$, respectively, until the conclusion (42) is reached.

Continuous dependence on the given data. According to (28) and Lemma 3, it holds

$$|u|_F^2 \leqq c_6^{-2}\|L\|^2\,.$$

Making use of (34′) and the definition of F, we derive

$$\|L\|^2 \leqq \beta\left(\int_0^T |f(t)|^2\,\mathrm{d}t + |v_0|^2\right),\quad (\beta = \text{const.})\,,$$

$$|u|_F^2 \geqq e^{-2\gamma T}\int_0^T (\|u(t)\|^2 + |u'(t)|^2)\,\mathrm{d}t\,. \tag{43}$$

Thus we are to the inequality (17), if $c = \beta e^{2\gamma T} c_6^{-2}$.

3. EXISTENCE AND UNIQUENESS THEOREM FOR THE PROBLEM $\mathscr{P}(u_0, v_0, f)$

Up to this time we have dealt only with the homogeneous initial condition $u(0) = \theta$. In the present section we shall introduce the complete non-homogeneous Cauchy problem and prove the existence and uniqueness of its solution.

Definition 2. *Assume* (1) *till* (14), *furthermore let* $B(t) \in \mathscr{L}(H, H)$ *and* $u_0 \in V$ *be given. We say that a function* u *is a solution of the Cauchy problem* $\mathscr{P}(u_0, v_0, f)$, *if*

$$u \in L_2(I, V)\,, \quad u' \in L_2(I, H)\,, \quad u(0) = u_0 \tag{44}$$

and (15) *holds for any* $\varphi \in D(I)$.

Remark 3. In case of the "convolution symmetry" from Remark 2, Definition 2 expresses the condition of the stationary value of the functional [1] (assuming moreover $C'(T) = \theta$)

$$\mathscr{F}_1(u) = \int_0^T ((Lu - 2f)(t)\,, \quad u(T - t))\,\mathrm{d}t + (u'(0) - 2v_0, C(0)\,u(T)) - \\ - (u_0, B^*(T)\,u(T) + C(T)\,u'(T))\,,$$

which is defined on the set of functions satisfying (44).

Let us set

$$w(t) = u_0\left(1 - \frac{t}{T}\right).$$

Obviously, w satisfies (44). Defining $U(t) = u(t) - w(t)$, we are led to the equivalent problem $\mathscr{P}_1(\theta, v_0, f_1)$ for $U(t)$. A function U will be called a solution of the Cauchy problem $\mathscr{P}_1(\theta, v_0, f_1)$, if

$$U \in L_2(I, V)\,, \quad U' \in L_2(I, H)\,, \quad U(0) = \theta$$

and

$$\int_0^T \Big\{a_0(t; U, \varphi) + a_1(t; U, \varphi) + \int_0^t a_2(t, \tau; U(\tau), \varphi(t))\,\mathrm{d}\tau + \\ + (U'(t), B(t)\,\varphi(t)) - (C(t)\,U'(t), \varphi'(t))\Big\}\,\mathrm{d}t = \\ = \int_0^T \Big\{\frac{1}{T}(u_0, B(t)\,\varphi(t)) - \frac{1}{T}(C(t)\,u_0, \varphi'(t)) - a_0(t; w(t), \varphi(t)) - \\ - a_1(t; w(t), \varphi(t)) - \int_0^t a_2(t, \tau; w(\tau), \varphi(t))\,\mathrm{d}\tau + (f(t), \varphi(t))\Big\}\,\mathrm{d}t + (C(0)\,v_0, \varphi(0)) \tag{45}$$

for every $\varphi \in D(I)$.

Theorem 3. *Assume* (1) *till* (14), $u_0 \in V$ *and* $B(t) \in \mathscr{L}(H, H)$. *Then the problem* $\mathscr{P}(u_0, v_0, f)$ *has precisely one solution* u *and*

$$(46) \qquad \int_0^T (\|u(t)\|^2 + |u'(t)|^2)\,\mathrm{d}t \leqq c\left(\|u_0\|^2 + |v_0|^2 + \int_0^T |f(t)|^2\,\mathrm{d}t\right)$$

holds.

Proof. *Existence.* We may proceed in the same way as in the proof of Theorem 1, choosing the same spaces F and Φ in Theorem 2. The only change will be in the definition of the functional $L(\varphi)$ which is now in accordance with (45),

$$(47) \qquad L(\varphi) = \int_0^T \left\{\frac{1}{T}(e^{-\gamma t}u_0, B(t)\, e^{-\gamma t}\varphi') - \frac{1}{T}(C(t)\, u_0, (e^{-2\gamma t}\varphi')') - \right.$$

$$- a_0(t; e^{-\gamma t}w, e^{-\gamma t}\varphi') - a_1(t; e^{-\gamma t}w, e^{-\gamma t}\varphi') -$$

$$\left. - \int_0^t a_2(t, \tau; e^{-\gamma t}\, w(\tau), e^{-\gamma t}\, \varphi'(t))\,\mathrm{d}\tau + (e^{-\gamma t}f, e^{-\gamma t}\varphi')\right\}\mathrm{d}t + (C(0)\, v_0, \varphi'(0))\,.$$

We can show that the new functional $L(\varphi)$ is continuous on Φ. First let us integrate several terms by parts, using (12) and (30)

$$(48) \quad \int_0^T (C(t)\, u_0, (e^{-2\gamma t}\varphi')')\,\mathrm{d}t = -\int_0^T (C'(t)\, u_0, e^{-2\gamma t}\varphi')\,\mathrm{d}t - (C(0)\, u_0, \varphi'(0))\,,$$

$$(49) \qquad \int_0^T a_0(t; e^{-\gamma t}w, e^{-\gamma t}\varphi')\,\mathrm{d}t = \int_0^T a_0(t; e^{-2\gamma t}w, \varphi')\,\mathrm{d}t =$$

$$= -\int_0^T \{a_0(t; (e^{-2\gamma t}w)', \varphi) + a_0'(t; e^{-2\gamma t}w, \varphi)\}\,\mathrm{d}t =$$

$$= \int_0^t \left\{e^{-\gamma t}a_0\left(t; \frac{1}{T}u_0 + 2\gamma w, e^{-\gamma t}\varphi\right) - a_0'(t; e^{-\gamma t}w, e^{-\gamma t}\varphi)\right\}\mathrm{d}t\,.$$

Estimating $L(\varphi)$ by means of (48), (49) and some inequalities, which have been used in the preceding sections, we obtain

$$(50) \qquad |L(\varphi)| \leqq \int_0^T \left\{c_7 \frac{1}{T}|e^{-\gamma t}u_0|\,|e^{-\gamma t}\varphi'| + c'\frac{1}{T}|e^{-\gamma t}u_0|\,|e^{-\gamma t}\varphi'| + \right.$$

$$+ Me^{-\gamma t}\left\|\frac{1}{T}u_0 + 2\gamma w\right\|\,\|e^{-\gamma t}\varphi\| + c_5\|e^{-\gamma t}w\|\,\|e^{-\gamma t}\varphi\| +$$

$$+ c_1 \|e^{-\gamma t} w\| \, |e^{-\gamma t} \varphi'| + c_4 |e^{-\gamma t} \varphi'| \left(\int_0^t |e^{-\gamma t} w(\tau)|^2 \, d\tau \right)^{1/2} +$$

$$+ |e^{-\gamma t} f| \, |e^{-\gamma t} \varphi'| \Big\} \, dt + \|C(0)\| \left(|u_0| \, |\varphi'(0)| + |v_0| \, |\varphi'(0)| \right) \leqq$$

$$\leqq \beta_1 \left(\int_0^T (\|e^{-\gamma t} \varphi\|^2 + |e^{-\gamma t} \varphi'|^2) \, dt \right)^{1/2} \left(\int_0^T \left\{ \left\| \frac{1}{T} u_0 + 2\gamma w \right\|^2 + \right. \right.$$

$$\left. \left. + \|e^{-\gamma t} w\|^2 + |e^{-\gamma t} u_0|^2 + |e^{-\gamma t} f|^2 + |e^{-\gamma t} w|^2 \right\} dt \right)^{1/2} +$$

$$+ \|C(0)\| \left(|u_0| + |v_0| \right) |\varphi'(0)| \leqq \beta \left(\|u_0\|^2 + |v_0|^2 + \int_0^T |f(t)|^2 \, dt \right)^{1/2} |\varphi|_\Phi ,$$

where β is independent of φ.

Theorem 2 says, that there exists $\tilde{u} \in F$ such that (27) and (28) holds. Choosing again any $\varphi \in D(I)$ and φ_0 according to (35), the relation (27) can be rewritten in the form of (45); consequently $\tilde{u}$ is a solution of the problem $\mathscr{P}_1(\theta, v_0, f_1)$. Then $\tilde{u} + w = u$ represents a solution of the problem $\mathscr{P}(u_0, v_0, f)$.

Uniqueness. The difference of any two solutions of the problem $\mathscr{P}(u_0, v_0, f)$ is a solution of the problem $\mathscr{P}(\theta, \theta, \theta)$. From Theorem 1 it follows, that the latter problem has only trivial solution.

Continuous dependence on the given data. By virtue of (28), Lemma 3 and (50), we may write

$$|\tilde{u}|_F^2 \leqq c_6^{-2} \|L\|^2 \leqq \beta c_6^{-2} \left(\|u_0\|^2 + |v_0|^2 + \int_0^T |f(t)|^2 \, dt \right).$$

Using also (43), the inequality (46) can be obtained.

References

[1] *Hlaváček I.:* Variational formulation of the Cauchy problem for equations with operator coefficients. Aplikace matematiky 16 (1971) 1, 46—63.

[2] *Lions J. L.:* Equations differentielles operationnelles et problèmes aux limites, Grundlehren Math. Wiss., Bd. 111, Springer 1961.

[3] *Ладыженская О. А.:* Смешанная загача для гиперболических уравнений. Москва 1953.

[4] *Люстерник Л. А., Соболев В. И.:* Элементы функционального анализа. Москва 1951.

[5] *Achieser N. I., Glasmann I. M.:* Theorie der linearen Operatoren im Hilbert-Raum. Akademie-Verlag Berlin 1958.

Výtah

EXISTENCE A JEDNOZNAČNOST ŘEŠENÍ CAUCHYOVY ÚLOHY PRO LINEÁRNÍ INTEGRO-DIFERENCIÁLNÍ ROVNICE S OPERÁTOROVÝMI KOEFICIENTY

IVAN HLAVÁČEK

V teorii neutronových polí vznikají úlohy, které lze popsat integro-diferenciálními rovnicemi s počátečními podmínkami. Cílem tohoto článku je definovat určitou třídu problémů zahrnující zmíněný fyzikální příklad, a dokázat korektnost těchto úloh.

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