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ELLIPTIC BOUNDARY VALUE PROBLEMS WITH NONVARIATIONAL PERTURBATION AND THE FINITE ELEMENT METHOD

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I. INTRODUCTION

We shall consider boundary value problems of the type $Pu = f$ on a bounded domain $\Omega \subset R_n$ with homogeneous boundary conditions $B_j u = 0$ on Γ (the boundary of Ω), $j = 1, \ldots, m$. B_j are linear differential operators and $P = L + K$ where L is an elliptic operator and K a "small" peturbation. A numerical solution of this problem by "classical" Galerkin's method is investigated e.g. in [3]. Furthermore, our problem without the perturbation K is solved in [2] (this means that $P = L$). The method described in [2] (the so called finite element method) will be generalized to the problem with the perturbation. We obtain a slightly more general method than that proposed in [3].

The task of this paper is:

a) to formulate the properties of K in such a way that they cover as many practical cases as possible (see paragraph IV)

b) to define a weak solution of our problem and to find a necessary and sufficient condition of solvability (see paragraph II)

c) to apply the finite element method and to prove some facts concerning the convergence of this method (see paragraph III). •

We start with some standard notation:

1) Ω is a bounded domain in R_n, Γ is the boundary of Ω;

2) $H^l(\Omega)$ (l is a real number) is a Sobolev's space – see [1];

3) $\|\cdot\|_l$ is a norm in $H^l(\Omega)$ – see [1];

4) if $u, v \in H^0(\Omega)$, then (u, v) is an inner product in $L_2(\Omega)$;

5) $H^{-l}(\Omega)$ is isomorphic with $(H^l(\Omega))'$;

6) $g \in H^{-l}(\Omega)$, $\varphi \in H^l(\Omega)$, $\tilde{g} \in (H^l(\Omega))'$ is isomorphic with g; we denote the value of the linear functional $\tilde{g}$ at the point φ by $[g, \varphi]$; if $g \in H^0(\Omega)$, then $[g, \varphi] = (g, \varphi)$ – see [1];

7) let $T: H^r(\Omega) \to H^s(\Omega)$ be linear and bounded. Then

$$\|T\|_{s,r} = \sup_{\|u\|_r \neq 0 u \in \mathscr{D}(T)} \frac{\|Tu\|_s}{\|u\|_r}$$

We shall solve problems (1) and (2):

$$Lu + Ku = f \quad \text{on} \quad \Omega\,, \tag{1}$$

where f is a distribution on Ω and $Lu = \sum_{|k|,|l| \leqq m} D^k(a_{kl} D^l u)$ and $K: H^m(\Omega) \to H^q(\Omega)$, $q > -m$ is linear, bounded, $a_{kl} \in C^\infty(\bar{\Omega})$;

$$B_i u = 0 \quad \text{on} \quad \Gamma\,, \qquad i = 0, \ldots, m-1 \tag{2}$$

where $B_i u = \sum_{|j| \leqq m_i} b_{ij} D^j u$, $0 \leqq m_i \leqq 2m - 1$, $b_{ij} \in C^\infty(\Gamma)$. Let us define the classical solution of (1) & (2) as a function $u \in C^{2m}(\Omega) \cap C^{2m-1}(\bar{\Omega})$ which obeys (1) in the sense of distributions and (2) simultaneously. Let $V = \{u \mid u \in C^{2m}(\Omega) \cap C^{2m-1}(\bar{\Omega}),\ B_j u = 0 \text{ on } \Gamma,\ j = 0, \ldots, m-1\}$ and $W = \bar{V}$ where the closure V is understood in the norm $\|\cdot\|_m$. We shall say that W is the space of weak solutions. W is well-known to be a Hilbert space with the norm $\|\cdot\|_m$ and $H_0^m(\Omega) \subseteq W \subseteq H^m(\Omega)$.

We keep two assumptions (A) and (B) which are used usually for solving (1) & (2) without a "perturbation" (i.e., when $Ku \equiv 0$ for all $u \in W$):

Assumption (A): If $u \in W \cap C^{2m}(\Omega) \cap C^{2m-1}(\bar{\Omega})$, then:

a) $a(u, \varphi) = \sum_{|k|,|l| \leqq m} \int (-1)^{|k|} a_{kl}(x) D^l u(x) D^k \varphi(x)\, \mathrm{dx} = (Lu, \varphi) = \int_\Omega Lu(x) \varphi(x)\, \mathrm{dx}$ for all $\varphi \in W$,

b) $B_j u = 0$ on Γ for $i = 0, \ldots, m-1$.

Assumption (B): There exist positive constants ϑ and θ such that:

a) $a(u, u) \geqq \vartheta \|u\|_m^2$,

b) $a(u, v) \leqq \theta \|u\|_m \|v\|_m$ for all $u, v \in W$.

II. WEAK SOLUTION OF THE PROBLEM. GALERKIN'S METHOD

Definition 1. *Let* $f \in H^{-m}(\Omega)$. *Then* u *is a weak solution of* (1) & (2) *if* u *is a solution of the following problem*:

$$(3) \qquad u \in W; \quad a(u, \varphi) + [Ku, \varphi] = [f, \varphi]$$

for all $\varphi \in W$.

Remark (the sense of generalization): if $u \in W \cap C^{2m}(\Omega) \cap C^{2m-1}(\bar{\Omega})$ is a solution of (3), then u is a classical solution of (1) & (2). Conversely, if u is a classical solution of (1) & (2), then u solves (3), too.

Let us recall two well-known theorems (see [1]):

Theorem 1. *Let* $\Gamma \in C^\infty$. *Then there exists an operator* T *such that*:

a) $T: H^{-m+\varepsilon}(\Omega) \to H^{m+\varepsilon}(\Omega) \cap W$ *is linear and bounded for each* $\varepsilon \geqq 0$;

b) *if* $\psi \in H^{-s}(\Omega)$ (where $s \leqq m$), *then* $u = T\psi \in W \cap H^{2m-s}(\Omega)$ *is a unique solution of the following problem*:

$$(4) \qquad u \in W; \quad a(u, \varphi) = [\psi, \varphi] \quad \text{for all} \quad \varphi \in W;$$

c) *there exists such a constant* C_0 *independent of* s *that* $\|T\psi\|_{2m-s} \leqq C_0 \|\psi\|_{-s}$;

d) *there exists* T^{-1}, *the inverse operator to* T, *and* $T^{-1}: H^{m+\varepsilon}(\Omega) \cap W \to H^{-m+\varepsilon}(\Omega)$ *is linear and bounded.*

Proof. The statement of Theorem 1 is in the case $\varepsilon = 0$ a consequence of the Lax-Milgram theorem. In the case $\varepsilon \geqq m$, a) and b) follows from Theorem 5.2 Ch. 2 [1]. To prove a), b) for $\varepsilon \in (0, m)$ we use Theorem 5.1. Ch. 1 [1]. According to this theorem the operator $T: [H^{-m}, H^0]_\theta \to [W, W \cap H^{2m}]_\theta$ is bounded for all $\theta \in (0, 1)$. This completes the proof.

Remark. The assumption about Γ is too strong. The assertions which follow make use only of the fact that the assertion of Theorem 1 holds and do not depend on the smoothness properties of Γ. Therefore, we may assume only that Γ has such properties that the assertion of Theorem 1 holds. In this way our results may be generalised.

Theorem 2. (Rellich): *The operator* $T: H^{-m+\varepsilon}(\Omega) \to H^{m+\varkappa}(\Omega)$ *where* $\varkappa < \varepsilon$ *is compact.* ●

Theorem 3. *The operator* TK *maps* W *into* W *and is linear and compact. The solution* u *of the problem* (3) *for an arbitrary* $f \in H^{-m}(\Omega)$ *exists and is unique if and only if* $-1 \notin P_\sigma(TK)$ (*the point spectrum of the operator* TK); *furthermore,* $u = (I + TK)^{-1} Tf$.

Proof. Evidently, the operator $K : W \to H^q(\Omega)$ is bounded. Since $T : H^{-m+\varepsilon}(\Omega) \to W$ is compact for $\varepsilon > 0$ (according to Theorem 2) and since $q > -m$ we conclude that $T: H^q(\Omega) \to W$ is compact. Hence $TK : W \to W$ is compact. Since (according to Theorem 1) $T: H^{-m}(\Omega) \to W$ is a one-to-one mapping, the rest of the statement follows as an evident consequence of the Fredholm alternative.

Remark (regularity of solution): Let $f \in H^t(\Omega)$, $t \geqq -m$. Then $u \in W \cap H^{2m+\min(t,q)}(\Omega)$ Indeed: u solves (3), hence $u + TKu = Tf$; if $f \in H^t(\Omega)$, then $Tf \in W \cap H^{2m+t}(\Omega)$ and if $u \in W$, then $TKu \in W \cap H^{2m+q}(\Omega)$.

Definition 2. *Let $S_h \subset W$ be a closed subspace of $H^{-m}(\Omega)$. We shall say that u_h is Galerkin's approximation of the solution u of problem* (3) *if*

$$u_h \in S_h\,; \quad a(u_h, \varphi) + [Ku_h, \varphi] = [f, \varphi] \tag{5}$$

for all $\varphi \in S_h$.

Remark: A remark on the actual calculation of the value $[Ku_h, \varphi]$ will be broght in paragraph 4. Now we formulate the problem (5) (similarly to the problem (3)) as the problem of solving an operator equation. The procedure used in [3] for the analysis of the "classical" Galerkin's method will be used now.

Definition 3. *Denote by $\tilde{P}_h$ and P_h the projection of W and $H^{-m}(\Omega)$ respectively onto S_h defined as follows: if $u \in W$ then $\tilde{P}_h u \in S_h$ and $a(u - \tilde{P}_h u, \varphi) = 0$ for all $\varphi \in S_h$ and if $u \in H^{-m}(\Omega)$, then $P_h u \in S_h$ and $[u - P_h u, \varphi] = 0$ for all $\varphi \in S_h$.*

Theorem 4. *Let $f \in H^{-m}(\Omega)$. Then $u_h \in W$ solves the equation*

$$(I + \tilde{P}_h TK)\, u_h = \tilde{P}_h T P_h f \tag{6}$$

if and only if it solves the problem (5).

Proof. Let $u_h \in W$ be such that $(I + \tilde{P}_h TK)\, u_h = \tilde{P}_h T P_h f$. Put $U = TP_h f$ and $U_1 = TKu_h$. From the definition of the operator T it follows that $a(U, \varphi) = [P_h f, \varphi]$ and $a(U_1, \varphi) = [Ku_h, \varphi]$ for all $\varphi \in W$. Since we assume that $u_h = \tilde{P}_h(TP_h f - TKu_h)$ we have $u_h = \tilde{P}_h(U - U_1)$. This implies

a) $u_h \in S_h$;

b) $a(u_h, \varphi) = a(U - U_1, \varphi) = a(U, \varphi) - a(U_1, \varphi) = [P_h f, \varphi] - [Ku_h, \varphi] = [f, \varphi] - [Ku_h, \varphi]$ for all $\varphi \in S_h$. Hence u_h solves the problem (5).

Proof of the converse assertion is analogous. ●

Galerkin's method of the solution of (3) is based on the following principle: Let $\{S_h\}_{h \in (0,1)}$ be a system of subspaces of W. For each $h \in (0, 1)$ let u_h be Galerkin's approximation of u on S_h. Let the system $\{S_h\}$ approximate W in a certain sense. Then we expect the convergence of u_h to u. The finite element method is a modification of Galerkin's method which is based on a special choice of the system $\{S_h\}$. We shall demand that the system $\{S_h\}$ should be in class $\mathscr{S}^r$ (see the following definition).

Remark. In our terminology the concept of Galerkin's method is slightly more general than the classical one shown e.g. in [3] but it coincides with the present terminology used e.g. in [2].

III. THE RATE OF CONVERGENCE

In this paragraph we suppose that $-1 \notin P_\sigma(TK)$.

Definition 4. *We say that the system $\{S_h\}_{h\in(0,1)}$ of subspaces S_h of the space W is in the class $\mathscr{S}^r$ where an integer $r > m$ if:*

a) *S_h is closed for each $h \in (0, 1)$ in $H^{-m}(\Omega)$;*

b) *there exists a constant C so that for any $w \in W$ there exists $w_h \in S_h$ such that if $w \in W \cap H^l(\Omega)$ where $l \geqq m$, then*

$$\|w - w_h\|_s \leqq C\|w\|_l \, h^{\min(r,l)-s} \tag{7}$$

for $-m \leqq s \leqq m$. ●

Let us suppose in the following that $\{S_h\}_{h\in(0,1)}$ is in the class $\mathscr{S}^r$.

Lemma 1. *There exists a constant C_1 such that*

$$\|\tilde{P}_h - I\|_{s,l} \leqq C_1 h^{\min(r-m,m-s)+\min(r-m,l-m)} \tag{8}$$

for $l > m$, $-m \leqq s \leqq m$.

Proof. Let $w \in W \cap H^l(\Omega)$. Since $a(w - \tilde{P}_h w, \varphi) = 0$ for all $\varphi \in S_h$, we have

$$\begin{aligned} a(w - \tilde{P}_h w, w) &= a(w - \tilde{P}_h w, w - \varphi) = \\ &= a(w - \tilde{P}_h w, w - \tilde{P}_h w) \quad \text{for all} \quad \varphi \in S_h . \end{aligned} \tag{9}$$

Using the condition (B) we can get

$$\|w - \tilde{P}_h w\|_m^2 \leqq \frac{\theta}{\vartheta} \|w - \tilde{P}_h w\|_m \|w - \varphi\|_m \tag{10}$$

for all $\varphi \in S_h$.

According to the assumption that $\{S_h\}_{h\in(0,1)}$ is in the class $\mathscr{S}^r$, there exists $w_h \in S_h$ such that $\|w - w_h\|_m \leqq C\|w\|_l\, h^{\min(r,l)-m}$. Hence

(11) $$\|w - \tilde{P}_h w\|_m \leqq C\frac{\theta}{\vartheta}\|w\|_l\, h^{\min(r,l)-m}$$

and

(12) $$\|I - \tilde{P}_h\|_{m,l} = \sup_{\substack{\|w\|_l \neq 0 \\ w\in W\cap H^l(\Omega)}} \frac{\|w - \tilde{P}_h w\|_m}{\|w\|_l} \leqq C\frac{\theta}{\vartheta}\, h^{\min(r,l)-m}.$$

Now we define a bilinear form $a^*(u, v)$ on W:

(13) $$a^*(u, v) = a(v, u)\,; \quad u, v \in W$$

Evidently it follows from the assumption (B) that

(B*) $$a^*(w, w) \geqq \vartheta\|w\|_m^2$$
$$a^*(v, w) \leqq \theta\|w\|_m\,\|v\|_m$$

for all $v, w \in W$.

We can say that the assumptions of Theorem 1 are satisfied and this implies the following assertion: There exists an operator T^* such that for each $\varepsilon \geqq 0$, $T^* : H^{-m+\varepsilon}(\Omega) \to H^{m+\varepsilon}(\Omega) \cap W$ is linear and bounded. If $\psi \in H^{-s}(\Omega)$, $s \leqq m$, then $z = T^*\psi \in W \cap H^{2m-s}(\Omega)$ is a unique solution of the problem

(14) $$a^*(z, \varphi) = [\psi, \varphi] \quad \text{for all} \quad \varphi \in W.$$

Simultaneously, there exists a constant C_0 independent of s such that

(15) $$\|T^*\psi\|_{2m-s} \leqq C_0\|\psi\|_{-s}.$$

We can use these facts as follows: Let us put $\varphi = w - \tilde{P}_h w$ in (14); we obtain

(16) $$a^*(T^*\psi, w - \tilde{P}_h w) = [\psi, w - \tilde{P}_h w].$$

Since $a(w - \tilde{P}_h w, \varphi) = 0$ for all $\varphi \in S_h$, it holds in accordance with (13)

$$a^*(T^*\psi, w - \tilde{P}_h w) = a(w - \tilde{P}_h w, T^*\psi) =$$
$$= a(w - \tilde{P}_h w, T^*\psi - \varphi) = a^*(T^*\psi - \varphi, w - \tilde{P}_h w)$$

for all $\varphi \in S_h$. If we substitute from this expression to the left hand side of the equation (16) we get

(17) $$[\psi, w - \tilde{P}_h w] = a^*(T^*\psi - \varphi, w - \tilde{P}_h w)$$

for all $\varphi \in S_h$. If we use (B*) we obtain

$$(18) \qquad [\psi, w - \tilde{P}_h w] \leqq \theta \|T^*\psi - \varphi\|_m \|w - \tilde{P}_h w\|_m \quad \text{for all} \quad \varphi \in S_h .$$

According to Definition 4 there exists such $w_h \in S_h$ (for each $h \in (0, 1)$) that $\|T^*\psi - w_h\|_m \leqq C\|T^*\psi\|_{2m-s} h^{\min(r,2m-s)-m}$. In accordance with (15) it is $\|T^*\psi - w_h\|_m \leqq CC_0\|\psi\|_{-s} h^{\min(r,2m-s)-m}$. We can estimate the right hand side of (18) using this inequality and (11). This yields

$$(19) \qquad [\psi, w - \tilde{P}_h w] \leqq CC_0 \frac{\theta}{\vartheta} \|\psi\|_{-s} \|w\|_l h^{\min(r-m,m-s)+\min(r-m,l-m)} .$$

We defined $[\psi, w - \tilde{P}_h w]$ as the value of the functional $\tilde{\psi}$ ($\tilde{\psi} \in (H^s(\Omega))'$ is isomorphic with $\psi \in H^{-s}(\Omega)$) at the point $w - \tilde{P}_h w \in W \subset H^s(\Omega)$. It means that $|[\psi, w - \tilde{P}_h w]| \leqq \|\psi\|_{-s} \|w - \tilde{P}_h w\|_s$. From this it follows immediately: if we denote by $z \in (H^{-s}(\Omega))'$ the functional which is isomorphic with $w - \tilde{P}_h w$, then $[\psi, w - \tilde{P}_h w]$ is the value of the functional z at the point ψ. The norm of z equals $\|w - \tilde{P}_h w\|_s$. Hence

$$(20) \quad \|w - \tilde{P}_h w\|_s = \sup_{\|\psi\|_{-s} \neq 0} \frac{[\psi, w - \tilde{P}_h w]}{\|\psi\|_{-s}} \leqq C_1 \|w\|_l h^{\min(r-m,m-s)+\min(r-m,l-m)}$$

where

$$C_1 = CC_0 \frac{\theta}{\vartheta} .$$

This inequality holds in accordance with (19). It follows from (20) that

$$(21) \qquad \|I - \tilde{P}_h\|_{s,l} = \sup_{\substack{\|w\|_l \neq 0 \\ w \in W \cap H^l(\Omega)}} \frac{\|(I - \tilde{P}_h) w\|_s}{\|w\|_l} \leqq C_1 h^{\min(r-m,m-s)+\min(r-m,l-m)}$$

which was to be proved.

The following lemma guarantees the existence and the uniqueness of a solution of (6) and (5).

Lemma 2. *There exists* $h_0 \in (0, 1)$ *such that for* $0 \leqq h \leqq h_0$ *the operator* $(I + \tilde{P}_h TK)^{-1} : S_h \to S_h$ *exists and*

$$(22) \qquad \|(I + \tilde{P}_h TK)^{-1}\|_{m,m} \leqq$$

$$\leqq \frac{\|(I + TK)^{-1}\|_{m,m}}{1 - \|(I + TK)^{-1}\|_{m,m} \|TK\|_{2m+q,m} \|\tilde{P}_h - I\|_{m,2m+q}} .$$

Proof: It is easy to verify that the following identity holds:

(23) $$I + \tilde{P}_h TK = (I + TK)(I + (I + TK)^{-1}(\tilde{P}_h - I)\, TK)\,.$$

We know that $(I + TK)^{-1}$ exists and is bounded; we can prove that $(I + (I + TK)^{-1} \cdot (\tilde{P}_h - I)\, TK)^{-1} : W \to W$ exists for h small enough. In fact, $TK : W \to W$ is bounded (compact), hence $\tilde{P}_h - I : W \to W$ is bounded (according to $h \in (0, 1)$) and $(I + TK)^{-1} : W \to W$ is bounded. Hence the operator $(I + TK)^{-1}(P_h - I)\, TK : W \to W$ is bounded. The following assertion is well-known: if

(24) $$\|(I + TK)^{-1}(\tilde{P}_h - I)\, TK\|_{m,m} < 1$$

then there exists $(I + (I + TK)^{-1}(\tilde{P}_h - I)\, TK)^{-1} : W \to W$, it is bounded and

(25) $$\|(I + (I + TK)^{-1}(\tilde{P}_h - I)\, TK)^{-1}\|_{m,m} \leqq \leqq (1 - \|(I + TK)^{-1}(\tilde{P}_h - I)\, TK\|_{m,m})^{-1}\,.$$

We can verify the validity of (24) for h sufficiently small:

(26) $$\|(I + TK)^{-1}(\tilde{P}_h - I)\, TK\|_{m,m} \leqq \leqq \|(I + TK)^{-1}\|_{m,m}\, \|\tilde{P}_h - I\|_{m,2m+q}\, \|TK\|_{2m+q,m}\,;$$

if we take h_0 small enough such that $\|\tilde{P}_h - I\|_{m,2m+q} < (\|(I + TK)^{-1}\|_{m,m} \cdot \|TK\|_{2m+q,m})^{-1}$ for $0 < h \leqq h_0$, then the condition (24) is satisfied for such h's. Let us turn back to the equation (23). Both operators on the right hand side of (23) have bounded inverses in W. Hence the operator $I + \tilde{P}_h TK$ has a bounded inverse in W. The estimate of its norm (22) follows from (13), (25) and (26). It is evident that $(I + \tilde{P}_h TK)^{-1} : S_h \to S_h$. ●

The following identity will be useful for the error estimates of Galerkin's approximations u_h of the exact solution of (3).

Lemma 3. *We have*

(27) $$u - u_h = (I + TK)^{-1}(T - \tilde{P}_h T P_h) f + + (I + TK)^{-1}(\tilde{P}_h - I)\, TK(I + \tilde{P}_h TK)^{-1}\, \tilde{P}_h T P_h f$$

where $u = (I + TK)^{-1}\, Tf$, $u_h = (I + \tilde{P}_h TK)^{-1}\, \tilde{P}_h T P_h f$, $h \leqq h_0$ (*see Lemma* 2).

Proof. $u - u_h = (I + TK)^{-1}\, Tf - (I + \tilde{P}_h TK)^{-1}\, \tilde{P}_h T P_h f = (I + TK)^{-1} \cdot (Tf - \tilde{P}_h T P_h f) + ((I + TK)^{-1} - (I + \tilde{P}_h T P_h)^{-1})\, \tilde{P}_h T P_h f$; $(I + TK)^{-1} - (I + \tilde{P}_h TK)^{-1} = (I + TK)^{-1}(\tilde{P}_h TK - TK)(I + \tilde{P}_h TK)^{-1}$. Hence $u - u_h = (I + TK)^{-1}(T - \tilde{P}_h T P_h) f + (I + TK)^{-1}(\tilde{P}_h TK - TK)(I + \tilde{P}_h TK)^{-1}\, \tilde{P}_h T P_h f$. ●

Lemma 4. *There exists a constant C_2 such that*

$$\|T - \tilde{P}_h T P_h\|_{s,l} \leqq C_2 h^{\min(r-m,m+l)+\min(r-m,m-s)} \tag{28}$$

where $l > -m$, $-m \leqq s \leqq m$.

Proof. In [2] the problem (3) has been solved without the perturbation K. Hence $u = Tf$, $u_h = \tilde{P}_h T P_h f$. The following estimate is proved there: if $f \in H^l(\Omega)$, $l > -m$, $-m \leqq s \leqq m$, then $\|u - u_h\|_s \leqq C_2 \|f\|_l \, h^{\min(r-m,m+l)+\min(r-m,m-s)}$. Hence

$$\|Tf - \tilde{P}_h T P_h f\|_s \leqq C_2 \|f\|_l \, h^{\min(r-m,m+l)+\min(r-m,m-s)} . \tag{29}$$

Since

$$\|T - \tilde{P}_h T P_h\|_{s,l} = \sup_{\substack{\|f\|_l \neq 0 \\ f \in W \cap H^l(\Omega)}} \frac{\|(T - \tilde{P}_h T P_h) f\|_s}{\|f\|_l} ,$$

the estimate (28) follows immediately from the inequality (29). ●

Now two error estimates are presented:

Theorem 5. *There is a constant C_3 such that*

$$\|u - u_h\|_m \leqq C_3 \|f\|_t \, h^{\min(m+t,m+q,r-m)} , \tag{30}$$

$f \in H^t(\Omega)$, $t > -m$; u, u_h are the corresponding solutions of (3) *and* (5).

Proof. From (27) we can conclude

$$u - u_h = (I + TK)^{-1} Qf \tag{31}$$

where

$$Q = (T - \tilde{P}_h T P_h) + (\tilde{P}_h - I) \, TK(I + \tilde{P}_h TK)^{-1} \tilde{P}_h T P_h . \tag{32}$$

We can show that

$$\|Q\|_{m,t} \leqq \tilde{C}_3 h^{\min(m+t,m+q,r-m)} \tag{33}$$

where $\tilde{C}_3$ is a positive constant. Indeed:

$$\begin{aligned} \|Q\|_{m,t} \leqq {} & \|T - \tilde{P}_h T P_h\|_{m,t} + \\ & + \|\tilde{P}_h - I\|_{m,2m+q} \| TK\|_{2m+q,m} \|(I + \tilde{P}_h TK)^{-1}\|_{m,m} \|\tilde{P}_h T P_h\|_{m,t} ; \end{aligned} \tag{34}$$

because $\|\tilde{P}_h T P_h\|_{m,t} \leqq \|T - \tilde{P}_h T P_h\|_{m,t} + \|T\|_{m,t}$ then $\|\tilde{P}_h T P_h\|_{m,t}$ is bounded according to $h \in (0, 1)$ ($\|T\|_{m,t} \leqq \|T\|_{m,m}$ and $\|T - \tilde{P}_h T P_h\|_{m,t}$ is bounded in accordance with (28)). We can use the inequalities (28), (8) and (22) to estimate $\|T - \tilde{P}_h T P_h\|_{m,t}$ and $\|\tilde{P}_h - I\|_{m,2m+q}$ and $\|(I + \tilde{P}_h TK)^{-1}\|_{m,m}$. Then the validity of

(33) is evident. Let us return to the equality (31). We can get: $\|u - u_h\|_m \leqq \|(I + TK)^{-1}\|_{m,m} \|Q\|_{m,t} \|f\|_t$. If we set $C_3 = \tilde{C}_3\|(I + TK)^{-1}\|_{m,m}$, then the proof is finished with regard to the inequality (33).

Remark. (optimality of estimate (30)): According to the remark concerning the regularity of the solution of problem (3), there is a solution $u \in W \cap H^{2m+\min(t,q)}$ for each $f \in H^t(\Omega)$; if the function f has this smoothness, the estimate (30) is of the order of the best approximation on the system $\{S_h\}_{h\in(0,1)}$ in the class $\mathscr{S}^r$ (see [2]).

Now we can slightly strengthen the assumptions on K. A possibility of error estimates in "worse" norms is one of the consequences.

Theorem 6. *Let us consider* $K : H^{m-\varepsilon}(\Omega) \to H^{q-\varepsilon}(\Omega)$ *for each* $0 \leqq \varepsilon \leqq m + q$ *Then there is a constant* C_4 *such that*

$$\|u - u_h\|_s \leqq C_4\|f\|_t\, h^{\min(r-m,m-s)+\min(r-m,m+t,m+q)} \tag{35}$$

where $f \in H^t(\Omega)$, $t \geqq -m$; $-q \leqq s < m$; *u and u_h are the corresponding solutions of the problem* (3) *and* (5).

Proof. We can show that there exists a constant $\tilde{C}_4$ such that

$$\|Q\|_{s,t} \leqq \tilde{C}_4 h^{\min(r-m,m-s)+\min(r-m,m+t,m+q)}. \tag{36}$$

Indeed:

$$\|Q\|_{s,t} \leqq \|T - \tilde{P}_h T P_h\|_{s,t} + \\ + \|\tilde{P}_h - I\|_{s,2m+q} \|TK\|_{2m+q,m} \|(I + \tilde{P}_h TK)^{-1}\|_{m,m} \|\tilde{P}_h T P_h\|_{m,t};$$

estimating the right hand side of this inequality by means of (28) and (8) and (22) we get (36) immediately.

If $m > s \geqq -q$, then $TK : H^s(\Omega) \to W$ is bounded and considered as an operator $TK : H^s(\Omega) \to H^s(\Omega)$ it is compact. Hence $(I + TK)^{-1} : H^s(\Omega) \to H^s(\Omega)$ which is linear and bounded exists if and only if $\{(I + TK)\, u = 0,\ u \in H^s(\Omega) \Rightarrow u \equiv 0\}$. However, if there exists $u \in H^s(\Omega)$ such that $(I + TK)\, u = 0$, then $u \in W$ because $TKu = -u$ and $TK : H^s(\Omega) \to W$. The operator $(I + TK)^{-1} : H^s(\Omega) \to H^s(\Omega)$ (linear, bounded) exists if and only if the operator $(I + TK)^{-1} : W \to W$ exists, which we suppose in thes whole paragraph. If we set $C_4 = \tilde{C}_4\|(I + TK)^{-1}\|_{s,s}$, then the estimate (35) follows from (31) and (36) immediately.

IV. EXAMPLES

Remark. A concrete application of the method can fail in calculating $[Ku_h, \varphi]$ where $u_h, \varphi \in S_h$. This complication can be easily avoided by introducing the assumption that $K : S_h \to L_2(\Omega)$. The following examples show that this assumption is realistic. We could use the procedure using besides the system $\{S_h\}$ another system $\{\tilde{S}_h\}$ too. We could require $\tilde{S}_h \subset W$ and if $u_h \in S_h$ and $\varphi \in \tilde{S}_h$, then $Ku_h\varphi \in L_2(\Omega)$. Under some assumptions on the relation of S_h and $\tilde{S}_h$ and of the bilinear form $a(\cdot, \cdot)$ – see [4] – it is possible to find the estimates similar to (30) and to (35) for Galerkin's approximations $u_h \in S_h$ defined for each $h \in (0, 1)$ as a solution of the following problem: $a(u_h, \varphi) + [Ku_h, \varphi] = [f, \varphi]$ for each $\varphi \in \tilde{S}_h$, where $[Ku_h, \varphi] = (Ku_h, \varphi)$. This approach, however, is not investigated in this paper.

Example 1. $-y''(x) - \lambda\, y(x)/x^{5/4} = f(x)$, $x \in (0,1) = \Omega$ $y(0) = y(1) = 0$. Let for each $f \in H^{-1}(\Omega)$ a weak solution of this problem exist for $\lambda = \lambda_0$. To faciliatete the application of our method let us note that $Ly = -y''$, $Ky = -\lambda_0\, y/x^{5/4}$, $W = H_0^1(\Omega)$, $a(u, v) = \int_0^1 u'(x)\, v'(x)\, dx$. Since $\lambda_0/x^{5/4} \in H^{-1/2}(\Omega)$ the operator $K : H_0^1(\Omega) \to H^{-1/2}(\Omega)$ is linear and bounded. Hence Theorem 5 applies; e.g. if we use piecewise linear splines on Ω (this system is in the class $\mathscr{S}^2$) we can get the estimate $\|u - u_h\|_1 = 0(h^{\min 1/2, 1+t})$ for $t > -1$, $f \in H^t(\Omega)$.

Example 2. Let us consider problem (3) where $Ku = \lambda \sum_{|\alpha| \leqq m} b_\alpha D^\alpha u$, $b_\alpha \in H^0(\Omega)$. Hence $K : H^m(\Omega) \to H^0(\Omega) = L_2(\Omega)$ is linear and bounded; $q = 0$. It is easy to show that the assumptions of Theorems 5, 6 are satisfied. We can obtain the estimate $\|u - u_h\|_s \leqq C_4 \|f\|_t\, h^{\min(r-m, m-s) + \min(r-m, m, m+t)}$ where $0 \leqq s \leqq m$; e.g. if $m = 1$ then means by of Guglielmo's splines (piecewise linear on the triangulation), which are in the class $\mathscr{S}^2$, we can obtain the estimate $\|u - u_h\|_s \leqq C_4 \|f\|_t\, h^{\min(1, 1-s) + \min(1, 1+t)}$ for $0 \leqq s \leqq 1$. We can see that a "smooth perturbation" does not affect the rate of convergence desired in [2] for the unperturbed case.

Example 3. Let us consider problem (3) where $[Ku](x) = \lambda \int_\Omega \mathscr{K}(x, y)\, u(y)\, dy$ where $\mathscr{K}(x, y)$ is a Hilbert-Schmidt kernel on $\Omega \times \Omega$. Then $K : H^{m-\varepsilon}(\Omega) \to H^0(\Omega)$ for $0 \leqq \varepsilon \leqq m$.

Hence the assumptions of Theorems 5,6 are satisfied.

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Souhrn

ELIPTICKÉ OKRAJOVÉ ÚLOHY S NEVARIAČNÍ PORUCHOU A METODA KONEČNÝCH PRVKŮ

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Nechť homogenní okrajová úloha lze převést na následující problém: Hledáme $u \in W$ tak, aby $a(u, \varphi) + [Ku, \varphi) = [f, \varphi]$ pro všechna $\varphi \in W$, kde: W je uzavřený podprostor $H^{-m}(\Omega)$ a $H_0^m(\Omega) \subset W \subset H^m(\Omega)$; $a(u, \varphi)$ je W – eliptická bilinární forma; $K : H^m(\Omega) \to H^q(\Omega)$, $q > -m$; $[z, \varphi]$ je hodnota lineárního funkcionálu isomorfního s funkcí $z \in H^{-m}(\Omega)$ v bodě $\varphi \in H^m(\Omega)$. Nechť uvedená úloha je jednoznačně řešitelná pro všechna $f \in H^{-m}(\Omega)$. Dále uvažujeme systém $\{S_h\}_{h \in (0,1)}$ podprostorů S_h prostoru W, který má jisté aproximační vlastnosti (patří do třídy $\mathscr{S}^r$). V Příkladech je ukázáno, že tyto vlastnosti mají také obvyklé „kopečkové" funkce užívané v metodě konečných prvků.

Článek se zabývá odhadem přesnosti Galerkinovy metody, aplikované na podprostorech S_h. Odhad závisí na řádu nejlepší aproximace prostoru W systémem $\{S_h\}_{h \in (0,1)}$ a na q, tj. „kvalitě" poruchového operátoru.

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