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PERIODIC SOLUTIONS OF A WEAKLY NONLINEAR HYPERBOLIC EQUATION IN E_2 AND E_3

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1. Introduction. For $n = 2$ and 3 I investigate the classical solutions of the equation with constant coefficients

$$(1.1^{(n)}) \qquad \Box_n u + 2au_t + 2(B, \nabla_n u) + cu = h(t, x) + \varepsilon f(t, x, u, \varepsilon).$$

The initial conditions are

$$(1.2^{(n)}) \qquad u(0, x) = p(x), \quad u_t(0, x) = q(x), \quad x \in E_n.$$

In $(1.1^{(n)})$

$$\Box_n \equiv \partial^2/\partial t^2 - \partial^2/\partial x_1^2 - \partial^2/\partial x_2^2 - \ldots - \partial^2/\partial x_n^2,$$

$$\nabla_n = (\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_n),$$

$$B = (b_1, b_2, \ldots, b_n),$$

$$x = (x_1, x_2, \ldots, x_n).$$

ε is a small parameter, $t \in E_1^+ = \langle 0, +\infty \rangle$.

The study of $(1.1^{(n)})$-type equations was initiated in [2] by F. A. Ficken and B. A. Fleishman. In [3] J. Havlová studied $(1.1^{(1)})$ with a more general right-hand side, viz. that of $f(t, x, u, u_x, u_t, \varepsilon)$.

2. Preliminaries. Denote for $n \geqq 2$ $E_{n+1}^+ = E_1^+ \times E_n$, $\omega^2 = \|B\|^2 + c - a^2 = b_1^2 + b_2^2 + \ldots + b_n^2 + c - a^2$,

$$(2.1) \qquad \omega = \begin{matrix} \sqrt{(\|B\|^2 + c - a^2)} \\ i\sqrt{(a^2 - \|B\|^2 - c)} \end{matrix} \quad \text{if} \quad \begin{matrix} \omega^2 \geqq 0, \\ \omega^2 < 0 \end{matrix}$$

$\Omega = |\omega|$, i.e.,

$$\omega = \begin{matrix} \Omega \\ i\Omega \end{matrix} \quad \text{if} \quad \begin{matrix} \omega^2 \geqq 0 , \\ \omega^2 < 0 \end{matrix} \tag{2.2}$$

$$\lambda = \begin{matrix} a \\ a - \Omega \end{matrix} \quad \text{if} \quad \begin{matrix} \omega^2 \geqq 0 . \\ \omega^2 < 0 \end{matrix} \tag{2.3}$$

Throughout the paper we assume

$$\lambda > 0 . \tag{2.4}$$

Let us note that this condition is equivalent to $a > 0$, $\|B\|^2 + c > 0$ in the case of $\omega^2 < 0$.

Further we denote

$$\Gamma^2(\tau, t; \xi, x) \equiv (t - \tau)^2 - \|x - \xi\|^2 , \tag{2.5}$$
$$\Gamma_0^2 = \Gamma^2(0, t; \xi, x) .$$

The facility of multiindex notation is used:

$$D_j = D_j^1 = \partial/\partial x_j , \quad D_x^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \dots D_n^{\alpha_n} , \quad |\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$$

with α_i non-negative integers.

If $k \geqq 2$ is an integer we denote by $C^k(E_{n+1}^+)$ or briefly C^k the Banach space of real functions $v(t, x)$ having continuous and bounded on E_{n+1}^+ partial derivatives $D_x^\alpha v$, $D_x^\beta v_t$, v_{tt}, $|\beta| < |\alpha| \leqq k$. Setting

$$|||v|||_{k,t} = \sup_{x \in E_n} \{|D_x^\alpha v(t, x)|, |D_x^\beta v_t(t, x)|, |v_{tt}(t, x)|; |\beta| < |\alpha| \leqq k\} , \tag{2.6}$$

we define the norm in C^k by

$$\sup_{t \geqq 0} |||v|||_{k,t} = \|v\|_{C^k} . \tag{2.7}$$

Further let us put

$$\|v\|_{k,t} = \sup_{x \in E_n} \{|D_x^\alpha v(t, x)|; |\alpha| \leqq k\} . \tag{2.8}$$

In $(1.1^{(n)})$, $(1.2^{(n)})$ we introduce the variables

$$u = w . \exp\{-at + (B, x)\} , \tag{2.9}$$
$$v = e^{-(B,x)} . u = e^{-at} . w .$$

We get

$$\Box_n w + \omega^2 w = e^{at}\{\delta(t, x) + \varepsilon f_1(t, x, w, \varepsilon)\} , \tag{$2.10^{(n)}$}$$

$$w(0, x) = \varphi(x), \quad w_t(0, x) = \psi(x)$$

and

$$(2.11^{(n)}) \qquad \Box_n v + 2av_t + (\omega^2 + a)\, v = \delta(t, x) + \varepsilon\, g(t, x, v, \varepsilon),$$
$$v(0, x) = \varphi(x), \quad v_t(0, x) = (\psi - a\varphi)(x)$$

with

$$(2.12) \qquad \delta(t, x) = e^{-(B,x)} . h(t, x),$$
$$g(t, x, v, \varepsilon) = f_1(t, x, w, \varepsilon) = e^{-(B,x)} . f(t, x, u, \varepsilon),$$
$$\varphi(x) = e^{-(B,x)} . p(x),$$
$$\psi(x) = e^{-(B,x)} . (q + ap)(x).$$

We shall assume

$(G^{(k,n)} . 1)$: For an arbitrary $R \geqq 0$ there exists a positive constant $A(R)$ such that for $(t, x, v, \varepsilon) \in E^+_{n+1} \times \langle -R, R\rangle \times \langle -\varepsilon_0, \varepsilon_0\rangle$ the partial derivatives

$$D^\alpha_x D^j_v\, g(t, x, v, \varepsilon), \quad |\alpha| + j \leqq k$$

are continuous and bounded:

$$|D^\alpha_x D^j_v\, g(t, x, v, \varepsilon)| \leqq A(R);$$

$(G^{(k,n)} . 2)$: For an arbitrary $R \geqq 0$ the partial derivatives $D^\alpha_x D^j_v\, g(t, x, v, \varepsilon)$, $|\alpha| + j \leqq k$ satisfy the Lipschitz condition in v with $B(R)$ as the Lipschitz constant;

$(G^{(k,n)} . 3)$: $D^\alpha_x\, \delta(t, x)$, $|\alpha| \leqq k$ are continuous and bounded on E^+_{n+1};

$(G^{(n)} . 4)$: The functions $\varphi(x)$ and $\psi(x)$ possess continuous and bounded on E_n partial derivatives up to the orders $[n/2] + 2$ and $[n/2] + 1$, respectively.

Specializing to the case $n = 2$, $(2.10^{(2)})$ is equivalent to an integral equation (cf. [1]; $D_t = \partial/\partial t$)

$$(2.13) \qquad w(t, x, y) = (2\pi)^{-1} \iiint_G \operatorname{ch} i\omega\Gamma . \Gamma^{-1} . e^{at} . [\delta(\tau, \xi, \eta) +$$
$$+ \varepsilon f_1(\tau, \xi, \eta, w(\tau, \xi, \eta), \varepsilon)]\, \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}\tau +$$
$$+ (2\pi)^{-1} \iint_B \operatorname{ch} i\omega\Gamma_0 . \Gamma_0^{-1} . \psi(\xi, \eta)\, \mathrm{d}\xi\, \mathrm{d}\eta +$$
$$+ (2\pi)^{-1} D_t \left(\iint_B \operatorname{ch} i\omega\Gamma_0 . \Gamma_0^{-1} . \varphi(\xi, \eta)\, \mathrm{d}\xi\, \mathrm{d}\eta \right).$$

G denotes the interior of the cone $\Gamma > 0$, $0 < \tau < t$ with B as the basis. The equivalence of $(2.10^{(2)})$ to (2.13) is meant in the sense that every solution of $(2.10^{(2)})$ solves (2.13) and vice versa. (Cf. [1].)

Introducing polar coordinates r, v in (2.13)

$$(2.14)\qquad \xi = x + \varrho r \,.\, \cos v\,, \quad \eta = y + \varrho r \,.\, \sin v\,,$$
$$0 \leqq v < 2\pi\,, \quad \varrho = t - \tau \text{ or } t\,,$$

we get with regard to (2.9)

$$(2.15)\qquad v(t, x, y) = (2\pi)^{-1} \,.\, \int_0^t \int_0^{2\pi} \int_0^1 \operatorname{ch}\left(i\omega \,.\, (t - \tau) \sqrt{(1 - r^2)}\right) . \,(1 - r^2)^{-1/2} \,.$$
$$.\, e^{-a(t-\tau)} \,.\, (t - \tau) \,.\, [\delta(\tau, x + (t - \tau)\, r \,.\, \cos v, y + (t - \tau)\, r \,.\, \sin v) +$$
$$+ \varepsilon\, g(\tau, x + (t - \tau)\, r \,.\, \cos v, y + (t - \tau)\, r \,.\, \sin v, v(\tau, x +$$
$$+ (t - \tau)\, r \,.\, \cos v, y + (t - \tau)\, r \,.\, \sin v), \varepsilon)] \,.\, r \,\mathrm{d}r\, \mathrm{d}v\, \mathrm{d}\tau +$$
$$+ (2\pi)^{-1}\, t e^{-at} \int_0^{2\pi} \int_0^1 \operatorname{ch} i\omega t \sqrt{(1 - r^2)} \,.\, (1 - r^2)^{-1/2} \,.\, (\psi + a\varphi) \,.$$
$$.\, (x + tr \,.\, \cos v, y + tr \,.\, \sin v) \,.\, r \,\mathrm{d}r\, \mathrm{d}v +$$
$$+ (2\pi)^{-1}\, D_t\left(t e^{-at} \int_0^{2\pi} \int_0^1 \operatorname{ch} i\omega t \sqrt{(1 - r^2)} \,.\, (1 - r^2)^{-1/2} \,.\right.$$
$$\left. .\, \varphi(x + tr \,.\, \cos v, y + tr \,.\, \sin v) \,.\, r \,\mathrm{d}r\, \mathrm{d}v\right) .$$

In the case $n = 3$ the analogue of (2.15) is (cf. [1])

$$(2.16)\qquad v(t, X) = \int_0^t e^{-at}(t - \tau)^{-1}\, D_t\left(\left(\int_0^{t-\tau} r^2\, J_0(\omega \Gamma) \,.\, Q(\delta + \varepsilon\, g(v))\,(\tau, X, r)\, \mathrm{d}r\right)\right) .$$
$$.\, \mathrm{d}\tau + e^{-at} \,.\, t^{-1} \,.\, D_t\left(\int_0^t r^2\, J_0(\omega \Gamma_0)\, Q(\psi + a\varphi)\,(X, r)\, \mathrm{d}r\right) +$$
$$+ D_t\left(e^{-at} \,.\, t^{-1} \,.\, D_t\left(\int_0^t r^2\, J_0(\omega \Gamma_0)\, Q(\varphi)\,(X, r)\, \mathrm{d}r\right)\right)$$

where $X = [x, y, z]$,

$$(2.17)\qquad Q(\delta + \varepsilon\, g(v))\,(\tau, X, r) = (4\pi)^{-1} \int_0^{2\pi} \int_0^{\pi} e^{at} \,.\, [\delta(\tau, x + r \,.\, \sin \vartheta \,.\, \cos v, y +$$
$$+ r \,.\, \sin \vartheta \,.\, \sin v, z + r \,.\, \cos \vartheta) + \varepsilon\, g(\tau, x + r \,.\, \sin \vartheta \,.\, \cos v, y +$$
$$+ r \,.\, \sin \vartheta \,.\, \sin v, z + r \,.\, \cos \vartheta, v(\tau, x + r \,.\, \sin \vartheta \,.\, \cos v, y +$$
$$+ r \,.\, \sin \vartheta \,.\, \sin v, z + r \,.\, \cos \vartheta), \varepsilon)] \,.\, \sin \vartheta \,\mathrm{d}\vartheta\, \mathrm{d}v\,,$$

$$Q(\mu)(X, r) = (4\pi)^{-1} \int_0^{2\pi} \int_0^{\pi} \mu(r, \vartheta, v) \, . \sin \vartheta \, d\vartheta \, dv \, ,$$

$$\mu(r, \vartheta, v) = \mu(x + r \, . \sin \vartheta \, . \cos v, y + r \, . \sin \vartheta \, . \sin v, z + r \, . \cos \vartheta) \, .$$

In what follows we make use of the standard contraction mapping principle:

Let $v \to V(v)$ be a mapping of a Banach space X into itself such that (i) $\|v\| \leqq R$, $R \geqq 0$ implies $\|V(v)\| \leqq R$; (ii) there exists a real number θ, $0 < \theta < 1$ such that for $\|v_1\|, \|v_2\| \leqq R$ the inequality $\|V(v_1) - V(v_2)\| \leqq \theta \, . \|v_1 - v_2\|$ is valid. Then there exists a unique $v \in X$ such that $v = V(v)$ and $\|v\| \leqq R$.

For the proof cf. e.g. [4].

3. $\boldsymbol{n = 2}$. The starting point is the equation (2.15). We introduce special ad hoc integral operators similarly as in [2]. For $s(t, x, y) \in C^3(E_3^+)$ put

$$(3.1) \qquad H_2(s)(\tau, t, x, y) = (2\pi)^{-1} (t - \tau) \, e^{-a(t-\tau)} \, . \int_0^{2\pi} \int_0^1 s(\tau, x + (t - \tau) \, r \, . \cos v, y + (t - \tau) \, r \, . \sin v) \, . \operatorname{ch} i\omega(t - \tau) \sqrt{(1 - r^2)} \, . (1 - r^2)^{-1/2} \, r \, dr \, dv \, ,$$

$$M_2(s)(t, x, y) = \int_0^t H_2(s)(\tau, t, x, y) \, d\tau \, ,$$

$$\Psi_2(\psi)(t, x, y) = H_2(s)(0, t, x, y), \psi(\xi, \eta) = s(0, \xi, \eta) \, ,$$

$$\Phi_2(\varphi)(t, x, y) = (D_t \Psi_2 + a \Psi_2)(\varphi)(t, x, y) \, .$$

Then (2.15) may be written as

$$(3.2) \qquad v(t, x, y) = (\varepsilon \, M_2(g(v)) + M_2(\delta) + \Phi_2(\varphi) + \Psi_2(\psi))(t, x, y) \equiv \equiv V_2(v)(g, \delta, \varphi, \psi)(t, x, y) \, .$$

To this equation we apply the contraction mapping principle. We choose $X = C^2(E_3^+)$ and

$$\|v\| = \|v\|_{C^2} = \sup_{(t,x,y) \in E_3^+} \{|D_1^{\alpha_1} D_2^{\alpha_2} v(t, x, y)|, |D_1^{\beta_1} D_2^{\beta_2} v_t(t, x, y)|, |v_{tt}(t, x, y)|; |\beta| < |\alpha| \leqq 2\} \, .$$

Using the inequalities

$$(3.3a) \qquad \operatorname{ch}(i\omega(t - \tau) \sqrt{(1 - r^2)}) = \operatorname{ch}(-\Omega(t - \tau) \sqrt{(1 - r^2)}) = \operatorname{ch}(\Omega(t - \tau) \sqrt{(1 - r^2)}) \leqq \leqq \exp(\Omega(t - \tau) \sqrt{(1 - r^2)}) \leqq \exp(\Omega(t - \tau)) \, ,$$

$$|\operatorname{sh}(\Omega(t - \tau) \sqrt{(1 - r^2)})| \leqq \exp(\Omega(t - \tau))$$

for the case $\omega^2 < 0$,

$$\text{(3.3b)} \quad \left|\operatorname{ch}\left(i\omega(t - \tau)\sqrt{(1 - r^2)}\right)\right| = \left|\operatorname{ch}\left(i\Omega(t - \tau)\sqrt{(1 - r^2)}\right)\right| = \\ = \left|\cos\left(\Omega(t - \tau)\sqrt{(1 - r^2)}\right)\right| \leqq 1\,, \\ \left|\sin\left(\Omega(t - \tau)\sqrt{(1 - r^2)}\right)\right| \leqq 1$$

for the case $\omega^2 \geqq 0$ and the identity

$$\int_0^1 r(1 - r^2)^{-1/2}\,\mathrm{d}r = 1$$

we get

$$\text{(3.4)} \quad \left|H_2(s)(\tau, t, x, y)\right| \leqq \|s\|_{0,\tau} \cdot (t - \tau) \,.\, e^{-\lambda(t-\tau)}, \\ \left|D^\alpha H_2(s)(\tau, t, x, y)\right| = \left|H_2(D^\alpha s)(\tau, t, x, y)\right| \leqq \\ \leqq \|D^\alpha s\|_{0,\tau} \cdot (t - \tau) \,.\, e^{-\lambda(t-\tau)} \leqq \|s\|_{2,\tau} \cdot (t - \tau) \,.\, e^{-\lambda(t-\tau)}\,, \quad |\alpha| \leqq 2\,.$$

(The symbol D^α denotes $D_x^{\alpha_1} D_y^{\alpha_2}$.) Differentiating with respect to t in (3.1) we get

$$\text{(3.5)} \quad \left|D^\beta D_t H_2(s)(\tau, t, x, y)\right| = \left|D_t H_2(D^\beta s)(\tau, t, x, y)\right| \leqq \\ \leqq \|D^\beta s\|_{1,\tau} \,.\, e^{-\lambda(t-\tau)} \,.\, P_{1,1}(t - \tau) \leqq \\ \leqq \|s\|_{2,\tau} \,.\, e^{-\lambda(t-\tau)} \,.\, P_{1,1}(t - \tau)\,, \quad |\beta| \leqq 1\,.$$

Here $P_{1,1}(\zeta)$ denote a polynomial of the first degree with non-negative coefficients. Similar estimates hold for $D_t^2 H_2(s)$ and $D_t^3 H_2(s)$.

From (3.4–5) it results

$$\text{(3.6)} \quad \left|M_2(s)(t, x, y)\right| \leqq \int_0^t \left|H_2(s)(\tau, t, x, y)\right| \mathrm{d}\tau \leqq \\ \leqq \int_0^t \|s\|_{0,\tau} \,.\, e^{-\lambda(t-\tau)} \,.\, (t - \tau)\,\mathrm{d}\tau \leqq \|s\|_0 \,.\, m_0\,,$$

where

$$m_0 = \sup_{t \geqq 0} \left(\lambda^{-2} - \lambda^{-2}(1 + \lambda t)\, e^{-\lambda t}\right) > 0\,.$$

Similarly for $D^\beta D_t M_2(s)$ and $D_t^2 M_2(s)$. Finally, we thus obtain

$$\text{(3.7)} \qquad \|M_2(s)\|_{C^2} \leqq m \,.\, \|s\|_2\,,$$

m being a positive constant. For Ψ_2 and Φ_2 we have

$$\text{(3.8)} \qquad \|\Psi(\psi)\|_{C^2} \leqq K \,.\, \|\psi\|_2\,,$$

$$\text{(3.9)} \qquad \|\Phi_2(\varphi)\|_{C^2} \leqq L \,.\, \|\varphi\|_3\,.$$

These estimates used in (3.2) give

$$\|V_2(v)(g, \delta, \varphi, \psi)\|_{C^2} \leqq |\varepsilon| \, . \, m\|g(v)\|_2 + m\|\delta\|_2 + L\|\varphi\|_3 + K\|\psi\|_2 \leqq \leqq |\varepsilon| \, m \, A_1(R) + m\|\delta\|_2 + L\|\varphi\|_3 + K\|\psi\|_2 \, . \tag{3.10}$$

In the last inequality the estimate $\|g(v)\|_2 \leqq A_1(R)$ has been used based on the $(G^{(2,2)} \, . \, 1)$ assumption; $A_1(R) = A(R) \, . \, (1 + 3R + R^2)$. Proceeding further we have

$$\|V_2(v_1)(g, \delta, \varphi, \psi) - V_2(v_2)(g, \delta, \varphi, \psi)\|_{C_2} \leqq |\varepsilon| \, m\|g(v_1) - g(v_2)\|_2 \, . \tag{3.11}$$

With the aid of $(G^{(2,2)} \, . \, 1, 2)$ we get first of all

$$\|g(v_1) - g(v_2)\|_{2,t} \leqq E(R) \, . \, |||v_1 - v_2|||_{2,t} \tag{3.12}$$

with

$$E(R) = B(R)(1 + 3R + R^2) + A(R)(3 + 2R) \, . \tag{3.13}$$

Taking the supremum over $t \geqq 0$ and inserting the result into (3.11) we have

$$\|V_2(v_1)(g, \delta, \varphi, \psi) - V_2(v_2)(g, \delta, \varphi, \psi)\|_{C^2} \leqq |\varepsilon| \, m \, E(R) \, \|v_1 - v_2\|_{C^2} \, . \tag{3.14}$$

If we now choose the numbers R, ε_1, ε from the conditions ($\varepsilon_0 > 0$ given)

$$R > m\|\delta\|_2 + L\|\varphi\|_3 + K\|\psi\|_2 \, , \quad 0 < \varepsilon_1 < \min\left(\varepsilon_0, (m \, E(R))^{-1}\right), \quad |\varepsilon| \leqq \min\left(\varepsilon_1, (R - m\|\delta\|_2 - L\|\varphi\|_3 - K\|\psi\|_2)(m \, A_1(R))^{-1}\right) \tag{3.15}$$

then from (3.10) and (3.14) we see that the (i) and (ii) assumptions of the Contraction lemma are fulfilled. Hence we conclude

Theorem 1. (*Existence and uniqueness of the solution.*) *Let the functions* g, δ *defined in* (2.12) *satisfy* $(G^{(2,2)} \, . \, 1-3)$. *Let* φ *and* ψ *satisfy* $(G^{(2)} \, . \, 4)$. *Assume further* (3.15). *Then the Cauchy problem* $(1.1^{(2)})$, $(1.2^{(2)})$ *has exactly one solution* $u = u(t, x, y) \in C^2(E_3^+)$, $\|e^{-b_1x - b_2y} \, . \, u\|_{C^2} \leqq R$.

Remark. Later on we shall need the following modification of Theorem 1. Assume that g, δ satisfy $(G^{(3,2)} \, . \, 1-3)$. Assume $\varphi = \psi = 0$. Let

$$R > m\|\delta\|_3 \, , \quad 0 < \varepsilon_1' < \min\left(\varepsilon_0, (m \, E_1(R))^{-1}\right), \quad |\varepsilon| \leqq \varepsilon_2 = \min\left(\varepsilon_1', (R - m\|\delta\|_3)/ \, m \, A_2(R)\right), \tag{3.16}$$

where

$$A_2(R) = A(R) \, . \, (1 + 7R + 6R^2 + R^3) \, , \quad E_1(R) = A(R)(7 + 12R + 3R^2) + + B(R)(1 + 7R + 6R^2 + R^3) \, .$$

Then the Cauchy problem $(1.1^{(2)})$ with zero initial conditions has exactly one solution

$$u \in C^3(E_3^+), \ \|e^{-b_1x-b_2y} . u\|_{C^3} \leqq R.$$

The proof closely follows the ideas developed in the proof of Theorem 1 and is therefore omitted.

Theorem 2. (*Continuous dependence of the solution on the initial conditions and on the right-hand side.*) *Let* $R \geqq 0$ *and* (3.15) *hold. Let the functions* $v_r(t, x, y)$, $\|v_r\|_{C^2} \leqq R$ $(r = 1, 2, \ldots)$ *satisfy the equation*

$$v_r = V_2(v_r)(g, \delta_r, \varphi_r, \psi_r) . \tag{3.17}$$

Suppose that there exist functions δ, φ, ψ *such that*

$$\|\delta_r - \delta\|_2 \to 0 , \quad \|\varphi_r - \varphi\|_3 \to 0 , \quad \|\psi_r - \psi\|_2 \to 0 \tag{3.18}$$

when $r \to +\infty$. *Then there exists a function* $v(t, x, y)$ *which is the solution of* (3.2) *such that* $\|v_r - v\|_{C^2} \to 0$ *for* $r \to +\infty$.

Proof. From (3.17), (3.7)–(3.9) and (3.12) we get

$$\|v_r - v_{r'}\|_{C^2} \leqq |\varepsilon| . m\, E(R)\, \|v_r - v_{r'}\|_{C^2} + m\|\delta_r - \delta_{r'}\|_{C^2} + L\|\varphi_r - \varphi_{r'}\|_3 + + K\|\psi_r - \psi_{r'}\|_2$$

or

$$(1 - |\varepsilon| . m\, E(R)) . \|v_r - v_{r'}\|_{C^2} \leqq m\|\delta_r - \delta_{r'}\|_2 + L\|\varphi_r - \varphi_{r'}\|_3 + + K\|\psi_r - \psi_{r'}\|_2 .$$

Now $1 - |\varepsilon|\, m\, E(R)$ is positive by the assumption and the right-hand side tends to zero because of (3.18). Therefore $\{v_r\}_{r=1}^{\infty}$ is a Cauchy sequence in C^2 and there exists a function $v(t, x, y)$, the C^2-limit of $\{v_r\}_{r=1}^{\infty}$,

$$\|v_r - v\|_{C^2} \to 0 \tag{3.19}$$

as $r \to +\infty$. Moreover, $\|v\|_{C^2} \leqq R$. On the other hand, letting $r \to +\infty$ in (3.17) and taking into account (3.18), (3.19) we see that $v(t, x, y)$ is a solution of (3.2), q.e.d.

Theorem 3. (*Asymptotic stability.*) *Assume* $(G^{(2,2)} . 1, 2)$. *Let the functions* $v_i(t, x, y) \in C^2(E_3^+)$, $i = 1, 2$, $\|v_i\|_{C^2} \leqq R$ *satisfy* (3.2) *with the initial conditions* $\varphi_i \in C^3$, $\psi_i \in C^2$,

$$\|\varphi_i\|_3 \leqq B , \quad \|\psi_i\|_2 \leqq B , \tag{3.20}$$

$B > 0$ *a constant. Let* $E(R)$ *be defined as in* (3.13). *Then there exist positive constants* K, α *such that for*

$$|\varepsilon| < \min(\varepsilon_0, \alpha K^{-1}, (2E(R))^{-1}) \tag{3.21}$$

the inequality

$$(3.22) \qquad |||v_1 - v_2|||_{2,t} \leqq K \,.\, \exp\left(-(\alpha - |\varepsilon| K)\, t\right)$$

is valid.

Proof. Performing similar estimates which led to Theorem 1 (cf. (3.12)) we have with respect to (3.20)

$$(3.23) \qquad |||v_1 - v_2|||_{2,t} \leqq 2|\varepsilon| \int_0^t |||v_1 - v_2|||_{2,\tau} \,.\, e^{-\lambda(t-\tau)} \,.\, P_1(t-\tau)\, \mathrm{d}\tau +$$
$$+ 2\{\|\varphi_1 - \varphi_2\|_3 \,.\, e^{-\lambda t} \,.\, R_1(t) + \|\psi_1 - \psi_2\|_2 \,.\, e^{-\lambda t} \,.\, P_1(t)\} \leqq$$
$$\leqq |\varepsilon| \,.\, K \int_0^t |||v_1 - v_2|||_{2,t} \,.\, e^{-\lambda(t-\tau)}\, \mathrm{d}\tau + K e^{-\alpha t} \,.$$

Applying Gronwall's lemma we get (3.22).

Theorem 4. (*Existence and uniqueness of a T-periodic solution.*) *Assume* $a \neq 0$, $b_1^2 + b_2^2 + c > 0$, $0 < T < +\infty$. *Let* h, f *in* $(1.1^{(2)})$ *be T-periodic in t and let* $(G^{(3,2)} \,.\, 1-3)$ *be valid. Then for* $|\varepsilon| \leqq \varepsilon_2$ *and for* $R > m\|\delta\|_3$ *(cf. (3.16)) the equation* $(1.1^{(2)})$ *has exactly one T-periodic in t solution* $u(t, x, y) \in C^2$, $\|e^{-b_1 x - b_2 y} \,.\, u\|_{C^2} \leqq R$.

Proof. Assume firstly $a > 0$. By Remark following Theorem 1, Eq. $(1.1^{(2)})$ has exactly one solution $u(t, x, y) \in C^3(E_3^+)$, $\|e^{-b_1 x - b_2 y} \,.\, u\|_{C^3} \leqq R$, $u(0, x, y) = u_t(0, x, y) = 0$. Put

$$v(t, x, y) = e^{-b_1 x - b_2 y} \,.\, u(t, x, y) \,.$$

We have $v \in C^3$, $\|v\|_{C^3} \leqq R$ and v solves (3.2) with zero initial conditions.

For $j = 1, 2, \ldots$ let us define

$$(3.24) \qquad \begin{aligned} \varphi_j(x, y) &= v(jT, x, y)\,, \\ (\psi_j - a\varphi_j)(x, y) &= v_t(jT, x, y)\,, \\ v_j(t, x, y) &= v(t + jT, x, y)\,. \end{aligned}$$

We have

$$(3.25) \qquad \|v_j\|_{C^3} \leqq R \,.$$

Besides, φ_j, ψ_j satisfy $(G^{(2)} \,.\, 4)$, $\|\varphi_j\|_3 \leqq B$, $\|\psi_j\|_2 \leqq B$. Because $v_j(t, x, y)$ is a solution of (3.2) with the initial conditions (3.24) we get by Theorem 3

$$|||v_j - v|||_{2,t} \to 0 \quad \text{as} \quad t \to +\infty \,.$$

In other words to every $\eta \geqq 0$ there exists $t_\eta \geqq 0$ in such a way that for $t \geqq t_\eta$ and for $j = 1, 2, \dots$ we have $|||v_j - v|||_{2,t} \leqq \eta$. Choosing $mT \geqq t_\eta$, m an integer and $t \geqq 0$ we have $mT + t \geqq t_\eta$. Therefore

$$|||v_j - v|||_{2,mT+t} \leqq \eta \,. \tag{3.26}$$

Choosing further integers $p > q$, pT, $qT \geqq t_\eta$ we get by (3.26)

$$|||v_p - v_q|||_{2,t} = |||v_{p-q} - v|||_{2,qT+t} \leqq \eta \,. \tag{3.27}$$

Therefore

$$\|v_p - v_q\|_{C^2} = \sup_{t \geqq 0} |||v_p - v_q|||_{2,t} \leqq \eta$$

which proves that $\{v_j(t, x, y)\}_{j=1}^{\infty}$ forms a Cauchy sequence in C^2. In this way the existence of a C^2-limit function $\bar{v}(t, x, y)$ is established, $\|\bar{v} - v_j\|_{C^2} \to 0$ as $j \to +\infty$. From here and (3.25) it results further $\|\bar{v}\|_{C^2} \leqq R$. By (3.27) and (3.23) applied with v_p, v_q, φ_p, φ_q, ψ_p, ψ_q instead of v_1, v_2, φ_1 φ_2, ψ_1, ψ_2, respectively, we get also the Cauchy character of $\{\varphi_p\}_{p=1}^{\infty}$, $\{\psi_p\}_{p=1}^{\infty}$ respectively in C^3, C^2. I. e., there exist $\bar{\varphi} \in C^3$ and $\bar{\varphi} \in C^2$ such that $\|\bar{\varphi} - \varphi_p\|_3 \to 0$ as $p \to +\infty$ and $\|\bar{\psi} - \psi_p\|_2 \to 0$ as $p \to +\infty$. Besides, $\|\bar{\varphi}\|_3 \leqq B$, $\|\bar{\psi}\|_2 \leqq B$. Further

$$\bar{v}(0, x, y) = \lim_{p \to +\infty} v_p(0, x, y) = \lim_{p \to +\infty} v(pT, x, y) = \lim_{p \to +\infty} \varphi_p(x, y) = \bar{\varphi}(x, y)$$

and similarly $\bar{v}_t(0, x, y) = (\bar{\psi} - a\bar{\varphi})(x, y)$. On the other hand, $\bar{v}(t, x, y)$ is a solution of (3.2) with $\bar{\varphi}$ and $\bar{\psi}$ as initial conditions. This follows from the identity

$$\bar{v} - V_2(\bar{v})(g, \delta, \bar{\varphi}, \bar{\psi}) = (\bar{v} - v_p) + (V_2(v_p)(g, \delta, \varphi_p, \psi_p) - V_2(\bar{v})(g, \delta, \bar{\varphi}, \bar{\psi}))$$

letting $p \to +\infty$ and using previous estimates.

T-periodicity of $\bar{v}$ is guaranteed by the relation

$$\bar{v}(t + T, x, y) = \lim_{p \to +\infty} v_p(t + T, x, y) = \lim_{p \to +\infty} v_{p+1}(t, x, y) = \bar{v}(t, x, y) \,.$$

Uniqueness: Would there exist another T-periodic solution of (3.2), say $p(t, x, y)$, $\|p\|_{C^2} \leqq R$, then reasoning as in the case of $v_j(t, x, y)$ we get that the initial conditions for $p_j(t, x, y) \overset{\text{df}}{=} p(t + jT, x, y)$, viz. $p(0, x, y) = p(jT, x, y) = p_j(0, x, y)$ do converge towards $\bar{\varphi}(x, y)$ as $j \to +\infty$. We conclude that $p(0, x, y) = \bar{v}(0, x, y)$ and similarly $p_t(0, x, y) = \bar{v}(0, x, y)$.

On the other hand, by Theorem 3

$$|||\bar{v} - p|||_{2,t} \to 0 \tag{3.28}$$

as $t \to +\infty$. Because the initial conditions of $\bar{v}$ and p coincide (3.28) cannot hold unless $\bar{v}$ and p are identical in the whole range of t. From $\bar{v}(t, x, y) = e^{-b_1x - b_2y} \,.$ $. \, \bar{u}(t, x, y)$ we get a unique T-periodic in t solution $\bar{u}(t, x, y) \in C^2$ of $(1.1^{(2)})$, $\|e^{-b_1x-b_2y} \,.\, \bar{u}\|_{C^2} \leqq R$.

II. The case $a < 0$ is changed to the former one by defining $f(t, x, y, u, \varepsilon)$ for negative values of t via T-periodicity. The substitution $t = -\tau$ in $(1.1^{(2)})$ then changes the u_t-coefficient sign and so the proof is reduced to part I.

4. $n = 3$. Following the ideas developed in Sec. 3 we define: for $s(t, X) \in C^2(E_4^+)$ $(X = [x, y, z])$

$$(4.1) \qquad H_3(s)(\tau, t, X) = e^{-at}(t - \tau)^{-1} \, . \, D_t\left(\int_0^{t-\tau} r^2 \, . \, J_0(\omega\Gamma) \, . \, Q(s)(\tau, X, r) \, dr\right)$$

with

$$Q(s)(\tau, X, r) = (4\pi)^{-1} \int_0^{2\pi} \int_0^{\pi} e^{a\tau} \, . \, s(r) \, . \sin \vartheta \, d\vartheta \, dv \, ,$$

$$s(r) = s(\tau, x + r \, . \sin \vartheta \, . \cos v, y + r \, . \sin \vartheta \, . \sin v, z + r \, . \cos \vartheta) \, ;$$

$$(4.2) \qquad M_3(s)(t, X) = \int_0^t H_3(s)(\tau, t, X) \, d\tau \, ,$$

$$\Psi_3(\psi)(t, X) = H_3(s)(0, t, X) \quad \text{with} \quad \psi(X) = s(0, X) \, ,$$

$$\Phi_3(\varphi)(t, X) = (D_t \Psi_3 + a\Psi_3)(\varphi)(t, X) \, .$$

After that (2.16) may be rewritten in the form

$$(4.3) \qquad v(t, X) = (\varepsilon \, M_3(g(v)) + M_3(\delta) + \Phi_3(\varphi) + \Psi_3(\psi))(t, X) \equiv$$
$$\equiv V_3(v)(g, \delta, \varphi, \psi)(t, X) \, .$$

Theorem 1'. *Assume* $(G^{(2,3)} \, . \, 1-3)$ *and* $(G^{(2)} \, . \, 4)$. *Let*

$$(4.4) \qquad R > K\|\delta\|_2 + \varrho\|\varphi\|_3 + \mu\|\psi\|_2$$

with suitable positive constants K, ϱ, μ. *Let*

$$(4.5) \qquad |\varepsilon| \leqq \min\left(\varepsilon_3, (R - K\|\delta\|_2 - \varrho\|\varphi\|_3 - \mu\|\psi\|_2)/K \, A_1(R)\right) ,$$

where $0 < \varepsilon_3 < \min\left(\varepsilon_0, (K \, E(R))^{-1}\right)$, $E(R)$ *and* $A_1(R)$ *is defined as in Sec. 3. Then the problem* $(1.1^{(3)})$, $(1.2^{(3)})$ *has exactly one solution* $u = u(t, X) \in C^2(E_4^+)$, $\|e^{-b_1x - b_2y - b_3z} \, . \, u\|_{C^2} \leqq R$.

Proof. Similarly as in the proof of Theorem 1, estimates of the integral operators are required in order to verify the (i) and (ii) assumptions of the Contraction mapping principle.

1) The counterpart of the inequalities (3.3) is the inequality

$$(4.6) \qquad 2^n n! |J_n(\omega\Gamma) \, . \, (\omega\Gamma)^{-n}| \leqq \begin{matrix} 1 \\ e^{\Omega(t-\tau)} \end{matrix} \quad \text{if} \quad \begin{matrix} \omega^2 \geqq 0 \, , \\ \omega^2 < 0 \, , \end{matrix}$$

$n = 1, 2, \ldots$ (Cf. [5].)

2) For the total derivatives $D_t^k s$, $k = 1, 2, 3$ we get $|D_t^k s| \leqq 3^k \,.\, \|s\|_{k,t}$.

3) For $Q(D_t^k s)(\tau, X, t-\tau)$ we have $|Q(D_t^k s)(\tau, X, t-\tau)| \leqq 3^k \,.\, e^{a\tau} \,.\, \|s\|_{k,\tau}$, $k = 0, 1, 2, 3$.

4) Using the known relations (cf. [5])

$$J_n(\omega\Gamma) \,.\, (\omega\Gamma)^{-n} \to (n!\,.\,2^n)^{-1} \quad \text{as} \quad \Gamma \to 0\,,$$

$$J_n'(z) = n \,.\, J_n(z) \,.\, z^{-1} - J_{n+1}(z)\,, \quad n = 0, 1, 2, \ldots$$

we get

$$|D_t^k\, J_0(\omega\Gamma)| \leqq R_k(t-\tau) \quad \text{if} \quad \omega^2 \geqq 0$$

and

$$|D_t^k\, J_0(\omega\Gamma)| \leqq R_k(t-\tau) \,.\, e^{\Omega(t-\tau)} \quad \text{if} \quad \omega^2 < 0\,,$$

$k = 1, 2, 3, 4$. $R_k(\tau)$ are polynomials of the k-th degree with non-negative coefficients.

5) Denoting $(k = 1, 2, 3, 4)$

$$I_k = \int_0^{t-\tau} r^2 D_t^k\, J_0(\omega\Gamma)\, Q(s)(\tau, X, r)\,\mathrm{d}r$$

we have

$$|I_k| \leqq S_{k+3}(t-\tau) \,.\, e^{a\tau} \,.\, \|s\|_{2,\tau} \quad \text{if} \quad \omega^2 \geqq 0$$

and

$$|I_k| \leqq S_{k+3}(t-\tau) \,.\, e^{a\tau + \Omega(t-\tau)} \,.\, \|s\|_{2,\tau} \quad \text{if} \quad \omega^2 < 0.$$

$S_{k+3}(t-\tau)$ are polynomials of the $(k+3)$-rd degree with non-negative coefficients.

With the aid of the above inequalities we easily deduce the estimates for the integral operators:

a)
$$|D^\alpha H_3(s)(\tau, t, X)| \leqq \|D^\alpha s\|_{0,\tau} \,.\, e^{-\lambda(t-\tau)} \,.\, P_3(t-\tau) \leqq$$
$$\leqq \|s\|_{2,\tau} \,.\, e^{-\lambda(t-\tau)} \,.\, P_3(t-\tau)\,, \quad |\alpha| \leqq 2\,;$$
$$|D^\beta D_t\, H_3(s)(\tau, t, X)| \leqq \|s\|_{2,\tau} \,.\, e^{-\lambda(t-\tau)} \,.\, P_4(t-\tau)\,, \quad |\beta| \leqq 1\,;$$
$$|D_t^2\, H_3(s)(\tau, t, X)| \leqq \|s\|_{2,\tau} \,.\, e^{-\lambda(t-\tau)} \,.\, P_5(t-\tau)\,;$$
$$|D_t^3\, H_3(s)(\tau, t, X)| \leqq \|s\|_{3,\tau} \,.\, e^{-\lambda(t-\tau)} \,.\, P_6(t-\tau)\,.$$

(P_k are polynomials of the k-th degree, $k = 3, 4, 5, 6$.)

b) Using the inequalities stated in a) we get for M_3

$$|D^\alpha M_3(s)(t, X)| \leqq \int_0^t \|s\|_{2,\tau} \,.\, e^{-\lambda(t-\tau)} \,.\, P_3(t-\tau)\,\mathrm{d}\tau \leqq$$
$$\leqq \|s\|_2 \,.\, (r_3 + e^{-\lambda t}\, Q_3(t)) \leqq \|s\|_2 \,.\, K_0\,, \quad |\alpha| \leqq 2\,.$$

Here

$$K_0 = \sup_{t \geqq 0} (r_3 + e^{-\lambda t} Q_3(t)) ;$$

r_3 is a positive constant, $Q_3(t)$ a polynomial of the third degree. Further we have

$$|D^\beta D_t M_3(s)(t, X)| \leqq \|s\|_2 . K_1 ,$$

$$|D_t^2 M_3(s)(t, X)| \leqq \|s\|_2 . K_2 .$$

Summarizing we have

(4.7) $$\|M(s)\|_{C^2} \leqq \|s\|_2 . K$$

with $K = \max(K_0, K_1, K_2)$.

c) $|D^\alpha \Psi_3(\psi)(t, X)| \leqq \|\psi\|_2 . e^{-\lambda t} . P_3(t) \leqq \|\psi\|_2 . \mu_0 , \quad |\alpha| \leqq 2 ;$
$|D^\beta D_t \Psi_3(\psi)(t, X)| \leqq \|\psi\|_2 . e^{-\lambda t} . P_4(t) \leqq \|\psi\|_2 . \mu_1 , \quad |\beta| \leqq 1 ;$
$|D_t^2 \Psi_3(\psi)(t, X)| \leqq \|\psi\|_2 . e^{-\lambda t} . P_5(t) \leqq \|\psi\|_2 . \mu_2 ;$

(4.8) $$\|\Psi_3(\psi)\|_{C^2} \leqq \mu . \|\psi\|_2$$

with $\mu = \max(\mu_0, \mu_1, \mu_2)$.

d) $|D^\alpha \Phi_3(\varphi)(t, X)| \leqq \|\varphi\|_3 . e^{-\lambda t} . (P_4(t) + a P_3(t)) \leqq \|\varphi\|_3 . (\mu_1 + a\mu_0) ,$
$|\alpha| \leqq 2 ;$
$|D^\beta D_t \Phi_3(\varphi)(t, X)| \leqq \|\varphi\|_3 . e^{-\lambda t} . (P_5(t) + a P_4(t)) \leqq \|\varphi\|_3 . (\mu_2 + a\mu_1) ,$
$|\beta| \leqq 1 ;$
$|D_t^2 \Phi_3(\varphi)(t, X)| \leqq \|\varphi\|_3 . e^{-\lambda t} . (P_6(t) + a P_5(t)) \leqq \mu_3 . \|\varphi\|_3 ;$

(4.9) $$\|\Phi_3(\varphi)\|_{C^2} \leqq \varrho . \|\varphi\|_3$$

with $\varrho = \max(\mu_1 + a\mu_0, \mu_2 + a\mu_1, \mu_3)$.

Using in (4.3) the derived estimates a) to d) we get

(4.10) $$\|V_3(v)(g, \delta, \varphi, \psi)\|_{C^2} \leqq |\varepsilon| . K . \|g(v)\|_2 + K . \|\delta\|_2 + \varrho . \|\varphi\|_3 + + \mu . \|\psi\|_2 \leqq |\varepsilon| . K . A_1(R) + K\|\delta\|_2 + \varrho\|\varphi\|_3 + \mu\|\psi\|_2$$

and

(4.11) $$\|V_3(v_1)(g, \delta, \varphi, \psi) - V_3(v_2)(g, \delta, \varphi, \psi)\|_{C^2} \leqq |\varepsilon| . K . \|g(v_1) - g(v_2)\|_2 \leqq \leqq |\varepsilon| . K . E(R) . \|v_1 - v_2\|_{C^2} .$$

These inequalities with respect to (4.4) and (4.5) imply both (i) and (ii) conditions of the Contraction mapping principle.

In the same way it would be possible to derive analogous theorems 2', 3' and 4'.

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Souhrn

PERIODICKÁ ŘEŠENÍ SLABĚ NELINEÁRNÍ HYPERBOLICKÉ ROVNICE V E_2 A E_3

VÁCLAV VÍTEK

Pro $n = 2$ a 3 se dokazuje existence a jednoznačnost klasického periodického řešení rovnice

$$\Box_n u + 2au_t + 2(B, \nabla_n u) + cu = h(t, x) + \varepsilon f(t, x, u, \varepsilon)$$

$(x = (x_1, x_2, \ldots, x_n))$ za předpokladu periodičnosti pravé strany.

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