Václav Alda; Vojtěch Kundrát; Miloš Václav Lokajíček

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EXPONENTIAL DECAY LAW AND IRREVERSIBILITY OF DECAY AND COLLISION PROCESSES

VÁCLAY ALDA, VOJTĚCH KUNDRÁT, MILOŠ VÁCLAV LOKAJÍČEK

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1. INTRODUCTION

A considerable attention has been devoted to the description of decay and collision processes in past years. A series of various assumptions and their consequences have been considered and confronted with the well-known experimental facts. However, in most cases only an approximate approach could be performed. It seems that a new period of rigorous mathematical formulation and solution of the whole problem has been started only by papers [1], [2], [3].

These papers start from the basic approach in which a given decay (or collision) process is described by the pair $\{\mathscr{H}, U(t)\}$ where the $\mathscr{H}$ is a corresponding Hilbert space of state vectors and the $U(t)$ is the evolution operator determining time evolution of the given physical system. Consequences of various assumptions about the structure of the space $\mathscr{H}$ and about properties of the operator $U(t)$ have been studied. It has been assumed in all these papers that the Hilbert space $\mathscr{H}$ can be divided into two mutually orthogonal subspaces $\mathscr{H}_A$ and $\mathscr{H}_D$ where the $\mathscr{H}_A$ corresponds to the unstable particle and the $\mathscr{H}_D$ to its decay products.

The collection of the basic assumptions, which have been taken into account, can be listed as follows:

(1) $$\mathscr{H} = \mathscr{H}_A \oplus \mathscr{H}_D ;$$

(2) $$U(0) = 1 , \quad U(t + t') = U(t)\, U(t') , \quad \forall t, t' \geqq 0 ;$$

(3) $$U^*(t)\, U(t) = 1 = U(t)\, U^*(t) , \quad \forall t \geqq 0$$

(4) $$A(t + t') = A(t)\, A(t') , \quad \forall t, t' \geqq 0 ,$$

where

$$A(t) = P_A\, U(t)\, P_A ,$$

and P_A is the projection operator into the subspace $\mathscr{H}_A$;

(5) $$\langle\psi|H\psi\rangle \geqq 0\,, \quad \forall\psi \in \mathscr{H}\,,$$

i.e. the spectrum of the operator H is bounded below and H is defined by

(5') $$U(t) = e^{-iHt}\,;$$

the condition (5) can be used, of course, only if the conditions (2) and (3) are taken into account at the same time.

If the particle described by the subspace $\mathscr{H}_A$ ought actually to decay, then the operator $A(t)$ must not be unitary in this subspace; it must hold

(6) $$\forall u \in \mathscr{H}_A,\ \exists t > 0 : \|A(t)\,u\| < \|u\|\,.$$

And finally, the system $\{\mathscr{H}, U(t)\}$ should be minimal, i.e. it should hold

(7) $$\mathscr{H} = \Big(\bigcup_{t \geqq 0} [U(t) + U^*(t)]\,\mathscr{H}_A\Big)^-\,.$$

The states having no relation to the unstable particle are not included in $\mathscr{H}_D$.

In the mentioned papers a series of consequences following from different combinations of the assumptions (1)–(7) and some other additional ones have been derived. In this paper we will deal with some further consequences which follow mainly from the assumption (4). They are contained in the following theorems.

2. WEISSKOPF-WIGNER CONDITION AND THE STRUCTURE OF $\mathscr{H}_D$

Theorem 1. *Let the system $\{\mathscr{H}, U(t)\}$ fulfil the conditions* (1) *and* (2); *then it follows from* (4) *that it is possible to write*

(8) $$\mathscr{H}_D = \mathscr{D}_+ \oplus \mathscr{D}_-\,,$$

where the two mutually orthogonal subspaces $\mathscr{D}_+$ and $\mathscr{D}_-$ have the following properties (for $\forall d_\pm \in \mathscr{D}_\pm$, $\forall u \in \mathscr{H}_A$, $\forall t \geqq 0$)

(9a) $$\langle d_- \mid U(t)\,u\rangle = 0\,,$$

(9b) $$\langle d_- \mid U(t)\,d_+\rangle = 0\,,$$

(9c) $$\langle u \mid U(t)\,d_+\rangle = 0\,;$$

and on the contrary, the condition (4) *follows from the existence of two mutually orthogonal subspaces $\mathscr{D}_+$ and $\mathscr{D}_-$ the elements of which fulfil the conditions* (9).

Proof. Let us introduce the following reduced evolution operators

(10) $$A(t) = P_A\,U(t)\,P_A\,, \quad B(t) = P_D\,U(t)\,P_A\,,$$
$$C(t) = P_A\,U(t)\,P_D\,, \quad D(t) = P_D\,U(t)\,P_D\,,$$

where

$$P_D = 1 - P_A .$$

It follows from (2) and (10)

$$A(t + t') = A(t)\,A(t') + C(t)\,B(t') , \tag{11a}$$

$$B(t + t') = B(t)\,A(t') + D(t)\,B(t') , \tag{11b}$$

$$C(t + t') = C(t)\,D(t') + A(t)\,C(t') , \tag{11c}$$

$$D(t + t') = D(t)\,D(t') + B(t)\,C(t') , \tag{11d}$$

and if we add the condition (4) we obtain at once

$$C(t)\,B(t') = 0 , \quad \forall t, t' \geqq 0 . \tag{12}$$

Now, let $\mathscr{D}_+$ be a subspace of $\mathscr{H}_D$ which is defined by the condition

$$\mathscr{D}_+ = \{d_+ \in \mathscr{H}_D : C(t)\,d_+ = 0 , \ \forall t \geqq 0\} ; \tag{13}$$

and let $\mathscr{D}_-$ be its orthogonal complement in $\mathscr{H}_D$,

$$\mathscr{D}_- = \mathscr{H}_D \ominus \mathscr{D}_+ ;$$

the mutually orthogonal subspaces $\mathscr{D}_+$ and $\mathscr{D}_-$ defined by (13) fulfil the condition (8).

It follows from (10), (12) and (13)

$$B(t)\,\mathscr{H} = B(t)\,\mathscr{H}_A \subset \mathscr{D}_+ , \tag{14}$$

and one obtains immediately that (9a) is fulfilled, as

$$\langle d_- | d_+ \rangle = 0 . \tag{15}$$

We get further from (11c) and (13)

$$C(t)\,D(t')\,d_+ = 0 , \quad \forall d_+ \in \mathscr{D}_+ , \quad \forall t, t' \geqq 0 ,$$

and therefore,

$$U(t)\,d_+ = D(t)\,d_+ \in \mathscr{D}_+ , \quad \forall t \geqq 0 , \tag{16}$$

which leads at once to (9bc) and the proof of the first part is finished.

On the contrary, from (9c) we obtain (13) and from (9a) and (15) also (14); the condition (4) is a direct consequence of (13) and (14).

From (9a) we can also derive that

$$B^*(t)\,\mathscr{D}_- = \{0\} , \quad \forall t \geqq 0 ; \tag{17}$$

and similarly that

(18a) $$U^*(t)\,\mathscr{D}_- \subset \mathscr{D}_-\,, \quad \forall t \geqq 0\,,$$

(18b) $$C^*(t)\,\mathscr{H}_A \subset \mathscr{D}_-\,, \quad \forall t \geqq 0\,;$$

these relations will be useful later on.

It follows from Theorem 1 that the exponential decay law (i.e. Eq. (4)) implies that such a decay process should be described as an irreversible one. We shall see from the following that the subspaces $\mathscr{D}_+$ and $\mathscr{D}_-$ have the properties identical with the outgoing and incoming subspaces D_+ and D_- of Lax and Phillips [4], p. 45.

The evolution operator $U(t)$ of Theorem 1 may be, of course, non-unitary. In the next theorems we will derive some additional consequence following from adding the assumption (3).

Theorem 2. *Let the system* $\{\mathscr{H}, U(t)\}$ *fulfil the conditions* (1)–(4); *then it holds*

(19a) $$U^*(t)\,U(t')\,\mathscr{D}_+ = U(t' - t)\,\mathscr{D}_+ \subset \mathscr{D}_+\,,$$

(19b) $$U(t)\,U^*(t')\,\mathscr{D}_- = U^*(t' - t)\,\mathscr{D}_- \subset \mathscr{D}_-\,,$$

for any $t' \geqq t$.

Proof. Using (2) it is possible to write

$$U(t + \tau)\,d_+ = U(t)\,U(\tau)\,d_+\,;$$

then (19a) follows immediately from conditions (3) and (16). Eq. (19b) can be derived in a similar way.

Let us mention the physical meaning of Eqs. (19). If we take a finite t, then it follows e.g. from (19b) that it is possible to define a subset $U^*(t)\,\mathscr{D}_-$ in $\mathscr{D}_-$, the states of which have the following properties: if we start from such a state it is not possible to get any state from $\mathscr{H}_A$ or $\mathscr{D}_+$ by a further evolution given by $U(t)$ before the given time t is over.

It is useful to introduce the following symbols.

(20a) $$\tilde{\mathscr{D}}_+(\tau) = U(\tau)\,\mathscr{D}_+ = D(\tau)\,\mathscr{D}_+\,, \qquad \forall \tau \geqq 0\,;$$

(20b) $$\tilde{\mathscr{D}}_-(\tau) = U^*(-\tau)\,\mathscr{D}_- = D(\tau)\,\mathscr{D}_-\,, \qquad \forall \tau \leqq 0\,.$$

Remark. All results of the Theorem 1 and 2 do not depend on the condition (6). If the (6) is not fulfilled the division of $\mathscr{H}$ into $\mathscr{D}_+$ and $\mathscr{D}_-$ may be quite arbitrary and may loose a reasonable physical meaning. However, much more can be said about the system $\{\mathscr{H}, U(t)\}$ if also the condition (6) is taken into account (see further theorems).

Theorem 3. *Let the system* $\{\mathscr{H}, U(t)\}$ *fulfil the conditions* (1)–(4) *and* (6); *then the subspaces* $\tilde{\mathscr{D}}_+(\tau)$ *and* $\tilde{\mathscr{D}}_-(\tau)$ *form strongly decreasing sequences with increasing* $|\tau|$:

(21a) $$\tilde{\mathscr{D}}_+(\tau) \supset \tilde{\mathscr{D}}_+(\tau'), \quad \tilde{\mathscr{D}}_+(\tau) \neq \tilde{\mathscr{D}}_+(\tau'), \quad |\tau'| > |\tau|;$$

(21b) $$\tilde{\mathscr{D}}_-(\tau) \supset \tilde{\mathscr{D}}_-(\tau'), \quad \tilde{\mathscr{D}}_-(\tau) \neq \tilde{\mathscr{D}}_-(\tau'), \quad |\tau'| > |\tau|.$$

Proof. The proof can be performed by contradiction. Suppose that for some fixed $\tau' \geqq 0$ the following holds:

$$\tilde{\mathscr{D}}_+(\tau + \tau') \supset \tilde{\mathscr{D}}_+(\tau');$$

using the condition (20a) and (19a) one gets

$$\mathscr{D}_+ \supset U^*(\tau)\, \mathscr{D}_+ .$$

And since

$$U^*(\tau)\, \mathscr{D}_+ = B^*(\tau)\, \mathscr{D}_+ + D^*(\tau)\, \mathscr{D}_+ ,$$

we should get

$$B^*(\tau)\, \mathscr{D}_+ = \{0\};$$

this combined with (17) gives $B^*(\tau)\, \mathscr{H}_D = \{0\}$, which contradicts the assumption (6) and the condition (21a) is proved. The condition (21b) can be proved in a similar way.

As the subspaces $\tilde{\mathscr{D}}_+(\tau)$ form a decreasing sequence, it is possible to introduce the non-empty subspaces

(22a) $$\tilde{\mathscr{D}}_+(\tau, \Delta\tau) = \tilde{\mathscr{D}}_+(\tau) \ominus \tilde{\mathscr{D}}_+(\tau + \Delta\tau), \quad \Delta\tau > 0;$$

the elements of which are denoted by $d_+(\tau, \alpha)$. And similarly

(22b) $$\tilde{\mathscr{D}}_-(\tau, \Delta\tau) = \tilde{\mathscr{D}}_-(\tau) \ominus \tilde{\mathscr{D}}_-(\tau - \Delta\tau), \quad \Delta\tau > 0;$$

with the elements $d_-(\tau, \alpha)$.

Corollary 1. *Let the system* $\{\mathscr{H}, U(t)\}$ *fulfil the conditions* (1)–(4) *and* (6); *then the vectors given by* (22a) *fulfil for any* $h \geqq \Delta\tau$

(23a) $$\langle d_+(\tau, \alpha) \mid d_+(\tau + h, \beta)\rangle = 0,$$

(23b) $$\langle d_-(\tau, \alpha) \mid d_-(\tau - h, \beta)\rangle = 0,$$

and further

(24a) $$U(t)\, d_+(\tau, \alpha) = d_+(\tau + t, \alpha),$$

(24b) $$U^*(t)\, d_-(\tau, \alpha) = d_-(\tau - t, \alpha).$$

Proof. As $\tilde{\mathscr{D}}_+(\tau, \Delta\tau)$ is orthogonal complement to $\tilde{\mathscr{D}}_+(\tau + \Delta\tau)$ in $\tilde{\mathscr{D}}_+(\tau)$, the states $d_+(\tau, \alpha)$ and $d_+(\tau + \Delta\tau, \beta)$ belong always to two mutually orthogonal subspaces of $\mathscr{D}_+$ and the condition (23a) is fulfilled. Further, it follows from (19a)

$$\tilde{\mathscr{D}}_+(\tau) = U^*(t)\, \tilde{\mathscr{D}}_+(\tau + t)\,, \quad \forall \tau, t \geqq 0\,. \tag{25}$$

Therefore, there exists one-to-one correspondence between the subspaces $\tilde{\mathscr{D}}_+(\tau, \Delta\tau)$ characterized by different τ and it is possible to introduce the parameter α which denotes a given evolution line with increasing t in agreement with Eq. (24a). Eqs. (23b) and (24b) can be derived in a similar manner.

Corollary 2. *Let the system $\{\mathscr{H}, U(t)\}$ fulfil the conditions of Theorem 3; then there exist the subspaces $\mathscr{P}_\pm$ in $\mathscr{H}_D$*

$$\mathscr{P}_+ = \bigcap_{\tau \geqq 0} \tilde{\mathscr{D}}_+(\tau)\,, \tag{26a}$$

$$\mathscr{P}_- = \bigcap_{\tau \leqq 0} \tilde{\mathscr{D}}_-(\tau)\,, \tag{26b}$$

with the following properties

$$B^*(t)\, \mathscr{P}_+ = C(t)\, \mathscr{P}_+ = \{0\}\,, \quad \forall t \geqq 0\,, \tag{27a}$$

$$B^*(t)\, \mathscr{P}_- = C(t)\, \mathscr{P}_- = \{0\}\,, \quad \forall t \geqq 0\,. \tag{27b}$$

Proof. The condition $C(t)\, \mathscr{P}_+ = \{0\}$ follows immediately from (13) and (26a). According to (25) we can write

$$B^*(t)\, \tilde{\mathscr{D}}_+(\tau) = \{0\}\,, \quad \forall t \leqq \tau\,. \tag{28}$$

Suppose now, that the condition (27a) is not fulfilled, i.e. that there exists $p \in \mathscr{P}_+$ for which

$$\langle u \mid B^*(t)\, p > \neq 0\,;$$

and according to (28)

$$p \in \mathscr{D}_+ \ominus \tilde{\mathscr{D}}_+(t)\,,$$

which contradicts (26a). The condition (27b) can be proved in a similar way.

Remark. According to Corollary 2 there exist subspaces $\mathscr{P}_+$ and $\mathscr{P}_-$ in $\mathscr{H}_D$ which develope quite separately from the other parts of the space $\mathscr{H}$. These subspaces need not be empty if the condition (7) is not required.

Corollary 3. *In the system given by Theorem 3 each state*

$$d_\pm(\tau, \alpha) \in \mathscr{H}_D\,,$$

is localized in time; *i.e. it holds*

$$\langle d_{\pm}(\tau, \alpha) \mid U(t)\, d_{\pm}(\tau, \alpha)\rangle = 0\,,$$

for any $t \geqq \Delta\tau$.

This result follows immediately from the Corollary 1 and the mutual orthogonality of the subspaces $\mathscr{H}_A$, $\mathscr{D}_+$ and $\mathscr{D}_-$.

Remark. If the condition (7) is added to the assumptions of Corollary 3 then the assertion is valid for any vector from $\mathscr{D}_+$ or $\mathscr{D}_-$ which represents a stronger result than it was derived e.g. (1) (see Lemma 1) under the same conditions.

Theorem 4. *Let the system* $\{\mathscr{H}, U(t)\}$ *fulfil the conditions* (1)–(4), (6) *and* (7); *then*

$$\mathscr{P}_+ = \{0\}\,, \quad \mathscr{P}_- = \{0\}\,; \tag{29}$$

$$\mathscr{H} = \Big(\bigcup_{t \geqq 0} U^*(t)\, \mathscr{D}_+\Big)^- = \Big(\bigcup_{t \geqq 0} U(t)\, \mathscr{D}_-\Big)^- \tag{30}$$

Proof. If $p \in \mathscr{P}_+ \subset \mathscr{D}_+$, then according to (18b)

$$\langle p, U^*(t)\, u\rangle = 0\,, \quad \forall u \in \mathscr{H}_A\,, \quad \forall t \geqq 0\,. \tag{31}$$

And similarly, using (27a) we obtain

$$\langle p, U(t)\, u\rangle = \langle p, B(t)\, u\rangle = \langle B^*(t)\, p, u\rangle = 0\,,$$

which together with (31) leads to

$$p \perp ([U(t) + U^*(t)]\, \mathscr{H}_A)^-\,, \quad \forall t, t' \geqq 0\,;$$

and owing to (7) we get the first part of (29). The second part for $\mathscr{P}_-$ can be obtained in the same way. The conditions (30) can be also easily derived.

Remark. If we combine the results of Theorems 1, 3 and 4 we can conclude that the subspaces $\mathscr{D}_+$ and $\mathscr{D}_-$ fulfil four assumptions on which the scattering theory of Lax and Phillips [4] is based.

Till now the condition (5) has not been taken into account. It will be considered in the following theorem.

Theorem 5. *Let us suppose that the restrictions* (1)–(5) *are made concerning the system* $\{\mathscr{H}, U(t)\}$; *then*

$$A^*(t)\, A(t) = P_A\,, \quad D^*(t)\, D(t) = P_D\,, \tag{32}$$

and the evolution in both the subspaces $\mathscr{H}_A$ *and* $\mathscr{H}_D$ *is quite separated and no transitions between them are possible.*

This theorem can be derived directly from the results of papers [1], [2], [3], [5]. However, for the sake of completeness we will briefly sketch a proof. We will start from Eq. (12). For a fixed t' this expression can be taken as a function $f(t)$.

Since

$$U(t) = \int_{-\infty}^{+\infty} e^{it\lambda} \, \mathrm{d}E_\lambda \,,$$

where E_λ is the spectral decomposition of unity corresponding to $U(t)$, we can conclude (with the help of assumption (5)) that the $f(t)$ is holomorphic in the half--plane $\operatorname{Im} t > 0$ and continuous in the half-plane $\operatorname{Im} t \geqq 0$. Being $f(t) = 0$ for real $t > 0$ we can define in a unique manner a holomorphic function φ in the half-plane $\operatorname{Re} t > 0$ which is the holomorphic continuation of f; and it must be $\varphi(t) = 0$. So we can conclude that $f(t) = 0$ for $\operatorname{Im} t > 0$ and hence $f(t) = 0$ for $-\infty < t < \infty$.

Therefore the condition (12) holds also for any $t < 0$, and as

$$B^*(t) = C(-t)\,,$$

we get immediately

$$B(t) = 0\,, \quad \forall t\,,$$

from which the condition (31) can be easily derived.

And we have obtained the known result that the first five basic assumptions are in a strict disagreement with the condition (6); or that a decay of an unstable particle cannot be described in a frame of the conditions (1)–(5).

Remark. The operator function given by (12) can be continued to negative values for t as well as for t'. Moreover, it is not necessary to start from the condition (12) holding for any positive t, t'. The proof and the result of Theorem 5 remains valid also in such a case, when the condition (4) (and (12) as well) are fulfilled only for t and $t + t'$ lying in an interval (t_1, t_2), where $t_2 > t_1 > 0$ (see also [3]).

3. CONCLUSION

The requirement of an exponential decay law in combination with other reasonable assumptions leads to the conclusion that the Hilbert space as well as the evolution operator have the structures quite identical with those which represent the basis of the scattering theory of Lax and Phillips [4, p. 45].

It is, of course, necessary to mention a result of some papers in which it is shown that from the formal mathematical point of view the exponential decay law in some definite time interval can be obtained in a sufficient (at least for the present) approximation even if the strict validity of Eq. (4) is not required and one limits oneself only to the conditions (1) – (3) and (5). On the other hand, it is necessary to take into account that the result of Theorem 5 cannot be avoided even if the validity of condition (4) is limited to a less interval of positive t.

All of the basic assumptions (1) – (5) can be reasoned to some extent. It is, of course, no doubt that all these assumptions cannot hold strictly at the same time (see Theorem 5) if a decay of an unstable particle is to be described. From an a-priori point of view the conditions (1) – (5) should be, however, regarded as a set of equally justified assumptions. A decision, which of them should be actullly released, ought to be done only after a careful comparison of individual consequences with the corresponding experimental facts is performed.

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Souhrn

EXPONENCIÁLNÍ ROZPADOVÝ ZÁKON A IRREVERSIBILITA ROZPADOVÝCH A KOLISNÍCH PROCESŮ

VÁCLAV ALDA, VOJTĚCH KUNDRÁT, MILOŠ VÁCLAV LOKAJÍČEK

Při popisu rozpadového procesu se používá Hilbertův prostor $\mathscr{H} = \mathscr{H}_A \oplus \mathscr{H}_D$, kde $\mathscr{H}_A$ odpovídá nestabilní částici a $\mathscr{H}_D$ rozpadovým produktům. V práci se odvozují důsledky pro strukturu prostoru $\mathscr{H}$ z postupně přidávaných předpokladů o procesu rozpadu.

Author's addresses: Václav Alda, Matematický ústav ČSAV, Žitná 25, 11 567 Praha 1; *Vojtěch Kundrát, Miloš Václav Lokajíček,* Fysikální ústav ČSAV, Na Slovance 2, 180 40 Praha 8.