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A MIXED FINITE ELEMENT METHOD CLOSE TO THE EQUILIBRIUM MODEL APPLIED TO PLANE ELASTOSTATICS

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INTRODUCTION

In the present paper we derive a new variational formulation of the displacement boundary value problem in linear plane elastostatics, following the idea of [1], [2], where the Dirichlet problem for an elliptic differential equation has been treated.

The new variational principle for the problem under consideration is established on the basis of a non-classical splitting of the system of the differential operators and the Friedrichs transformation. The principle is justified by proving the existence, uniqueness and a continuous dependence of the solution of the variational problem on the given data. Then we show a possible application of the variational principle, establishing a mixed finite element model and deriving an a priori estimate of error. As a result, two components of the approximate vector-field converge to the real displacements and the third tends to the shear stress.

The new method seems to represent a model in between the compatible and mixed models. In fact, the new model has three unknowns (u_1, u_2, τ), whereas the compatible and mixed models have two (u_1, u_2) and five $(u_1, u_2, \sigma_x, \sigma_y, \tau)$ unknowns, respectively.

1. DERIVATION OF A VARIATIONAL PRINCIPLE FOR PLANE ELASTOSTATICS

Let us consider a bounded domain $\Omega \subset E_2$ with Lipschitz boundary Γ, occupied by a homogeneous elastic body and a Cartesian coordinate system $x = (x_1, x_2)$.

For simplicity, we assume the material to be isotropic, with the Lamé's constants $\lambda_0 > 0$, $\mu_0 > 0$. Denoting $v_{,i} = \partial v / \partial x_i$, the system of Lamé's equations (cf. e.g. [3]) can be written in the form

$$(1.1) \qquad (\lambda_0 + \mu_0)\, u_{j,ji} + \mu_0 u_{i,jj} + F_i = 0 \quad (i = 1, 2)\,,$$

where $\boldsymbol{u} = (u_1, u_2)$ is the displacement vector and F_i the body force components. Henceforth a repeated index implies summation over the range (1, 2). Let us consider the homogeneous boundary conditions

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma .$$

Moreover, let $\boldsymbol{F} \in [L_2(\Omega)]^2$. The weak solution of the problem under consideration is defined as an element $\boldsymbol{u} \in [W_0^{1,2}(\Omega)]^2$ such that

$$A(\boldsymbol{u}, \boldsymbol{v}) = (F_i, v_i) \quad \forall v_i \in W_0^{1,2}(\Omega) , \tag{1.2}$$

where $(\varphi, \psi) = \int_\Omega \varphi\psi \, \mathrm{d}x$,

$$A(\boldsymbol{u}, \boldsymbol{v}) = \lambda_0(u_{i,i}, v_{j,j}) + \tfrac{1}{2}\mu_0(u_{i,j} + u_{j,i} , \quad v_{i,j} + v_{j,i}) .$$

Note that (1.2) expresses the zero variation of the potential energy

$$\mathscr{L}(\boldsymbol{u}) = \tfrac{1}{2}A(\boldsymbol{u}, \boldsymbol{u}) - (F_i, u_i)$$

on $[W_0^{1,2}(\Omega)]^2$. Setting

$$\mathscr{N}_1 = N_1^0(\boldsymbol{u}) , \quad \mathscr{N}_2 = N_2^0(\boldsymbol{u}) , \quad \mathscr{N}_3 = N_3^0(\boldsymbol{u}) ,$$
$$N_j^0(\boldsymbol{u}) = u_{j,j} \quad (j = 1, 2, \text{ no sum}) ,$$
$$N_3^0(\boldsymbol{u}) = u_{1,2} + u_{2,1}$$

and applying the Friedrichs transformation[1]) to the problem $\mathscr{L}(\boldsymbol{u}) = \mathscr{L}(\boldsymbol{u}, \mathscr{N}_j) = $ $=$ min., we obtain the dual variational formulation, namely the so called principle of minimum complementary energy (Castigliano). Because of some difficulties in the construction of admissible stress fields in the above principle (cf. e.g. [4]), we choose here a different approach.

Let us set

$$\mathscr{N}_k = N_k(\boldsymbol{u}) = N_k^0(\boldsymbol{u}) + \alpha_{kj}u_j , \quad (k = 1, 2, 3) \tag{1.4}$$
$$\mathscr{N}_4 = N_4(\boldsymbol{u}) = \beta_j u_j ,$$
$$\mathscr{N}_5 = N_5(\boldsymbol{u}) = \gamma_j u_j ,$$

where α_{kj}, β_j, γ_j are constants. Define[2])

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[1]) See e.g. [6], chpt. IV. § 9.

[2]) Note that $\boldsymbol{k}$ is the matrix of the stress-strain relations:

$$\boldsymbol{\tau} = \boldsymbol{k}\boldsymbol{\varepsilon},$$

where

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{12} \end{bmatrix} , \quad \boldsymbol{\varepsilon} = \begin{bmatrix} N_1^0(\boldsymbol{u}) \\ N_2^0(\boldsymbol{u}) \\ N_3^0(\boldsymbol{u}) \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} ,$$

$\boldsymbol{\tau}$, $\boldsymbol{\varepsilon}$ being the stress and strain tensor, respectively.

$$(1.5) \qquad \boldsymbol{k} = \begin{bmatrix} \lambda_0 + 2\mu_0, & \lambda_0, & 0 \\ \lambda_0, & \lambda_0 + 2\mu_0, & 0 \\ 0, & 0, & \mu_0 \end{bmatrix},$$

$$K(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \{ \sum_{i,j=1}^{3} k_{ij} N_i(\boldsymbol{u}) N_j(\boldsymbol{v}) + K_4 N_4(\boldsymbol{u}) N_4(\boldsymbol{v}) + K_5 N_5(\boldsymbol{u}) N_5(\boldsymbol{v})\} \, \mathrm{d}x.$$

Let us choose such constants α_{kj}, β_j, γ_j, K_4, K_5, that

$$(1.6) \qquad A(\boldsymbol{u}, \boldsymbol{v}) = K(\boldsymbol{u}, \boldsymbol{v}) \quad \forall \boldsymbol{u}, \boldsymbol{v} \in [W_0^{1,2}(\Omega)]^2 .$$

It is easy to find that (1.6) is satisfied if

$$(1.7) \qquad \alpha_{3j} = 0 \quad (j = 1, 2)$$

$$(1.8) \qquad \boldsymbol{k}^0 \boldsymbol{\alpha} = \boldsymbol{d} ,$$

where

$$\boldsymbol{k}^0 = \begin{bmatrix} \lambda_0 + 2\mu_0, & \lambda_0 \\ \lambda_0, & \lambda_0 + 2\mu_0 \end{bmatrix}, \quad \boldsymbol{\alpha} = \begin{bmatrix} \alpha_{11}, & \alpha_{12} \\ \alpha_{21}, & \alpha_{22} \end{bmatrix}$$

and $\boldsymbol{d}$ is a diagonal matrix,

$$(1.9) \qquad \boldsymbol{\alpha}^T \boldsymbol{k}^0 \boldsymbol{\alpha} + K_4 \boldsymbol{\beta}\boldsymbol{\beta}^T + K_5 \gamma\gamma^T = 0 ,$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2)^T$, $\gamma = (\gamma_1, \gamma_2)^T$.

In fact, using (1.3), we may write

$$(1.10) \qquad A(\boldsymbol{u}, \boldsymbol{v}) = \sum_{i,j=1}^{3} (k_{ij} N_i(\boldsymbol{u}), N_j(\boldsymbol{v}))$$

and inserting (1.7)–(1.10) into (1.5), we obtain

$$K(\boldsymbol{u}, \boldsymbol{v}) = A(\boldsymbol{u}, \boldsymbol{v}) + \int_\Omega \{k_{ij}^0 (N_i^0(\boldsymbol{u}) \, \alpha_{jm} v_m + \alpha_{ik} u_k \, N_j(\boldsymbol{v})) +$$

$$+ (k_{ij}^0 \alpha_{ik} \alpha_{jm} + K_4 \beta_k \beta_m + K_5 \gamma_k \gamma_m) \, u_k v_m\} \, \mathrm{d}x .$$

By virtue of the symmetry of $\boldsymbol{k}^0$, (1.8) and (1.9), the last integral reduces to

$$\int_\Omega \sum_{i=1}^{2} d_{ii} (N_i^0(\boldsymbol{u}) \, v_i + u_i \, N_i^0(\boldsymbol{v})) \, \mathrm{d}x = \sum_{i=1}^{2} d_{ii} \int_\Omega (u_{i,i} v_i + u_i v_{i,i}) \, \mathrm{d}x =$$

$$= \sum_{i=1}^{2} d_{ii} \int_\Gamma u_i v_i \nu_i \, \mathrm{d}\Gamma = 0$$

for $v_i \in W_0^{1,2}(\Omega)$, consequently (1.6) holds.

Let us apply the Friedrichs transformation to the problem

$$\mathscr{L}(\boldsymbol{u}) = \tfrac{1}{2}K(\boldsymbol{u}, \boldsymbol{u}) - (F_i, u_i) = \min., \quad \boldsymbol{u} \in [W_0^{1,2}(\Omega)]^2 .$$

Thus we obtain

$$\mathscr{F}(\lambda_j, \mu_i, u_i, \mathscr{N}_j) = \tfrac{1}{2}\int_\Omega \Big(\sum_{i,j=1}^{3} k_{ij}\mathscr{N}_i\mathscr{N}_j + K_4\mathscr{N}_4^2 + K_5\mathscr{N}_5^2 - F_j u_j\Big)\,\mathrm{d}x +$$

$$+ \sum_{j=1}^{5}\int_\Omega \lambda_j(N_j(\boldsymbol{u}) - \mathscr{N}_j)\,\mathrm{d}x + \int_\Gamma \mu_j u_j \,\mathrm{d}\Gamma .$$

Define the subscript transformation $j \to (j+1)$ as follows:

$$(j+1) = 2 \quad \text{for} \quad j = 1, (j+1) = 1 \quad \text{for} \quad j = 2 .$$

Then the integration by parts yields

$$\sum_{j=1}^{5}(\lambda_j, N_j(\boldsymbol{u})) = -\sum_{j=1}^{2}(\lambda_{j,j} + \lambda_{3,j+1} - \alpha_{ij}\lambda_i - \lambda_4\beta_j - \lambda_5\gamma_j, u_j) +$$

$$+ \int_\Gamma \sum_{j=1}^{2}(\lambda_j\nu_j + \lambda_3\nu_{j+1})\,u_j\,\mathrm{d}\Gamma .$$

Denote

$a_j(\lambda) = \lambda_{j,j} + \lambda_{3,j+1} - \alpha_{ij}\lambda_i$ ($j = 1, 2,$ no sum over j) where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$.
Then

$$\mathscr{F} = \tfrac{1}{2}\int_\Omega \Big(\sum_{i,j=1}^{3} k_{ij}\mathscr{N}_i\mathscr{N}_j + K_4\mathscr{N}_4^2 + K_5\mathscr{N}_5^2\Big)\,\mathrm{d}x - \int_\Omega \Big\{\sum_{j=1}^{2}(a_j(\lambda) + F_j - \lambda_4\beta_j -$$

$$- \lambda_5\gamma_j)\,u_j - \sum_{j=1}^{5}\lambda_j\mathscr{N}_j\Big\}\,\mathrm{d}x + \int_\Gamma \sum_{j=1}^{2}(\mu_j + \lambda_j\nu_j + \lambda_3\nu_{j+1})\,u_j\,\mathrm{d}\Gamma .$$

The variations of $\mathscr{F}$ with respect to $\mathscr{N}_j$ and u_j read

$$\delta_{\mathscr{N}_j}\mathscr{F} = \Big(\sum_{i=1}^{3} k_{ij}\mathscr{N}_i - \lambda_j, \delta\mathscr{N}_j\Big) \quad (j = 1, 2, 3) \tag{1.11}$$

$$\delta_{\mathscr{N}_j}\mathscr{F} = (K_j\mathscr{N}_j - \lambda_j, \delta\mathscr{N}_j) \quad (j = 4, 5, \text{no sum})$$

$$\delta_{u_j}\mathscr{F} = -(a_j(\lambda) + F_j - \lambda_4\beta_j - \lambda_5\gamma_j, \delta u_j) +$$

$$+ \int_\Gamma (\mu_j + \lambda_j\nu_j + \lambda_3\nu_{j+1})\,\delta u_j\,\mathrm{d}\Gamma .$$

Following the Friedrichs transformation, we have to eliminate $\lambda_4, \lambda_5, \boldsymbol{u}, \mu$ from $\mathscr{F}$ by means of the zero variations of (1.11). To this end we assume

$$\beta_1\gamma_2 - \beta_2\gamma_1 \neq 0 , \quad K_4K_5 \neq 0 \tag{1.12}$$

which enables to find λ_4, λ_5 from the two equations

$$\beta_j\lambda_4 + \gamma_j\lambda_5 = a_j(\lambda) + F_j \quad (j = 1, 2)\,. \tag{1.13}$$

Moreover, we set

$$\mu_j + \lambda_j\nu_j + \lambda_3\nu_{j+1} = 0 \quad \text{on} \quad \Gamma\,, \quad (j = 1, 2) \tag{1.14}$$

$$\lambda_j = \sum_{i=1}^{3} k_{ij}\mathcal{N}_i \quad (j = 1, 2, 3)$$

$$\lambda_j = K_j\mathcal{N}_j \quad (j = 4,5)\,(\text{no sum})\,.$$

Substituting for $\mathcal{N}_j$, λ_4, λ_5, μ_j from (1.13), (1.14) into $\mathcal{F}$, we are led to the functional

$$\mathcal{S}(\lambda) = \tfrac{1}{2}\int_\Omega \{\sum_{i,j=1}^{3} k_{ij}^{-1}\lambda_i\lambda_j + K_4^{-1}[\sum_{j=1}^{2} C_{1j}^{-1}(a_j(\lambda) + F_j)]^2 +$$

$$+ K_5^{-1}[\sum_{j=1}^{2} C_{2j}(a_j(\lambda) + F_j)]^2\}\,\mathrm{d}x\,,$$

where k_{ij}^{-1} and C_{ij}^{-1} are entries of the matrices inverse to $\boldsymbol{k}$ and

$$\boldsymbol{C} = \begin{bmatrix} \beta_1, & \gamma_1 \\ \beta_2, & \gamma_2 \end{bmatrix},$$

respectively.

From (1.8), (1.9) we obtain e.g.

$$d_{11}^2k_{11}^{-1} + K_4\beta_1^2 + K_5\gamma_1^2 = 0\,. \tag{1.15}$$

As $k_{11}^{-1} > 0$, (1.15) and (1.12) imply that at least one of K_4, K_5 must be negative.

Obviously, the conditions (1.8), (1.9) and (1.12) do not suffice to determine the parameters α, β, γ, K_4, K_5 uniquely. Let us choose

$$K_4^{-1} = K_5 < 0\,, \tag{1.16}$$

$$\boldsymbol{\alpha} = \begin{bmatrix} \xi_1\left(\dfrac{k_{22}^0}{|k^0|}\right)^{1/2}, & \xi_2 = \dfrac{k_{12}^0}{(k_{11}^0|k^0|)^{1/2}} \\ -\xi_1\dfrac{k_{12}^0}{(k_{22}^0|k^0|)^{1/2}}, & -\xi_2\left(\dfrac{k_{11}^0}{|k^0|}\right)^{1/2} \end{bmatrix},$$

where ξ_1, ξ_2 are arbitrary non-zero real numbers, $|k^0|$ denotes the determinant of $\boldsymbol{k}^0$. By an easy calculation we derive

$$\boldsymbol{\alpha}^T\boldsymbol{k}^0\boldsymbol{\alpha} = \begin{bmatrix} \xi_1^2, & -\mathcal{A}\xi_1\xi_2 \\ -\mathcal{A}\xi_1\xi_2, & \xi_2^2 \end{bmatrix}, \tag{1.17}$$

where

$$\mathscr{A} = -\frac{k_{12}^0}{(k_{11}^0 k_{22}^0)^{1/2}} = -\frac{\lambda_0}{\lambda_0 + 2\mu_0}.$$

Calculating the variation of $\mathscr{S}(\lambda)$ and making use of (1.16), (1.17), the following statement is justified: the stationary value of $\mathscr{S}(\lambda)$ is characterized by the condition

(1.18)

$$B(\lambda, \mu) \equiv \sum_{i,j=1}^{3} (k_{ij}^{-1}\lambda_i, \mu_j) - (1 - \mathscr{A}^2)^{-1} \sum_{j=1}^{2} \left((\xi_j^{-2} a_j(\lambda) + \frac{\mathscr{A}}{\xi_1 \xi_2} a_{j+1}(\lambda), \quad a_j(\mu) \right) =$$

$$= (1 - \mathscr{A}^2)^{-1} \sum_{j=1}^{2} \left(\xi_j^{-2} F_j + \frac{\mathscr{A}}{\xi_1 \xi_2} F_{j+1}, \quad a_j(\mu) \right)$$

for all $\mu \in [L_2(\Omega)]^3$ such that $a_j(\mu) \in L_2(\Omega)$, $j = 1, 2$.

We may expect that a vector λ, satisfying (1.18), will be related to the solution $\boldsymbol{u}$ of the primal problem (1.1) according to (1.14) and (1.4), i.e.

$$\lambda_i = \sum_{j=1}^{3} k_{ij} N_j(\boldsymbol{u}), \quad (i = 1, 2, 3). \tag{1.19}$$

Really, in the next section, we shall justify the variational formulation (1.18), using the expected relations.

Remark 1.1 The Euler's equations, associated with the variational problem (1.18), are (1.19), where $\boldsymbol{u} = \boldsymbol{u}(\lambda)$,

$$u_j(\lambda) = -(1 - \mathscr{A}^2)^{-1} \left[\xi_j^{-2}(a_j(\lambda) + F_j) + \frac{\mathscr{A}}{\xi_1 \xi_2} (a_{j+1}(\lambda) + F_{j+1}) \right],$$

$$(j = 1, 2, \text{no sum}).$$

These relations can be interpreted as equations of compatibility for quasi-deformations $\mathscr{N}_j$ or quasi-stresses λ_j, respectively.

2. CORRECTNESS OF THE NEW VARIATIONAL FORMULATION

In order to study the variational principle (1.18), we introduce the following

Definition 2.1. *Let the linear space*

$$\mathscr{H} = \{\lambda \in [L_2(\Omega)]^3, \quad a_j(\lambda) \in L_2(\Omega), \quad j = 1, 2\}$$

(*where the differential operators in* $a_j(\lambda)$ *are taken in the sense of distributions*), *be furnished with the norm* [1])

$$\|\lambda\|_{\mathscr{H}} = [\sum_{i=1}^{3} \|\lambda_i\|^2 + \sum_{j=1}^{2} \|a_j(\lambda)\|^2]^{\frac{1}{2}} .$$

Let the bilinear form $B(\lambda, \mu)$ be defined on $\mathscr{H} \times \mathscr{H}$ by means of (1.18), (1.16). *The form $B(\lambda, \mu)$ is obviously symmetric and continuous on $\mathscr{H} \times \mathscr{H}$.*

Theorem 2.1. *The variational problem to find $\hat{\lambda} \in \mathscr{H}$ such that*

$$B(\hat{\lambda}, \mu) = (1 - \mathscr{A}^2)^{-1} \sum_{j=1}^{2} \left(\xi_j^{-2} F_j + \frac{\mathscr{A}}{\xi_1 \xi_2} F_{j+1}, a_j(\mu)\right) \quad \forall \mu \in \mathscr{H} \tag{2.1}$$

has a unique solution $\hat{\lambda}$ in $\mathscr{H}$.

The solution $\hat{\lambda}$ is related to the solution **u** *as follows:*

$$\hat{\lambda}_i = k_{ij}^0(N_j^0(\boldsymbol{u}) + \alpha_{jk} u_k) , \quad (i = 1, 2) , \tag{2.2}$$

$$\hat{\lambda}_3 = \mu_0 N_3^0(\boldsymbol{u}) ,$$

$$u_j = -(1 - \mathscr{A}^2)^{-1} \left[\xi_j^{-2}(a_j(\hat{\lambda}) + F_j) + \frac{\mathscr{A}}{\xi_1 \xi_2} (a_{j+1}(\hat{\lambda}) + F_{j+1})\right], \quad (j = 1, 2) \tag{2.2'}$$

Moreover, it holds

$$\|\hat{\lambda}\|_{\mathscr{H}} \leqq C \sum_{i=1}^{2} \|F_i\| . \tag{2.3}$$

Proof. Existence. We can show that the vector $\hat{\lambda}$, defined in (2.2), belongs to $\mathscr{H}$ and satisfies (2.1). In fact, we may write

$$\sum_{i=1}^{3} k_{ji}^{-1} \hat{\lambda}_i = N_j^0(\boldsymbol{u}) + \alpha_{jk} u_k \quad (j = 1, 2, 3) , \tag{2.4}$$

$$\begin{aligned} a_j(\hat{\lambda}) = \hat{\lambda}_{j,j} + \hat{\lambda}_{3,j+1} - \alpha_{ij}\hat{\lambda}_i &= k_{js}^0 (N_s^0(\boldsymbol{u})_{,j} + \alpha_{sk} u_{k,j}) + \\ &+ \mu_0 N_3^0(\boldsymbol{u})_{,j+1} - \alpha_{ij} k_{is}^0(N_s(\boldsymbol{u}) + \alpha_{sk} u_k) = \\ &= -F_j + d_{jj} u_{j,j} - d_{jj} N_j^0(\boldsymbol{u}) - (\boldsymbol{\alpha}^T \mathbf{k} \boldsymbol{\alpha})_{jk} u_k = \\ &= -F_j - \xi_j^2 u_j + \mathscr{A} \xi_1 \xi_2 u_{j+1} \quad (j = 1, 2, \text{no sum over } j) , \end{aligned} \tag{2.5}$$

where we used (1.8), (1.17) and the Lamé's equations (1.1) in the form

$$k_{js}^0 N_s^0(\boldsymbol{u})_{,j} + \mu_0 N_3^0(\boldsymbol{u})_{,j+1} + F_j = 0 , \quad (j = 1, 2, \text{no sum}) .$$

[1]) Henceforth $\|w\|_k$ and $\|w\|$ denote the usual norms in $W^{k,2}(\Omega)$ and in $L_2(\Omega)$, respectively.

Inserting (2.4), (2.5) into (1.18), we obtain

$$(2.6)\quad B(\hat{\lambda}, \mu) = (1 - \mathscr{A}^2)^{-1} \sum_{j=1}^{2} \left((\xi_j^{-2} F_j + \frac{\mathscr{A}}{\xi_1 \xi_2} F_{j+1}, a_j(\mu) \right) + (u_j, a_j(\mu)) + \\ + \sum_{j=1}^{3} (N_j^0(\mathbf{u}), \mu_j) + (\alpha_{jk} u_k, \mu_j) .$$

Integration by parts yields

$$(2.7)\qquad (u_j, a_j(\mu)) = \sum_{j=1}^{2} (u_j, \mu_{j,j} + \mu_{3,j+1} - \alpha_{kj}\mu_k) = - \sum_{j=1}^{2} (u_{j,j}, \mu_j) - \\ - (u_{1,2} + u_{2,1}, \mu_3) - (\alpha_{kj} u_j, \mu_k) = - \sum_{j=1}^{3} (N_j^0(\mathbf{u}), \mu_j) - (\alpha_{jk} u_k, \mu_j) .$$

Finally, from (2.6) and (2.7) it follows that $\hat{\lambda}$ is a solution of (2.1).

Uniqueness. Let λ', λ'' be two solutions of (2.1) in $\mathscr{H}$. By subtraction we find that

$$(2.8)\qquad B(\tilde{\lambda}, \mu) = 0 \quad \forall \mu \in \mathscr{H} ,$$

where $\tilde{\lambda} = \lambda' - \lambda'' \in \mathscr{H}$.

Let us consider the solution $\mathbf{w}(\tilde{\lambda}) \in [W^{1,2}(\Omega)]^2$ of the problem (1.1) for the body forces $F_i = a_i(\tilde{\lambda})$, $(i = 1, 2)$ and the corresponding vector $\lambda(\tilde{\lambda}) \in \mathscr{H}$ with the components

$$\lambda_i(\tilde{\lambda}) = \sum_{j=1}^{3} k_{ij}(N_j^0(\mathbf{w}(\tilde{\lambda})) + \alpha_{jk}\, \mathbf{w}(\tilde{\lambda})_k) \quad (i = 1, 2, 3) .$$

From the proof of existence we conclude that

$$B(\lambda(\tilde{\lambda}), \mu) = (1 - \mathscr{A}^2)^{-1} \sum_{j=1}^{2} \left(\xi_j^{-2}\, a_j(\tilde{\lambda}) + \frac{\mathscr{A}}{\xi_1 \xi_2}\, a_{j+1}(\tilde{\lambda}), a_j(\mu) \right) \quad \forall \mu \in \mathscr{H} .$$

Inserting $\mu = \tilde{\lambda}$ and using (2.8), we obtain

$$0 = B(\lambda(\tilde{\lambda}), \tilde{\lambda}) = (1 - \mathscr{A}^2)^{-1} \left[\xi_j^{-2} \| a_j(\tilde{\lambda}) \|^2 + 2 \frac{\mathscr{A}}{\xi_1 \xi_2} (a_1(\tilde{\lambda}), a_2(\tilde{\lambda})) \right] \geqq \\ \geqq \frac{1 - |\mathscr{A}|}{1 - \mathscr{A}^2}\, \xi_j^{-2} \| a_j(\tilde{\lambda}) \|^2 ,$$

consequently $a_j(\tilde{\lambda}) = 0$, $(j = 1, 2)$.

Therefore inserting $\mu = \tilde{\lambda}$ into (2.8) and using the latter result, we derive

$$0 = B(\tilde{\lambda}, \tilde{\lambda}) = \sum_{i,j=1}^{3} (k_{ij}^{-1} \tilde{\lambda}_i, \tilde{\lambda}_j) \Rightarrow \tilde{\lambda} = \Theta .$$

The formulae (2.2') follow from (2.5), (2.6).

The estimate (2.3) is a consequence of (2.2), (2.5) and of the well-known a priori estimate

$$\sum_{i=1}^{2} \|u_i\|_1^2 \leqq C \sum_{j=1}^{2} \|F_i\|^2 .$$

3. A MIXED FINITE ELEMENT MODEL AND ITS CONVERGENCE

Following the approach of Section 3 of [1], we set

$$\xi_1 = \xi_2 = \xi = \xi_0 h^{-1-\varepsilon}, \tag{3.1}$$

where

$$0 < h \leqq 1 , \quad \xi_0 > 0 , \quad \varepsilon > 0$$

are parameters and transform λ_1, λ_2 into

$$\bar{\lambda}_j = d_{jj}^{-1} \lambda_j \quad (j = 1, 2, \text{no sum}) \tag{3.2}$$

where

$$d_{jj} = \sum_{s=1}^{2} k_{js}^0 \alpha_{sj} = \xi d_j^0 , \quad d_1^0 = \left(\frac{|k^0|}{k_{11}^0} \right)^{1/2} , \quad d_2^0 = -d_1^0 . \tag{3.3}$$

Substituting (3.2) into (1.18), introducing

$$\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \lambda_3) , \quad \bar{\mu} = (\bar{\mu}_1, \bar{\mu}_2, \mu_3)$$

and using (3.3), (1.17), we have

$$\begin{aligned} a_j(\lambda) = \bar{a}_j(\bar{\lambda}) &= d_{jj}\bar{\lambda}_{j,j} + \lambda_{3,j+1} - \sum_{i=1}^{2} \alpha_{ij} d_{ii} \bar{\lambda}_i = \\ &= d_{jj}\bar{\lambda}_{j,j} + \lambda_{3,j+1} - (\boldsymbol{\alpha}^T \mathbf{k}^0 \boldsymbol{\alpha})_{ji} \, \bar{\lambda}_i = \\ &= \xi d_j^0 \bar{\lambda}_{j,j} + \lambda_{3,j+1} - \xi^2 \bar{\lambda}_j + \mathscr{A} \xi^2 \bar{\lambda}_{j+1} \quad (j = 1, 2, \text{no sum}) , \end{aligned} \tag{3.4}$$

$$\begin{aligned} B(\lambda, \mu) = \bar{B}(\bar{\lambda}, \bar{\mu}) &= \sum_{i,j=1}^{2} \xi^2 (k_{ij}^0 d_i^0 \bar{\lambda}_i, d_j^0 \bar{\lambda}_j) + \mu_0^{-1} (\lambda_3, \mu_3) - \\ &- \sum_{j=1}^{2} (1 - \mathscr{A}^2)^{-1} \xi^{-2} (\xi d_j^0 \bar{\lambda}_{j,j} + \lambda_{3,j+1} - \xi^2 \bar{\lambda}_j + \mathscr{A} \xi^2 \bar{\lambda}_{j+1} + \\ &+ \mathscr{A} (\xi d_{j+1}^0 \bar{\lambda}_{j+1,j+1} + \lambda_{3,j} - \xi^2 \bar{\lambda}_{j+1} + \mathscr{A} \xi^2 \bar{\lambda}_j) , \\ &\xi d_j^0 \bar{u}_{j,j} + \mu_{3,j+1} - \xi^2 \bar{\mu}_j + \mathscr{A} \xi^2 \bar{\mu}_{j+1}) . \end{aligned}$$

After some calculations, we come to the formulae

$$(3.5)\qquad \bar{B}(\bar{\lambda}, \bar{\mu}) = \sum_{i,j=1}^{2} \mathscr{A}_{ij}(\bar{\lambda}_i, \bar{\mu}_j) + \sum_{j=1}^{2} \mathscr{A}_{3j}(\lambda_3, \bar{\mu}_j) + \sum_{j=1}^{2} \mathscr{A}_{j3}(\bar{\lambda}_j, \mu_3) + \mathscr{A}_{33}(\lambda_3, \mu_3),$$

where

$$(3.6)\qquad \mathscr{A}_{jj}(\bar{\lambda}_j, \bar{u}_j) = -(1 - \mathscr{A}^2)^{-1} (d_j^0)^2 (\bar{\lambda}_{j,j}, \bar{\mu}_{j,j}) + \xi d_j^0 [(\bar{\lambda}_j, \bar{\mu}_{j,j}) + (\bar{\lambda}_{j,j}, \bar{\mu}_j), \quad (j = 1, 2, \text{no sum}),$$

$$\mathscr{A}_{12}(\bar{\lambda}_1, \bar{\mu}_2) = -\mathscr{A}(1 - \mathscr{A}^2)^{-1} d_1^0 d_2^0 (\bar{\lambda}_{1,1}, \bar{\mu}_{2,2}),$$

$$\mathscr{A}_{21}(\bar{\lambda}_2, \bar{\mu}_1) = \mathscr{A}_{12}(\bar{\mu}_1, \bar{\lambda}_2)$$

$$\mathscr{A}_{3j}(\lambda_3, \bar{\mu}_j) = -(1 - \mathscr{A}^2)^{-1} \xi^{-1} \{\mathscr{A} d_j^0 (\lambda_{3,j}, \bar{\mu}_{j,j}) + (\lambda_{3,j+1}, d_j^0 \bar{\mu}_{j,j} - \xi(1 - \mathscr{A}^2)\bar{\mu}_j), \quad (j = 1, 2, \text{no sum}),$$

$$\mathscr{A}_{j3}(\bar{\lambda}_j, \bar{\mu}_3) = \mathscr{A}_{3j}(\bar{\mu}_3, \bar{\lambda}_j).$$

$$\mathscr{A}_{33}(\lambda_3, \mu_3) = \mu_0^{-1}(\lambda_3, \mu_3) - (1 - \mathscr{A}^2)^{-1} \xi^{-2} \sum_{j=1}^{2} (\lambda_{3,j+1} + \mathscr{A}\lambda_{3,j}, \mu_{3,j+1}).$$

Let us define another bilinear form

$$(3.7)\qquad \tilde{B}(\bar{\lambda}, \bar{\mu}) = B(\bar{\lambda}_1, \bar{\lambda}_2, \lambda_3; -\bar{\mu}_1, -\bar{\mu}_2, \mu_3) = - \sum_{i,j=1}^{2} \mathscr{A}_{ij}(\bar{\lambda}_i, \bar{\mu}_j) - \sum_{j=1}^{2} \mathscr{A}_{3j}(\lambda_3, \bar{\mu}_j) + \sum_{j=1}^{2} \mathscr{A}_{j3}(\bar{\lambda}_j, \mu_3) + \mathscr{A}_{33}(\lambda_3, \mu_3).$$

Defining

$$(3.8)\qquad \mathscr{H}_0 = \{\bar{\lambda} \in [L_2(\Omega)]^3, \xi d_j^0 \bar{\lambda}_{j,j} + \lambda_{3,j+1} \in L_2(\Omega), j = 1, 2\},$$

$$\mathscr{H}_0^- = \{\bar{\lambda} \in [L_2(\Omega)]^3, -\xi d_j^0 \bar{\lambda}_{j,j} + \lambda_{3,j+1} \in L_2(\Omega), j = 1, 2\},$$

it is easy to see that the problem (2.1) is equivalent with the problem to find $\bar{\lambda} \in \mathscr{H}_0$ such that

$$(3.9)\qquad \tilde{B}(\bar{\lambda}, \bar{\mu}) = (1 - \mathscr{A}^2) \xi^{-2} \sum_{j=1}^{2} (F_j + \mathscr{A} F_{j+1}, -\xi d_j^0 \bar{\mu}_{j,j} + \mu_{3,j+1} + \xi^2 \bar{\mu}_j - \mathscr{A}\xi^2 \bar{\mu}_{j+1}) \quad \forall \bar{\mu} \in \mathscr{H}_0^-.$$

To define a *Galerkin procedure*, we shall *assume* that two families of finite-definite subspaces V_h, V_{h_0} exist[1]) such that for any $0 < h \leqq 1$, $0 < h_0 \leqq 1$

(i) $\qquad V_h \subset W^{1,2}(\Omega), \quad V_{h_0} \in W_0^{1,2}(\Omega);$

[1]) All standard finite element spaces satisfy (i)—(iii).

(ii) (*Approximability*) integers $\varkappa \geqq 2$, $\varkappa_0 \geqq 2$ and a constant C exist such that

$$\forall v \in W^{\varkappa,2}(\Omega)\, \exists \chi \in V_h : \|v - \chi\|_i \leqq C h^{\varkappa - i} \|v\|_\varkappa \quad (i = 0, 1),$$

$$\forall w \in W^{\varkappa_0,2}(\Omega) \cap W_0^{1,2}(\Omega)\, \exists \psi \in V_h : \|w - \psi\|_i \leqq C h^{\varkappa_0 - i} \|w\|_{\varkappa_0}$$

$$(i = 0, 1);$$

(iii) (*Inverse inequality*) a constant C exists such that

$$\|\chi\|_1 \leqq C h^{-1} \|\chi\|$$

holds for any $\chi \in V_h$ and sufficiently small h.

Denoting $V(h_0, h) = [V_{h_0}]^2 \times V_h$,
we deduce from (i) that

$$V(h_0, h) \subset [W^{1,2}(\Omega)]^3 \subset \mathscr{H}_0\,, \quad V(h_0, h) \subset \mathscr{H}_0^-\,.$$

Next we shall prove the following

Lemma 1. *For any $\bar{\lambda} \in V(h_0, h)$ and sufficiently small h*

$$\tilde{B}(\bar{\lambda}, \bar{\lambda}) \geqq C\Big(\sum_1^2 \|\bar{\lambda}_{j,j}\| + \|\lambda_3\|\Big)^2 \tag{3.10}$$

holds, where C is independent of h, h_0 and $\bar{\lambda}$.

Proof. From (3.6) it follows that

$$\mathscr{A}_{j3}(\bar{\lambda}_j, \lambda_3) = \mathscr{A}_{3j}(\lambda_3, \bar{\lambda}_j) \quad (j = 1, 2)\,.$$

Further we have

$$\mathscr{A}_{jj}(\bar{\lambda}_j, \bar{\lambda}_j) = -(1 - \mathscr{A}^2)^{-1} (d_j^0)^2 \|\bar{\lambda}_{j,j}\|^2 \quad (j = 1, 2)\,,$$

because

$$2(\bar{\lambda}_j, \bar{\lambda}_{j,j}) = \int_\Omega (\bar{\lambda}_j^2)_{,j}\, \mathrm{d}x = \int_\Gamma \bar{\lambda}_j^2 v_j\, \mathrm{d}\Gamma = 0 \quad \forall \bar{\lambda}_j \in V_{h_0}\,.$$

Altogether we may write

$$\begin{aligned}(3.11)\ \tilde{B}(\bar{\lambda}, \bar{\lambda}) &= (1 - \mathscr{A}^2)^{-1} (d_1^0)^2 \sum_1^2 \|\bar{\lambda}_{j,j}\|^2 + 2\mathscr{A}(1 - \mathscr{A}^2)^{-1} d_1^0 d_2^0 (\bar{\lambda}_{1,1}, \bar{\lambda}_{2,2}) + \\ &+ \mu_0^{-1} \|\lambda_3\|^2 - (1 - \mathscr{A}^2)^{-1} \xi^{-2} (\|\lambda_{3,1}\|^2 + \|\lambda_{3,2}\|^2 + 2\mathscr{A}(\lambda_{3,1}, \lambda_{3,2})) \geqq \\ &\geqq (1 - \mathscr{A})^{-1} (d_1^0) \sum_1^2 \|\bar{\lambda}_{j,j}\|^2 + \mu_0^{-1} \|\lambda_3\|^2 - (1 - \mathscr{A})^{-1} \xi^{-2} \sum_{j=1}^2 \|\lambda_{3,j}\|^2\,.\end{aligned}$$

The inverse inequality (iii) yields

$$\begin{aligned}(3.12)\qquad & \mu_0^{-1} \|\lambda_3\|^2 - (1 - \mathscr{A})^{-1} \xi_0^{-2} h^{2+2\varepsilon} \sum_{j=1}^2 \|\lambda_{3,j}\|^2 \geqq \\ &\geqq \|\lambda_3\|^2 (\mu_0^{-1} - (1 - \mathscr{A})^{-1} \xi_0^{-2} C h^{2\varepsilon}) \geqq C \|\lambda_3\|^2\end{aligned}$$

for sufficiently small h. Inserting (3.12) into (3.11) we obtain (3.10).

Lemma 2. *Let us introduce the norms*

$$\|\mu\|_* = \|\mu_3\| + h\sum_{j=1}^{2}\|\mu_{3,j}\| + \sum_{j=1}^{2}\|\bar{\mu}_{j,j}\|,$$

$$\|\bar{\lambda}\|_h = \|\lambda_3\| + h^{-1}\sum_{j=1}^{2}\|\bar{\lambda}_j\| + \sum_{j=1}^{2}(\|\lambda_{3,j}\| + \|\bar{\lambda}_{j,j}\|).$$

Then it holds

$$|\tilde{B}(\bar{\lambda}, \bar{\mu})| \leqq C\|\bar{\lambda}\|_h \|\bar{\mu}\|_* \quad \forall\bar{\mu} \in V(h_0, h),\ \bar{\lambda} \in [W^{1,2}(\Omega)]^3 \tag{3.13}$$

where C is independent of $\bar{\lambda}$, $\bar{\mu}$, h, h_0.

Proof. The inequality (3.13) can be proved for each term of (3.7) separately, inserting (3.1) for ξ and using also the equality

$$(\bar{\lambda}_j, \bar{\mu}_{j,j}) + (\bar{\lambda}_{j,j}, \bar{\mu}_j) = 0 \quad \forall\bar{\mu}_j \in V_{h_0}.$$

For the estimate for $\mathscr{A}_{3j}(\lambda_3, \bar{\mu}_j)$ we make use of the Friedrichs inequality

$$\|v\| \leqq C\|v_{,j}\| \quad \forall v \in W_0^{1,2}(\Omega) \quad (j = 1, 2).$$

Definition 3.1. *We say that $\bar{\lambda}^h \in V(h_0, h)$ is a Galerkin approximation to the solution $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \lambda_3)$ of (3.9) if*

$$\tilde{B}(\bar{\lambda}^h, \bar{\mu}) = (1 - \mathscr{A}^2)^{-1}\xi^{-2}(F_j + \mathscr{A}F_{j+1}, \bar{a}_j(-\bar{\mu}_1, -\bar{\mu}_2, \mu_3)\ \forall\bar{\mu} \in V(h_0, h). \tag{3.14}$$

Theorem 3.1. *Let the solution **u** of (1.1) belong to $[W^{m,2}(\Omega)]^2$, where $m \geqq \max(\varkappa + 1, \varkappa_0)$.*

Then, for sufficiently small h, the Galerkin approximation $\bar{\lambda}^h$ is determined uniquely and it holds

$$\sum_{i=1}^{2}\|\bar{\lambda}_i^h - u_i\| + \|\lambda_3^h - \mu_0 N_3^0(\boldsymbol{u})\| + \sum_{i=1}^{2}\|\sum_{j=1}^{2}(k_{ij}\bar{\lambda}_{j,j}^h - k_{ij}N_j^0(\boldsymbol{u}))\| \leqq \tag{3.15}$$

$$\leqq C(h^{\varkappa-1} + h^{-1}h_0^{\varkappa_0} + h_0^{\varkappa_0-1} + h^{\varepsilon})\sum_{i=1}^{2}\|u_i\|_m.$$

Proof. Note that (3.14) is equivalent with (3.9), where $\mathscr{H}_0$ and $\mathscr{H}_0^-$ is replaced by $V(h_0, h)$. Then Lemma 1 and the Friedrichs inequality imply the existence and uniqueness of $\bar{\lambda}^h$.

By virtue of the equivalence of (3.9) with (2.1), the problem (3.9) has a unique solution $\bar{\lambda} \in \mathscr{H}$ for any fixed h, as follows from Theorem 2.1. The definition 3.1 and (3.9) imply that

$$\tilde{B}(\bar{\lambda}^h - \bar{\lambda}, \bar{\mu}) = \quad \forall\bar{\mu} \in V(h_0, h). \tag{3.16}$$

Henceforth let $\chi \in V(h_0, h)$ be arbitrary. Denote for brevity

$$\|\bar{\lambda}\|_B = \sum_1^2 \|\bar{\lambda}_{j,j}\| + \|\lambda_3\| .$$

By virtue of Lemma 1, (3.16) and Lemma 2 we may write

$$(3.17) \qquad C\|\bar{\lambda}^h - \chi\|_B^2 \leqq \tilde{B}(\bar{\lambda}^h - \bar{\lambda}, \bar{\lambda}^h - \chi) + \tilde{B}(\bar{\lambda} - \chi, \bar{\lambda}^h - \chi) \leqq \leqq C\|\bar{\lambda} - \chi\|_h \|\bar{\lambda}^h - \chi\|_* .$$

From the inverse inequality (iii) for $\mu_j \in V_h$ we conclude that

$$(3.18) \qquad \|\bar{\mu}\|_* \leqq \|\mu_3\| (1 + 2C) + \sum_1^2 \|\bar{\mu}_{j,j}\| \leqq C\|\bar{\mu}\|_B \quad \forall \bar{\mu} \in V(h_0. h) .$$

Inequalities (3.17), (3.18) result in

$$(3.19) \qquad \|\bar{\lambda}^h - \chi\|_B \leqq C\|\bar{\lambda} - \chi\|_h \quad \forall \chi \in V(h_0, h)$$

For the error of the Galerkin approximation, we may write, making use of (3.

$$(3.20) \qquad \|\bar{\lambda} - \bar{\lambda}^h\|_B \leqq \|\bar{\lambda} - \chi\|_B + \|\chi - \bar{\lambda}^h\|_B \leqq C\|\bar{\lambda} - \chi\|_h \quad \forall \chi \in V(h_0, h)$$

From Theorem 2.1 and (3.1)–(3.3) we deduce that

$$(3.21) \qquad \bar{\lambda}_i = u_i + (\xi_0 d_i^0)^{-1} h^{1+\varepsilon} k_{ij}^0 N_j^0(\boldsymbol{u}) \quad (i = 1, 2, \text{no sum over } i)$$
$$\bar{\lambda}_3 = \mu_0 N_3^0(\boldsymbol{u}).$$

Thus using the assumptions (ii) and (3.12), we have

$$(3.22) \qquad \|\bar{\lambda}_i - \chi_i\|_k \leqq \|u_i - \chi_i\|_k + Ch^{1+\varepsilon} \sum_{j=1}^2 \|u_{j,j}\|_k \leqq$$
$$\leqq C(h_0^{\varkappa_0 - k}\|u_i\|_{\varkappa_0} + h^{1+\varepsilon} \sum_{j=1}^2 \|u_j\|_2) \quad (i = 1, 2, k = 0, 1) ,$$

$$(3.23) \qquad \|\bar{\lambda} - \chi\|_h = \|\bar{\lambda}_3 - \chi_3\| + h^{-1} \sum_{i=1}^2 \|\bar{\lambda}_i - \chi_i\| + \sum_{j=1}^2 (\|\lambda_{3,j} - \chi_{3,j}\| +$$
$$+ \|\bar{\lambda}_{j,j} - \chi_{j,j}\|) \leqq C\{h^{\varkappa}\|\lambda_3\|_{\varkappa} + h^{\varkappa-1}\|\lambda_3\|_{\varkappa} + h^{-1} \sum_{i=1}^2 (h_0^{\varkappa_0}\|u_i\|_{\varkappa_0} +$$
$$+ h^{1+\varepsilon} \sum_{j=1}^2 \|u_j\|_2) + \sum_{j=1}^2 (h_0^{\varkappa_0 - 1}\|u_j\|_{\varkappa_0} + h^{1+\varepsilon} \sum_{i=1}^2 \|u_i\|_2)\} \leqq CQ ,$$

where

$$Q = h^{\varkappa-1} \sum_{i=1}^2 \|u_i\|_{\varkappa+1} + (h^{-1}h_0^{\varkappa_0} + h_0^{\varkappa_0 - 1} + h^{\varepsilon}) \sum_{i=1}^2 \|u_i\|_{\varkappa_0} .$$

From (3.21), (3.20), (3.23) we obtain

$$(3.24) \qquad \|\bar{\lambda}^h_{i,i} - u_{i,i}\| \leqq \|\bar{\lambda}^h_{i,i} - \bar{\lambda}_{i,i}\| + (\xi_0 d_i^0)^{-1} h^{1+\varepsilon} \sum_{j=1}^{2} k^0_{ij} \|N^0_j(\boldsymbol{u})_{,i}\| \leqq$$

$$\leqq \|\bar{\lambda}^h - \bar{\lambda}\|_B + Ch^{1+\varepsilon} \sum_{j=1}^{2} \|u_j\|_2 \leqq$$

$$\leqq C(\|\bar{\lambda} - \chi\|_h + h^{1+\varepsilon} \sum_{j=1}^{2} \|u_j\|_{\varkappa_0}) \leqq CQ \quad (i = 1, 2, \text{no sum}).$$

Using (3.20), (3.23), the Friedrichs inequality for $(\bar{\lambda}^h_i - u_i)$ and also the estimate

$$\|\sum_{j=1}^{2} (k^0_{ij} \bar{\lambda}^h_{j,j} - k^0_{ij} N^0_j(\boldsymbol{u}))\| \leqq C \sum_{j=1}^{2} \|\bar{\lambda}^h_{j,j} - u_{j,j}\| \quad (i = 1, 2),$$

we derive (3.15).

Remark 3.1. Choosing the linear triangular elements for V_{h_0}, V_h, we have $\varkappa = \varkappa_0 = 2$. Then it is suitable to set $\varepsilon = 1$, $h_0 = Ch$ to obtain from Theorem 3.1, that the right hand side of (3.15) is $O(h)$ if $m \geqq 3$.

In case of a smooth boundary Γ, the curved elements along the boundary may be employed (cf. [2].

Remark 3.2. The choice (3.1) with equal parameters ξ_1 and ξ_2 corresponds with the situation that both u_1 and u_2 are of the same importance. The method can be adjusted to the case that e.g. u_1 is more interesting than u_2, setting $\xi_1 > \xi_2$.

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Souhrn

SMÍŠENÁ METODA KONEČNÝCH PRVKŮ, BLÍZKÁ ROVNOVÁŽNÉMU MODELU, V ROVINNÉ PRUŽNOSTI

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Na základě postupu, uvedeného v článcích [1], [2], odvozuje se nová variační formulace druhé základní okrajové úlohy rovinné statické pružnosti. Výchozím bodem je neklasický rozklad diferenciálních operátorů a Friedrichsova transformace. Nový variační princip je ověřen důkazem existence, jednoznačnosti a spojité závislosti řešení na daných veličinách. Dále je ukázána možnost aplikace principu k sestrojení smíšeného modelu konečných prvků a odvozeny odhady chyb. Dvě složky vektoru přibližného řešení konvergují ke složkám posunutí, zatímco třetí složka konverguje k smykovému napětí. Nová metoda představuje tedy model mezi skupinou kompatibilních a hybridních modelů konečných prvků. Vskutku, nový model má tři neznámé (u_1, u_2, τ), zatímco kompatibilní, resp. hybridní modely mají dvě (u_1, u_2), resp. pět $(u_1, u_2, \sigma_x, \sigma_y, \tau)$ neznámých funkcí.

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