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SOME METHODS OF NUMERICAL INTEGRATION OVER A SEMI-INFINITE INTERVAL

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INTRODUCTION

The evaluation of a definite integral over a finite interval using Chebyshev polynomials was in the center of interest of a number of authors.

Clenshaw and Curtis (1960) obtained the polynomial approximation of a function over the zeros of $T_{N+1}(x) - T_{n-1}(x)$, i.e. over the points $x_s = \cos(\pi s/N)$, $s = 0, 1, \ldots, N$, in the form $f(x) \approx \sum_{r=0}^{N}{}'' a_r T_r(x)$ and then evaluated the definite integral performing term by term integration. The double primes in the above summation indicate that the first and last terms are halved and $T_n(x) = \cos n\theta$, $x = \cos\theta$ is the Chebyshev polynomial of the first kind. Filippi (1964) estimated $f(x)$ by a polynomial in $T_n'(x)$ of the form $f(x) \approx \sum_{r=1}^{N} b_r T_r'(x)$, over the zeros of $T_{N+1}'(x)$, i.e. over the points $x_s = \cos(\pi s/N + 1)$, $s = 1, 2, \ldots, N$ and then obtained the value of the definite integral.

Basu (1971) approximated $f(x)$ by a polynomial in $T_n(x)$ over the points $x_s = \cos(2s - 1)\pi/2N$, $s = 1, 2, \ldots, N$, i.e. over the zeros of $T_N(x)$ and then converted the expression in a series of $T_n'(x)$ by means of a conversion formula expressing $T_n(x)$ in terms of $T_n'(x)$. This final expression for $f(x)$, viz. $f(x) \approx \sum_{r=1}^{N}{}' c_r T_r'(x)$, where the prime indicates that the last term is halved, was utilized to find the value of the definite integral.

The problem of evaluating the integral $\int_0^\infty e^{-x} f(x)\,dx$ using a variant of Chebyshev polynomials $T_m^*(e^{-x}) = \cos m\theta$ with $2e^{-x} - 1 = \cos\theta$ was solved by Basu and Kundu (1975). They approximated $f(x)$ by a polynomial $\phi(x)$ by collocation over the zeros of $T_{N+1}^*(e^{-x})$, i.e. over the points $x_s = \log\sec^2(\theta_s/2)$, where $\theta_s = (2s + 1) \,.\, \pi/2(N + 1)$, $s = 0, 1, \ldots, N$, in the form $f(x) \approx \phi(x) = \sum_{m=0}^{N}{}' a_m T_m^*(e^{-x})$ and then evaluated the above integral performing term by term integration. In the present note the above integral is evaluated using $T^*_m(e^{-x})$, the variant Chebyshev polynomial, by several methods analogous to those of Clenshaw and Curtis (1960), Filippi (1964) and Basu (1971) and the methods are verified by numerical examples.

Our main interest lies in the exploration of some new methods of handling the quadrature problem over the semi-infinite interval by exploiting the existing methods for the similar problem over a finite interval. It may be remarked that none of the methods can claim to be the best one including the method proposed by Basu and Kundu (1975). The numerical results reveal that for a certain problem with a pre-assigned number of points, a particular method may offer rapid convergence but the desired accuracy in the result is not achieved while another method in the same case may provide better accuracy at the cost of a few more points.

ANALOGUE OF THE CLENSHAW & CURTIS METHOD

Let a function $f(x)$ defined over $(0, \infty)$ be expanded in a convergent series of the form

$$f(x) = \frac{A_0}{2} + A_1\, T^*_1(e^{-x}) + A_2\, T^*_2(e^{-x}) + \dots . \tag{1}$$

The polynomials $T^*_r(e^{-x})$ being orthogonal in the range with respect to the weight function $\sqrt{[e^{-x}/(1 - e^{-x})]}$, the coefficients are given by

$$A_r = \frac{2}{\pi} \int_0^\infty \sqrt{\left(\frac{e^{-x}}{1 - e^{-x}}\right)}\, T^*_r(e^{-x})\, f(x)\, \mathrm{d}x\, . \tag{2}$$

A useful polynomial approximation to $f(x)$ can be obtained by truncating the infinite series in (1). However, in attempting to find a suitable polynomial approximation to a general function $f(x)$, the integral occurring in (2) cannot be evaluated explicitly and recourse has to be made to approximate methods for evaluating A_r. The most widely used method is the "curve fitting" method. There are several variations of the method. In all these methods we construct the interpolation polynomial $\phi(x)$ by collocation with $f(x)$ at a specified set of points spread out over the semi--infinite interval. These polynomials are then used to evaluate the required integral.

Let the interpolation polynomial to $f(x)$ be expressed in the form

$$f(x) \approx \phi(x) = \frac{a_0}{2} + a_1\, T^*_1(e^{-x}) + a_2\, T^*_2(e^{-x}) + \dots + \frac{a_N}{2}\, T^*_N(e^{-x}) \tag{3}$$

where the interpolation is effected over the zeros of $T^*_{N+1}(e^{-x}) - T^*_{N-1}(e^{-x})$, i.e. over the points $x_s = \log \sec^2 (\theta_s/2)$ with $\theta_s = \pi\, s/N$, $s = 0, 1, \dots, N$.

Now since $T_m^*(e^{-x}) = \cos m\theta$, $2e^{-x} - 1 = \cos\theta$, we get form (3)

$$(4) \qquad f(\log\sec^2(\theta/2)) \approx \frac{a_0}{2} + a_1\cos\theta + a_2\cos 2\theta + \ldots + \frac{a_N}{2}\cos N\theta\,.$$

By using the orthogonality relation

$$(5) \qquad \begin{aligned} \sum_{s=0}^{N} \cos j\theta_s \cos i\theta_s &= 0 && \text{for} \quad i \neq j \\ &= N && \text{for} \quad i = j = 0 \quad \text{or} \quad N \\ &= N/2 && \text{for} \quad i = j \neq 0 \quad \text{or} \quad N \end{aligned}$$

the coefficients are obtained as

$$(6) \qquad a_r \approx \frac{2}{N}\sum_{s=0}^{N}{}'' f(\log\sec^2(\theta_s/2))\cos r\theta_s\,.$$

Hence from (3), we can write

$$(7) \qquad \int_0^\infty e^{-x} f(x)\,\mathrm{d}x \approx \sum_{r=0}^{N}{}'' a_r \int_0^\infty e^{-x}\, T_r^*(e^{-x})\,\mathrm{d}x\,.$$

Since

$$(8) \qquad \begin{aligned} \int_0^\infty e^{-x}\, T_r^*(e^{-x})\,\mathrm{d}x &= 0\,, && \text{for} \quad r \quad \text{odd} \\ &= \frac{1}{1-r^2}\,, && \text{for} \quad r \quad \text{even} \end{aligned}$$

we obtain

$$(9) \qquad \int_0^\infty e^{-x} f(x)\,\mathrm{d}x \approx \sum_{p=0}^{[N/2]}{}'' \frac{a_{2p}}{1-4p^2}$$

where $[N/2]$ means the largest integer contained in $N/2$ for a given N. The double primes indicate that the first term is halved and the last term is also halved if N is even.

ANALOGUE OF THE FILIPPI METHOD

In this method we express $f(x)\,e^{-x}$ in a series of $T_r^{*\prime}(e^{-x})$ and the equations similar to (1) and (2) are obtained as

$$(10) \qquad f(x)\,e^{-x} = A_1\, T_1^{*\prime}(e^{-x}) + A_2\, T_2^{*\prime}(e^{-x}) + \ldots$$

where

$$(11) \qquad A_r = \frac{2}{\pi r^2}\int_0^\infty \sqrt{(e^x - 1)}\; T_r^{*\prime}(e^{-x})\, f(x)\, e^{-x}$$

and $\sqrt{(e^x - 1)}$ is the corresponding weight function.

Now the interpolation polynomial is given by

$$(12) \qquad f(x)\, e^{-x} \approx \phi(x) = a_1\, T_1^{*\prime}(e^{-x}) + a_2\, T_2^{*\prime}(e^{-x}) + \ldots + a_N\, T_N^{*\prime}(e^{-x}),$$

the expansion is effected over the zeros of $e^x\, T_{N+1}^{*\prime}(e^{-x})$, i.e. over the points $x_s = \log \sec^2 (\theta_s/2)$ where $\theta_s = \pi\, s/(N+1)$, $s = 1, 2, \ldots, N$.

Again changing the variable, we get

$$(13) \qquad -1/2 \sin\theta\, f(\log \sec^2 (\theta/s)) \approx a_1 \sin\theta + 2a_2 \sin 2\theta + \ldots + N a_N \sin N\theta .$$

In consequence of the relations

$$(14) \qquad \sum_{s=1}^{N} \sin i\theta_s \sin j\theta_s = 0 \quad \text{if} \quad i \neq j\,, \quad i = j = 0$$
$$= \frac{N+1}{2} \quad \text{if} \quad i = j$$

the coefficients in (12) are given by

$$(15) \qquad a_r \approx -\frac{1}{r(N+1)} \sum_{s=1}^{N} f(\log \sec^2 (\theta_s/2)) \sin\theta_s \sin r\theta_s$$

and so from (12), we get

$$(16) \qquad \int_0^\infty e^{-x} f(x)\, \mathrm{d}x \approx \sum_{r=1}^{N} a_r \int_0^\infty T_r^{*\prime}(e^{-x})\, \mathrm{d}x .$$

Now since

$$(17) \qquad \int_0^\infty T_r^{*\prime}(e^{-x})\, \mathrm{d}x = -2 \quad \text{for} \quad r \quad \text{odd}$$
$$= 0 \quad \text{for} \quad r \quad \text{even},$$

(16) reduces to

$$(18) \qquad \int_0^\infty e^{-x} f(x)\, \mathrm{d}x \approx -2 \sum_{p=1}^{[(N+1)/2]} a_{2p-1} .$$

ANALOGUE OF THE BASU METHOD

Let $f(x)$ be approximated by a polynomial

$$(19) \qquad f(x) \approx \phi(x) = \frac{a_0}{2} + a_1\, T_1^*(e^{-x}) + \ldots + a_{N-1}\, T_{N-1}^*(e^{-x})$$

by collocation over the zeros of $T_N^*(e^{-x})$, i.e. over the points $x_s = \log \sec^2 (\theta_s/2)$ with $\theta_s = (2s-1)\,\pi/2N$, $s = 1, 2, \ldots, N$.

By virtue of the conversion formula

$$(20) \qquad T_j^*(e^{-x}) = \frac{e^x}{4}\left[\frac{T_{j-1}^{*\prime}(e^{-x})}{j-1} - \frac{T_{j+1}^{*\prime}(e^{-x})}{J+1}\right],$$

(19) reduces to

$$(21) \qquad f(x)\, e^{-x} \approx b_1\, T_1^{*\prime}(e^{-x}) + b_2\, T_2^{*\prime}(e^{-x}) + \ldots + \frac{b_N}{2}\, T_N^{*\prime}(e^{-x})$$

By making a change in the variable, (21) reduces to

$$(22) \qquad -\frac{1}{2}\sin\theta\, f(\log\sec^2(\theta/2)) \approx b_1 \sin\theta + 2b_2 \sin 2\theta + \ldots + N\,\frac{b_N}{2}\sin N\theta$$

and using the orthogonality relation

$$(23) \qquad \begin{aligned} \sum_{s=1}^{N} \sin i\theta_s \sin j\theta_s &= 0 \quad &&\text{if} \quad i \neq j\,, \quad i = j = 0 \\ &= N \quad &&\text{if} \quad i = j = N \\ &= N/2 \quad &&\text{if} \quad i = j \neq N \end{aligned}$$

we get

$$(24) \qquad b_r = -\frac{1}{rN}\sum_{s=1}^{N} f(\log\sec^2(\theta_s/2))\sin\theta_s \sin r\theta_s$$

and so (21) and (17) give

$$(25) \qquad \int_0^\infty e^{-x} f(x)\,\mathrm{d}x = -2\sum_{p=1}^{[(N+1)/2]}{}' \; b_{2p-1}$$

where the prime indicates that the last term is halved if N is odd.

Table 1

	Basu Method	Filippi Method	Clenshaw-Curtis Method
N	I	I	I
8	·206319673	·206507253	·205787725
10	·206331066	·206444653	·206009810
12	·206336468	·206411808	·206123124
14	·206339410	·206392622	·206188262
16	·206341171	·206380526	·206228940
⋮			
64		·206346244	·206340246

NUMERICAL EXAMPLES

We consider the following numerical examples:

(a) $$I = \int_0^\infty \frac{e^{-x}}{x+4}\,dx = 0{\cdot}206346\,,$$

(b) $$I = \int_0^\infty e^{-x}\sin x\,dx = 0{\cdot}5$$

(c) $$I = \int_{-\infty}^\infty e^{-x^2}\cos x\,dx = 1{\cdot}3803884\,.$$

Table 2

	Basu Method	Filippi Method	Clenshaw-Curtis Method
N	I	I	I
8	·495135092	·493546576	·512665465
9	·497000511	·491899248	·509475562
10	·499664788	·491786136	·504458705
11	·500990525	·491959675	·503475114
⋮			
33			·499996306
34		·499989332	

Table 3

	Basu Method	Filippi Method	Clenshaw-Curtis Method
N	I	I	I
12	1·38037380	1·380379575	1·380398500
13	1·38045055	1·380447705	1·380389740
14	1·38038243	1·380430232	1·380386691
15	1·38037246	1·380415192	1·380383993
16	1·38038750	1·380402806	1·380387698
17	1·38037767	1·380391688	1·380387160
⋮			
20			1·38038844
⋮			
26		1·3803884	

The numerical details of the above examples are contained in Tables 1, 2 and 3 respectively.

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Souhrn

NĚKTERÉ METODY NUMERICKÉ INTEGRACE NA POLOPŘÍMCE

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V článku se využívají některé metody numerické integrace na ohraničeném intervalu (Clenshaw a Curtis 1960, Filippi 1964, Basu 1971) k navržení metod numerické integrace na polopřímce. Užívá se přitom podobné techniky jako v dřívějším článku obou autorů. (Apl. Mat. 20 (1975), pp. 216—221).

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