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AN ALGEBRAIC ADDITION-THEOREM FOR WEIERSTRASS' ELLIPTIC FUNCTION AND NOMOGRAMS

AKIRA MATSUDA

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1. INTRODUCTION

As is well known, a determinantal form of the addition-theorem for Weierstrass' $\wp$ function represents a nomogram for $u + v + w = 0$. In this paper, the author uses another form of the addition-theorem for $\wp$ function involving no derivative $\wp'$ [1].

By a dual transformation, concurrent charts are transformed into an alignment chart where three scales coincide and a tangential contact chart consisting of a family of circles, which represent the relation $u + v + w = 0$. In this case the addition-theorem for $\wp$ function stated above is used.

2. DUAL TRANSFORMATION METHOD FOR CONSTRUCTING NOMOGRAM WITH A COMMON BASE

Consider the cubic equation in t

$$t^3 + u(x, y)\, t^2 + v(x, y)\, t + w(x, y) = 0\,, \tag{2.1}$$

where $u(x, y)$, $v(x, y)$ and $w(x, y)$ are functions of real variables x and y, and of class C^1 with respect to x, y. One of the functions $u(x, y)$, $v(x, y)$ and $w(x, y)$ may be a constant. Furthermore, we assume that the equation (2.1) is not separated into a function of x, y only and that of t only, that is, it does not take the form $f_1(x, y) = f_2(t)$.

Regarding t as a parameter, (2.1) represents a family of curves or, in a special case, a family of straight lines in xy-plane. We now consider a region of points $P(x, y)$ at which (2.1) has three distinct real roots t, and we denote the region by D.

For a given point $P(x, y)$ in D, let three distinct real roots of (2.1) be t_i $(i = 1, 2, 3)$. By the relations between roots and coefficients of a cubic equation, we have

$$(2.2) \qquad \begin{aligned} t_1 + t_2 + t_3 &= -u(x, y)\,, \\ t_1t_2 + t_2t_3 + t_3t_1 &= v(x, y)\,, \\ t_1t_2t_3 &= -w(x, y)\,. \end{aligned}$$

Assuming that x and y can be eliminated from the above expressions, we obtain an expression

$$(2.3) \qquad F(t_1 + t_2 + t_3,\ t_1t_2 + t_2t_3 + t_3t_1,\ t_1t_2t_3) = 0\,.$$

A given point $P(x, y)$ in D determines three distinct values t_i $(i = 1, 2, 3)$, corresponding to which we consider three curves c_i $(i = 1, 2, 3)$ represented by the following equations

$$t_i^3 + u(X, Y)\,t_i^2 + v(X, Y)\,t_i + w(X, Y) = 0 \quad (i = 1, 2, 3)\,,$$

where X, Y denote current coordinates. Then the curves c_i $(i = 1, 2, 3)$ pass through the point $P(x, y)$. Furthermore, the curves are different from each other; indeed, the curves are identical if and only if (2.1) takes the form $f_1(x, y) = f_2(t)$, but this does not occur by the assumption. Hence (2.1) forms a concurrent chart satisfying the functional relation (2.3) by itself.

Next, according to the envelope method developed by the author and K. Morita [2], we transform the curves (2.1) in xy-plane into a figure in $\bar{x}\bar{y}$-plane by the transformation

$$(2.4) \qquad (ax + hy + g)\,\bar{x} + (hx + by + f)\,\bar{y} + gx + fy + c = 0$$

where

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0\,,$$

which is an equation of a polar with respect to the general conic.

Assuming that (2.1) can be solved for y, we have $y = y(x, t)$, and then substituting this into (2.4) we get

$$(2.5) \qquad \begin{aligned} &\{ax + y(x, t)\,h + g\}\,\bar{x} + \{hx + y(x, t)\,b + f\}\,\bar{y} + \\ &\qquad + gx + y(x, t) f + c = 0\,. \end{aligned}$$

Differentiating this expression partially with respect to x we have

$$\left(a + h\frac{\partial y}{\partial x}\right)\bar{x} + \left(h + b\frac{\partial y}{\partial x}\right)\bar{y} + g + f\frac{\partial y}{\partial x} = 0\,,$$

and we eliminate x from (2.5) and the above expression.

Then we obtain, generally, an equation in the form

$$\bar{f}(\bar{x}, \bar{y}, t) = 0 ,$$

which expresses a tangential contact chart consisting of one family of curves in $\bar{x}\bar{y}$-plane. In the special case when (2.1) represents a family of straight lines, we obtain a pair of equations in the form

$$\bar{x} = \bar{x}(t) , \quad \bar{y} = \bar{y}(t) ,$$

which expresses an alignment chart where three scales coincide in $\bar{x}\bar{y}$-plane. Both the charts represent the relation (2.3).

3. ALIGNMENT CHART FOR $u_1 + u_2 + u_3 = 0$

We shall consider the equation

$$t^3 - \frac{x^2}{4}t^2 + \frac{2xy - g_2}{4}t - \frac{y^2 + g_3}{4} = 0 \tag{3.1}$$

where g_2 and g_3 are real constants, which is a special case of (2.1). Solving (3.1) for y, we obtain

$$y = tx \pm \sqrt{(4t^3 - g_2 t - g_3)} . \tag{3.2}$$

Here we assume that t takes real values satisfying

$$4t^3 - g_2 t - g_3 > 0 . \tag{3.3}$$

Regarding t as a parameter, the equation (3.1), which is equivalent to (3.2), represents a family of straight lines in xy-plane.

From (2.2) we have

$$t_1 + t_2 + t_3 = \frac{x^2}{4} ,$$

$$t_1 t_2 + t_2 t_3 + t_3 t_1 = \frac{xy}{2} - \frac{g_2}{4} ,$$

$$t_1 t_2 t_3 = \frac{y^2}{4} + \frac{g_3}{4} .$$

Eliminating x and y from the expressions we obtain

$$4(t_1 + t_2 + t_3)\left(t_1 t_2 t_3 - \frac{g_3}{4}\right) = \left(t_1 t_2 + t_2 t_3 + t_3 t_1 + \frac{g_2}{4}\right)^2 . \tag{3.4}$$

As we have discussed in § 2, the expression (3.1) represents a concurrent chart satisfying the relation (3.4).

Next, we transform (3.1) into a figure in $\bar{x}\bar{y}$-plane by the transformation expression

$$x\bar{x} - \bar{y} - y = 0, \tag{3.5}$$

which is an equation of a polar with respect to the parabola $x^2 = 2y$. Substituting (3.2) into (3.5) we have

$$x\bar{x} - \bar{y} - tx \mp \sqrt{(4t^3 - g_2 t - g_3)} = 0.$$

Differentiating the above expression partially with respect to x, we get $\bar{x} = t$; and substituting this into the above expression we obtain together with the last equation

$$\bar{x} = t, \quad \bar{y} = \mp \sqrt{(4t^3 - g_2 t - g_3)}. \tag{3.6}$$

Eliminating t from the equations we have

$$\bar{y}^2 = 4\bar{x}^3 - g_2 \bar{x} - g_3.$$

The pair of equations (3.6) represents an alignment chart satisfying the functional relation (3.4) with the restriction (3.3).

Here we use a form of the addition-theorem for Weierstrass' $\wp$ function [1]: when $u_1 \pm u_2 \pm u_3 \equiv 0 \pmod{2\omega_1, 2\omega_3}$, then

$$4\{\wp(u_1) + \wp(u_2) + \wp(u_3)\}\left\{\wp(u_1)\wp(u_2)\wp(u_3) - \frac{g_3}{4}\right\} = \tag{3.7}$$

$$= \left\{\wp(u_1)\wp(u_2) + \wp(u_2)\wp(u_3) + \wp(u_3)\wp(u_1) + \frac{g_2}{4}\right\}^2.$$

It is clear that the converse of this theorem is true.

Now, we put

$$t = \wp(u), \tag{3.8}$$

which is equivalent to $u = \int_t^\infty dx/\sqrt{(4x^3 - g_2 x - g_3)}$, and mark the value of u instead of t on the scale (3.6). Setting $t_i = \wp(u_i)$ $(i = 1, 2, 3)$, we obtain (3.7) from (3.4). Hence in the addition-theorem stated above the relation (3.4) can be replaced without loss of generality by the condition that one of the following relations holds:

$$u_1 + u_2 + u_3 = 0 \text{ or period}, \tag{3.9}$$

$$u_1 + u_2 - u_3 = 0 \text{ or period}. \tag{3.10}$$

In what follows, we continue under the initial condition that the value of u starts from zero at $\bar{x} = \infty$. Since the scale (3.6) is symmetrical with respect to the $\bar{x}$-axis, two points whose abscissas are equal have the same value of u. Hence we can state the following facts about values u_i $(i = 1, 2, 3)$ marked at three points which are intersections of the scale and a straight line:

When all the three points lie to the same side of the $\bar{x}$-axis, then (3.9) holds; and when one of them lies on the opposite side than the others, then (3.10) holds. This can be easily seen by considering the limit case $u_1 \to 0$ when the straight line passing through the three points becomes perpendicular to the $\bar{x}$-axis. Indeed, in Fig. 1, in case (a) we have $u_1 + u_2 + u_3 = 2u_3 =$ period and in case (b) we have $u_1 + u_2 - u_3 = 0$. Therefore, if we mark the value of u on the curve so that $u > 0$ when $\bar{y} > 0$ and $u < 0$ when $\bar{y} < 0$, then the relation (3.9) always holds.

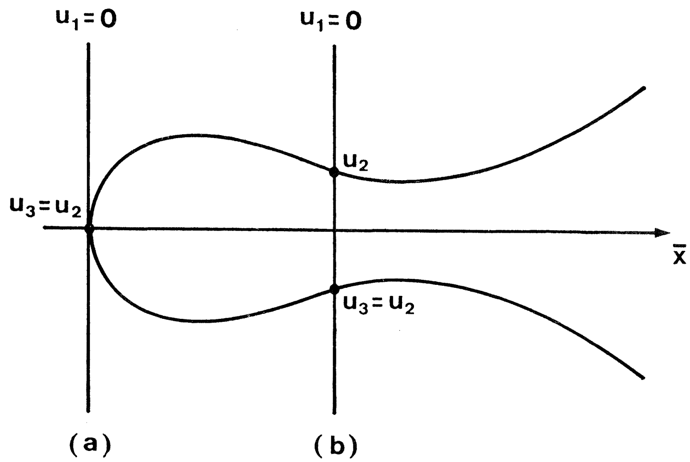

Fig. 1.

The nomogram thus obtained is the same as that found in Epstein's work [3], in which two examples are illustrated.

4. TANGENTIAL CONTACT CHART FOR $u_1 + u_2 + u_3 = 0$

In this section, we shall consider the following equation instead of (3.1):

$$(4.1) \qquad t^3 - \frac{x^2}{4a^2(x^2+1)} t^2 + \left\{\frac{xy}{2a^2(x^2+1)} - \frac{g_2}{4}\right\} t - \left\{\frac{y^2}{4a^2(x^2+1)} + \frac{g_3}{4}\right\} = 0\,,$$

where a (>0) is a constant. Multiplying both sides of the above equation by $4a^2(x^2 + 1)$ and rearranging with respect to y, we have

$$(4.2) \qquad y^2 - 2txy + t^2x^2 - a^2(4t^3 - g_2 t - g_3)(x^2 + 1) = 0\,.$$

Here we assume that t takes real values satisfying (3.3); setting

$$a^2(4t^3 - g_2 t - g_3) = r^2 \quad (r > 0),$$

then (4.2) becomes

$$y^2 - 2txy + t^2x^2 - r^2(x^2 + 1) = 0.$$

Solving this expression for y we obtain

$$y = tx \pm r\sqrt{(x^2 + 1)}. \tag{4.3}$$

Regarding t as a parameter, the equation (4.1), which is equivalent to (4.3), represents a family of hyperbolas in xy-plane.

From (2.2) we have

$$t_1 + t_2 + t_3 = \frac{x^2}{4a^2(x^2 + 1)},$$

$$t_1t_2 + t_2t_3 + t_3t_1 = \frac{xy}{2a^2(x^2 + 1)} - \frac{g_2}{4},$$

$$t_1t_2t_3 = \frac{y^2}{4a^2(x^2 + 1)} + \frac{g_3}{4}.$$

Eliminating x and y from the expressions we again have the relation (3.4) and, likewise (3.1), the equation (4.1) also represents a concurrent chart satisfying (3.4).

Next, we transform (4.1) into a figure in $\bar{x}\bar{y}$-plane by the expression (3.5). Substituting (4.3) into (3.5) we have

$$(\bar{x} - t)\,x - \bar{y} = \pm r\sqrt{(x^2 + 1)}.$$

Squaring both sides and rearranging with respect to x, we obtain

$$\{(\bar{x} - t)^2 - r^2\}\,x^2 - 2(\bar{x} - t)\,\bar{y}x + \bar{y}^2 - r^2 = 0. \tag{4.4}$$

Differentiating this expression partially with respect to x, we have

$$x = \frac{(\bar{x} - t)\,\bar{y}}{(\bar{x} - t)^2 - r^2}.$$

Then we substitute this into (4.4), after some calculations we cancel the factor r^2 and obtain

$$(\bar{x} - t)^2 + \bar{y}^2 = r^2$$

or

$$(\bar{x} - t)^2 + \bar{y}^2 = \{a\sqrt{(4t^3 - g_2t - g_3)}\}^2, \tag{4.5}$$

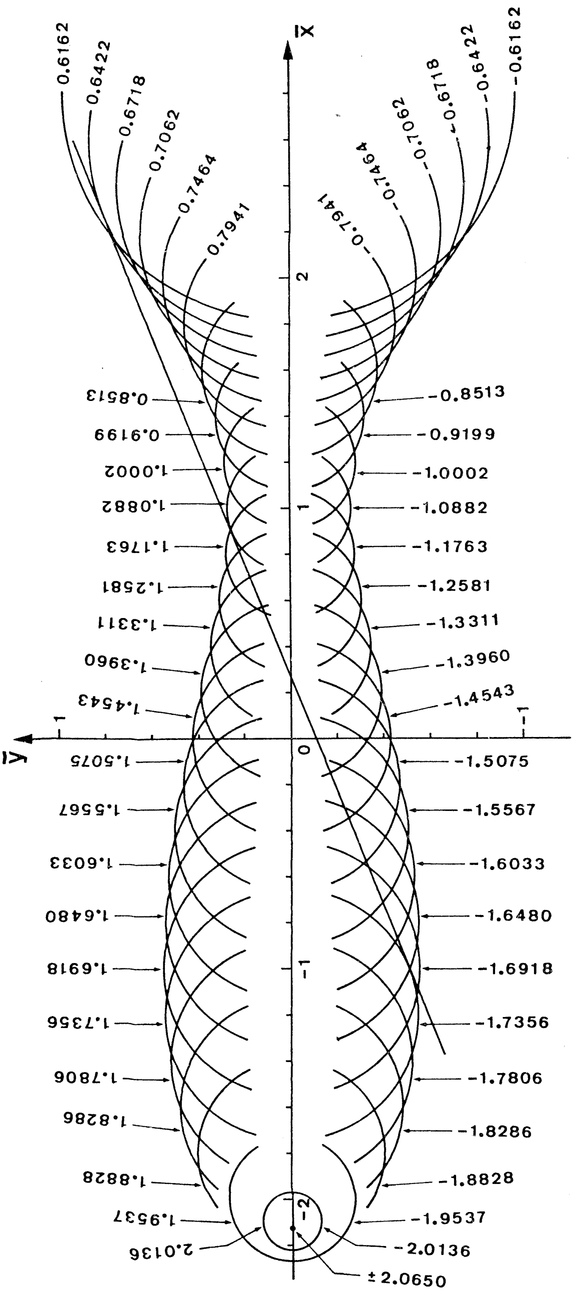

Fig. 2. Chart of $u_1 + u_2 + u_3 = 0$. The figure shows that $u_1 = 0{\cdot}6422$, $u_2 = 1{\cdot}0882$, $u_3 = -1{\cdot}7356 \Rightarrow u_1 + u_2 + u_3 \doteq 0$.

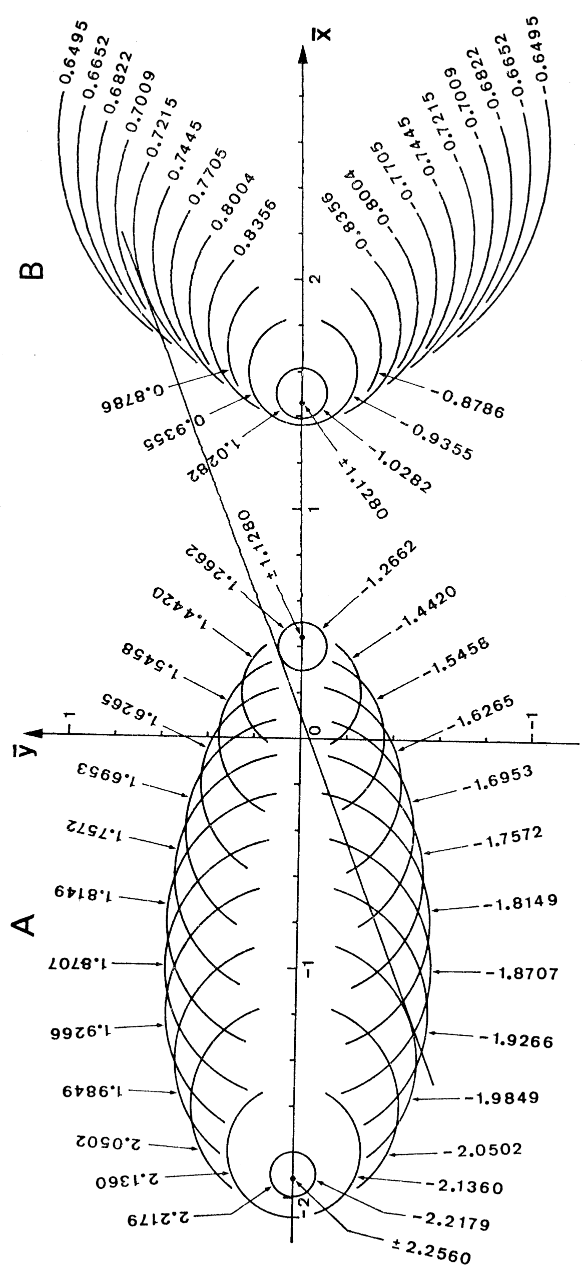

Fig. 3. Chart of $u_1 + u_2 + u_3 = 0$. The figure shows that $u_1 = 0{\cdot}7215$, $u_2 = 1{\cdot}2662$, $u_3 = -1{\cdot}9849 \Rightarrow u_1 + u_2 + u_3 \doteq 0$.

which expresses a family of circles with the center on the $\bar{x}$-axis and represents a tangential contact chart satisfying the functional relation (3.4) with the restriction (3.3).

In this chart, as in the case of § 3, we replace t by u according to the expression (3.8), and mark the value of u on a semicircle so that $u > 0$ when $\bar{y} > 0$ and $u < 0$ when $\bar{y} < 0$; then the relation (3.9) holds.

As in the case of the aligment chart there are two cases according as whether the equation

$$4x^3 - g_2 x - g_3 = 0 \tag{4.6}$$

has one real root or three real roots, and we shall show them in the following examples.

Example 1. When $g_2 = 12$ and $g_3 = -13$, (4.6) has only one real root $-2{\cdot}12777$ and the period is $4{\cdot}1300$. The chart with $a = 0{\cdot}12$ is shown in Fig. 2. Of course, each of semi-circles has many values of u, but in this figure only one value is marked, under the initial condition that u starts from zero at $\bar{x} = \infty$.

Example 2. When $g_2 = 12$ and $g_3 = -5$, (4.6) has three distinct real roots the largest of which is $1{\cdot}46523$, and the period is $2{\cdot}2560$. The chart with $a = 0{\cdot}16$ is shown in Fig. 3. The group of circles B alone forms a complete nomogram, and it is possible to construct such a nomogram by choosing a smaller value of a.

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Souhrn

ALGEBRAICKÁ ADIČNÍ VĚTA PRO WEIERSTRASSOVU ELIPTICKOU FUNKCI A NOMOGRAMY

AKIRA MATSUDA

Vyšetřuje se duální transformace, převádějící průsečíkový nomogram zobrazující jedinou rovnici buď na spojnicový nomogram nebo na nomogram s dotykovými

kontakty. Pomocí této transformace je sestrojen spojnicový nomogram, v němž tři stupnice splývají, a nomogram s dotykovými kontakty složený ze soustavy kružnic, které zobrazují vztah $u + v + w = 0$. V tomto případě je použita jistá forma adiční věty pro Weierstrassovu funkci $\wp$, která neobsahuje derivaci $\wp'$.

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